Existence of a phase transition of the interchange process on the Hamming graph

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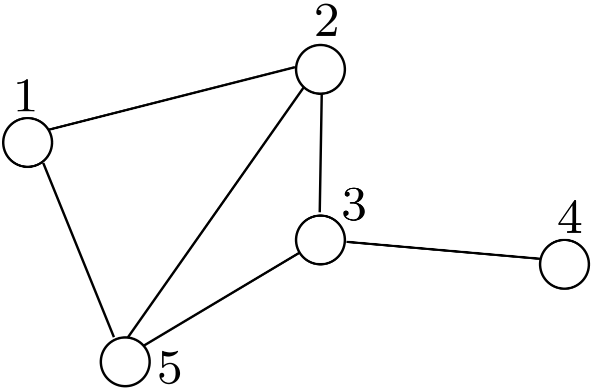
Place a particle on each vertex v. At rate 1 select an edge uniformly at random and swap the two particles across that edge.



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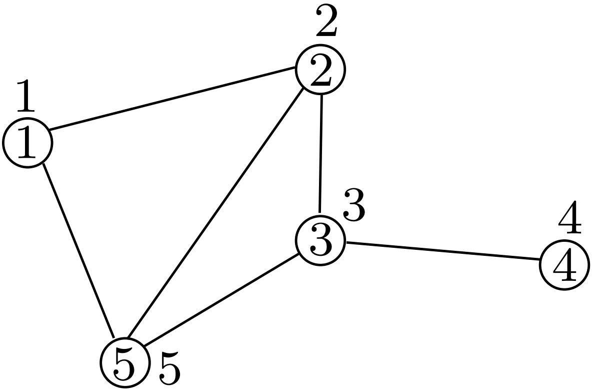

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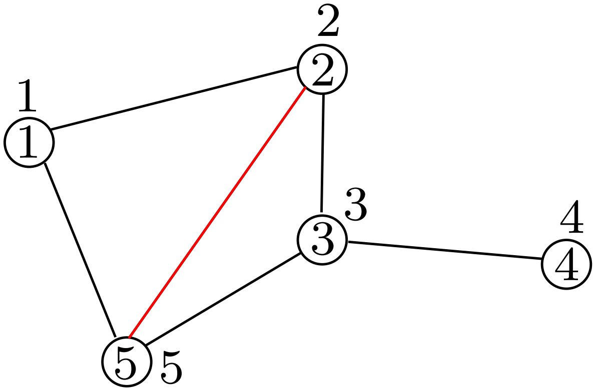

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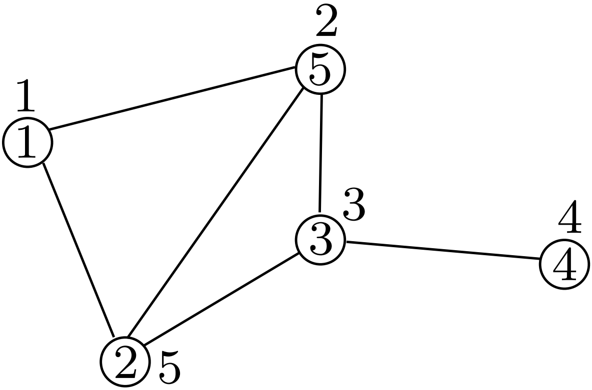

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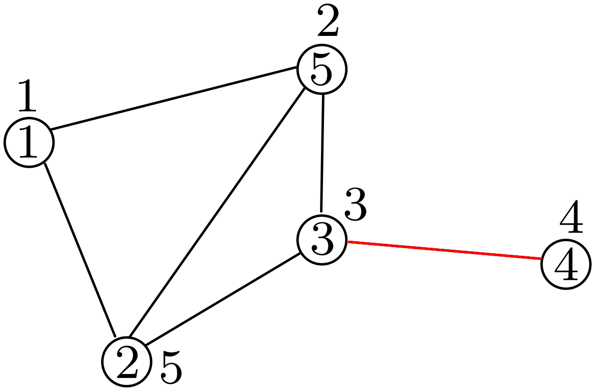

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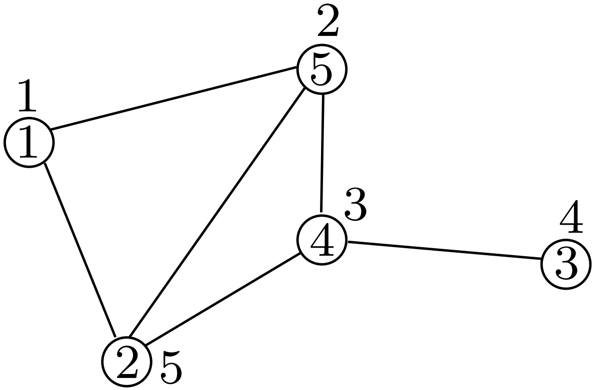

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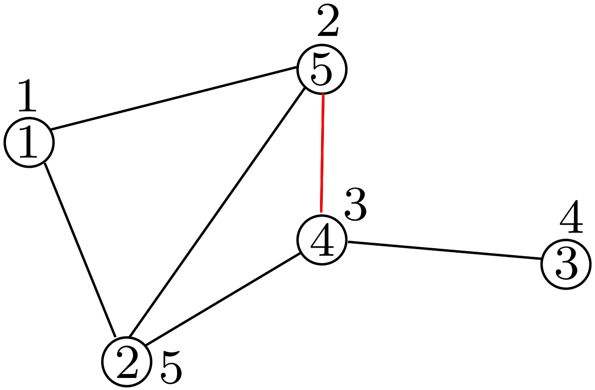

