Terrorists never congregate in even numbers (or: Some strange results in fragmentation-coalescence[1])

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[1] Joint work with Steven Pagett, Tim Rogers

Terrorists, consensus and biological clustering

- Consider a collection of n identical particles (terrorists/opinions), grouped together into some number of clusters (cells/consensus). We define a stochastic dynamical process as follows:
- Every k-tuple of clusters coalesces at rate $\alpha(k)n^{1-k}$, independently of everything else that happens in the system. The coalescing cells are merged to form a single cluster with size equal to the sum of the sizes of the merged clusters.
- Clusters fragment (terrorist cells are dispersed/consensus breaks) at constant rate $\lambda > 0$, independently of everything else that happens in the system. Fragmentation of a cluster of size ℓ results in ℓ 'singleton' clusters of size one.

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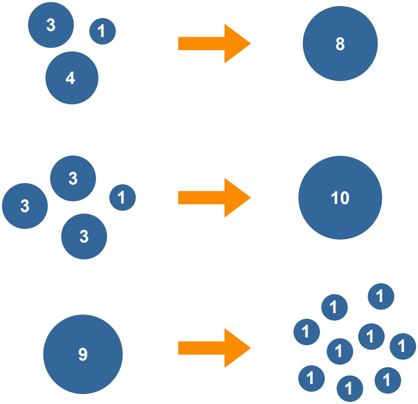

Model history (but only for dyadic coalescence)

- This model is a variant of the one presented in:
  Bohorquez, Gourley, Dixon, Spagat & Johnson (2009) Common ecology quantifies human insurgency *Nature* **462**, 911-914.
- It is also related to:
  Ráth and Tóth (2009) Erdős-Rènyi random graphs + forest fires = self-organized criticality, **14** Paper no. 45, 1290-1327.
- Without fragmentation, the model falls within the domain of study of Smoluchowski coagulation equations, originally devised to consider chemical processes occurring in polymerisation, coalescence of aerosols, emulsication, flocculation.
- In all cases: one is interested in the macroscopic behaviour of the model (large n), in particular in exploring universality properties.

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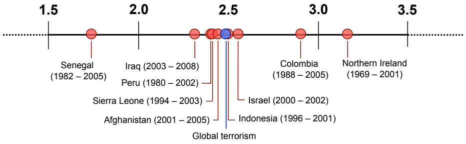
Heavy-tailed terrorism

- In the insurgency model, two blocks merge if a terrorist in each block make a connection, which they do at a fixed rate. This means that coalescence is more likely for a big terrorist cell.
- The macroscopic-scale, large time limit of the insurgency model for a "slow rate of fragmentation" shows that the distribution of block size is heavy tailed:

$$\text{``}\mathbb{P}(\text{typical block} = x) \approx \text{const.} \times x^{-\alpha}, \qquad x \to \infty.\text{''}$$

- Taken from Bohorquez, Gourley, Dixon, Spagat & Johnson (2009):



Back to our model: Generating function

- For each $n \in \mathbb{N}$, and $k \in \{1, \dots, n\}$, the state of the system is specified by the number of clusters of size k at time t.
- Introduce the random variables

$$w_{n,k}(t) := \frac{1}{n} \#\{\text{clusters of size } k \text{ at time } t\}, \quad 1 \leq k \leq n.$$

- Rather than working with these quantities directly, use the empirical generating function

$$G_n(x,t) = \sum_{k=1}^{n} x^k w_{n,k}(t), \qquad n \geq 1, x \in (0,1), t \geq 0$$

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Theorem 1

Theorem

Suppose that the coalescence rates $\alpha : \mathbb{N} \to \mathbb{R}^+$ satisfy

$$\alpha(k) \leq \exp(\gamma k \ln \ln(k)), \qquad \forall k,$$

where $\gamma < 1$ is an arbitrary constant. Let $G : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ be the solution of the deterministic initial value problem

$$G(x,0) = x,$$

$$\frac{\partial G}{\partial t}(x,t) = \lambda(x - G(x,t)) + \sum_{k=2}^{\infty} \frac{\alpha(k)}{k!} \left(G(x,t)^k - kG(1,t)^{k-1} G(x,t) \right).$$

Then $G_n(x,t)$ converges to $G(x,t)$ in L^2, uniformly in x and t, as $n \to \infty$, that is

$$\sup_{x \in [0,1], t \geq 0} \mathbb{E}\left[(G(x,t) - G_n(x,t))^2 \right] \to 0, \quad as \quad n \to \infty.$$

- The next theorem deals with the stationary cluster size distribution.
- Let

$$p_{n,k}(t) := \frac{\#\{\text{clusters of size } k \text{ at time } t\}}{\#\{\text{clusters at time } t\}}, \quad 1 \leq k \leq n.$$

- Define

$$p_k := \lim_{t\to\infty} \lim_{n\to\infty} p_{n,k}(t)\,,$$

as a distributional limit, which exists thanks to the previous theorem and that

$$\sum_{k=1}^{n} x^k p_{n,k}(t) = \frac{G_n(x,t)}{G_n(1,t)}, \qquad n \geq 1, x \in (0,1), t \geq 0\,.$$

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Theorem 2

Theorem

If α satisfies

$$\alpha(k) \leq \exp(\gamma k \ln\ln(k))\,, \qquad \forall k\,,$$

and m is the smallest integer such that $\alpha(m) > 0$, then the stationary cluster size distribution obeys

$$\lim_{\lambda \searrow 0} p_k = \begin{cases} \frac{1}{k}\left(\frac{m-1}{m}\right)^k \left(\frac{1}{m}\right)^{\frac{k-1}{m-1}} \binom{m\left(\frac{k-1}{m-1}\right)}{\frac{k-1}{m-1}} & \text{if } m-1 \text{ divides } k-1 \\ 0 & \text{otherwise} \end{cases}$$

and in particular, as $k \to \infty$

$$\lim_{\lambda \searrow 0} p_k \approx \begin{cases} k^{-3/2} & \text{if } m-1 \text{ divides } k-1 \\ 0 & \text{otherwise.} \end{cases}$$

Terrorists never congregate in even numbers

- In the large n and small λ limit we will see no clusters of even size whatsoever in the stationary distribution.
- The model has the apparently paradoxical feature that clusters of even size are vanishingly rare, despite the fact that $\lim_{\lambda \searrow 0} p_1 \approx 2/3$.
- This is a consequence of the weight of the tail of the cluster size distribution.
- The universal exponent 3/2 suggests a typical cluster size $\sum_1^n kp_k \approx O(n^{1/2}) \Rightarrow \sharp$ clusters $\approx O(n^{1/2})$.
  Coalescence of triples: $\binom{n^{1/2}}{3} \times \alpha(3)n^{1-3} \approx O(n^{-1/2})$
  Coalescence of quadruples: $\binom{n^{1/2}}{4} \times \alpha(4)n^{1-4} \approx O(n^{-1})$
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Some more strange results for exchangeable fragmentation-coalescence models[2]

[2] Joint work with Steven Pagett, Tim Rogers and Jason Schweinsberg.

Kingman n-coalescent

- The Kingman n-coalescent is an (exchangeable) coalescent process on the space of partitions of $\{1, \cdots, n\}$ denoted by

$$\Pi^{(n)}(t) = (\Pi_1^{(n)}(t), \cdots, \Pi_{N(t)}^{(n)}(t)), \qquad t \geq 0,$$

where $N(t)$ is the number of blocks at time t and $\Pi_i^{(n)}(t)$ is the elements of $\{1, \cdots, n\}$ that belong to the i-th block.

- Blocks merge in pairs, with a fixed rate c of any two blocks merging.
- Both $N(t)$, $t \geq 0$, is a Markov process and $\Pi^{(n)}$ is a Markov process.
- The notion of the Kingman coalescent can be mathematically extended in a consistent way to the space of partitions on $\mathbb{N}$. That is to say the pathwise limit

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- Included in this statement is the ability of Π to "come down from infinity".
- (Slighly) more precisely: if the initial configuration is the trivial partition

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(so that $N(0) = \infty$) then $N(t) < \infty$ almost surely, for all $t > 0$.
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Kingman insurgents meet counter terrorism

- At rate μ, each block in the system is shattered into singletons.
- When there are a finite number of blocks, each block must contain an infinite number of integers and hence when a block shatters, the system jumps back up to "infinity".
- If started with a finite number of blocks, the resulting process is still a Markov process on the space of partitions of $\mathbb{N}$ until the arrival of the first fragmentation.
- Can process be "extended" to a Markov process on $\mathbb{N} \cup \{+\infty\}$? Can the process "come down from infinty"?
- This would allow us to consider the process as recurrent on $\mathbb{N} \cup \{+\infty\}$.

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A remarkable phase transition

- We can continue to use the same notation as before with

$$\Pi(t) = (\Pi_1(t), \cdots \Pi_{N(t)}), \qquad t \geq 0,$$

  as a partitioned-valued process.
- A little thought (exchangeability!) shows that both $N(t)$ and $M(t) := 1/N(t)$, $t \geq 0$, are Markov process (with a possible absorbing state at $+\infty$ resp. 0).
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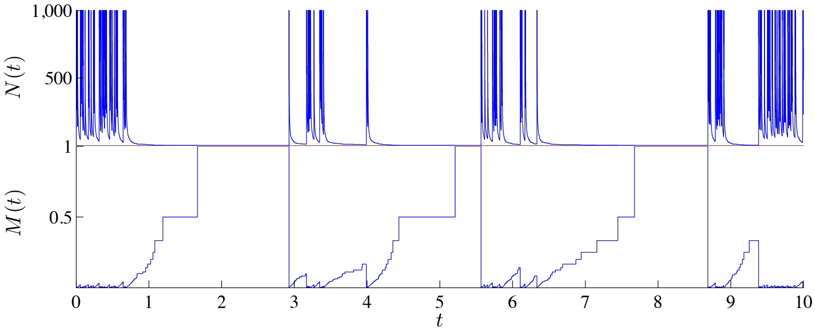
A remarkable phase transition

Theorem

If $\mu/c < 1/2$, then M is a recurrent strong Markov process on $\{1/n : n \in \mathbb{N}\} \cup \{0\}$.
(Comes down from infinity.)

If $\mu/c \geq 1/2$, then 0 is an absorbing state for M.
(Does not come down from infinity.)

Coming down from infinity



Stationary distribution

Theorem

Let $\lambda/c < 1/2$, then M has stationary distribution given by

$$\rho_M(1/k) = \frac{(1-2\lambda/c)}{\Gamma(2\lambda/c)} \frac{\Gamma(k-1+2\lambda/c)}{\Gamma(k+1)}, \quad k \in \mathbb{N}.$$

In particular $\rho_M(0) = 0$.

Thank you!