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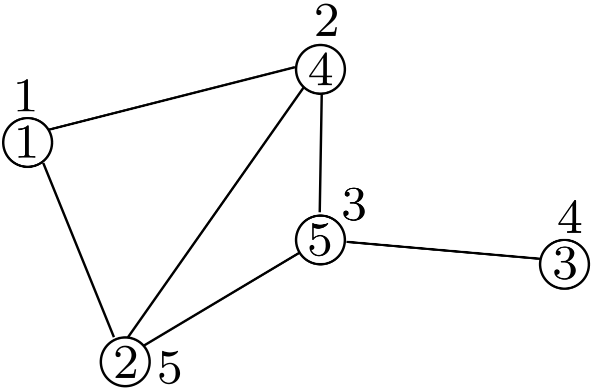

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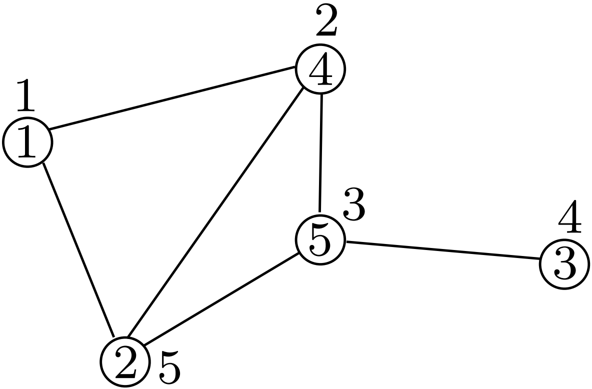

$\sigma_t(v) =$ particle at vertex v at time t

Cyclic notation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 3 & 2 & 6 \end{pmatrix}$$

we write $\sigma = (1,4,3)(2,5)(6)$ and call the bits inside *cycles*.

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Theorem (Tóth (1993))

Various quantities associated to the $\frac{1}{2}$-spin quantum Heisenberg ferromagnet in terms of the cycle lengths of $\tilde{\sigma}_t$, where

$$\mathbb{P}(\tilde{\sigma}_t = \sigma) = \frac{1}{\mathbb{E}[2^{\#\text{cycles of } \sigma_t}]} 2^{\#\text{cycles of } \sigma} \mathbb{P}(\sigma_t = \sigma).$$

We only look at σ_t in this talk.

What is known?

Theorem (Schramm(2005))

Let G be the complete graph and suppose that $t = \beta n$.

(i) **Subcritical phase,** $\beta < 1/2$: *all the cycles have length* $O(\log n)$

(ii) **Supercritical phase,** $\beta > 1/2$: *a positive proportion of vertices lie on cycles of length comparable to n*

Moreover, in the supercritical phase, the cycle lengths rescaled appropriately converge to a Poisson-Dirichlet distribution.

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Theorem (Berestycki (2011), Berestycki, Kozma (2015))

The phase transition of Schramm with (1) a different proof and (2) using representation theory.

Theorem (Kotecký, Miłoś, Ueltschi (2016))



Let G be the hypercube $\{0,1\}^n$ and suppose that $t = \beta 2^n$.

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Should be comparable to 2^n in the supercritical phase.

Theorem (Angel (2003), Hammond (2013), Hammond (2015))

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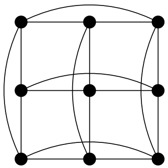
Conjecture

The interchange process on $\mathbb{Z}^d$ has finite cycles for all times when $d = 2$ and has a sharp phase transition between finite cycles and infinite cycles when $d \geq 3$.
When $d \geq 3$, on the graph $\{-n, \ldots, n\}^d$ there is a phase transition between cycles of length $O(\log n)$ and cycles of length comparable to n^d.

Our result

Hamming graph: $V = \{1, \dots, n\}^2$, edge between any two vertices on same row or column:



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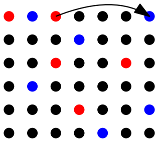

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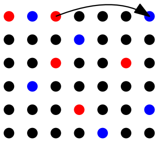
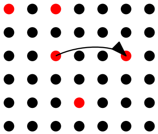


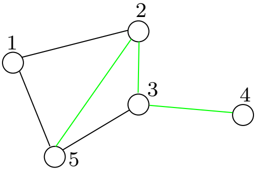
Split: When v and w are in the same cycle of σ_{t-}, e.g. $\sigma_{t-} = (1, 3, 4)(2, 5)$, $e = (1, 4)$

$$(1, 3, 4)(2, 5) \circ (1, 4) = (1)(2, 5)(3, 4).$$



Coupling with percolation

Obtained by ignoring the splits:
Each time an edge e rings, declare it to be open.



This results in a bond percolation G_t with parameter $p_t = 1 - e^{-t/|E|}$ (where $|E| = \#\{\text{of edges}\}$).

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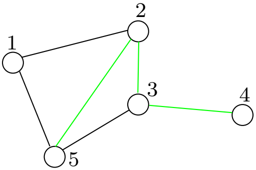

This results in a bond percolation G_t with parameter $p_t = 1 - e^{-t/|E|}$ (where $|E| = \#\{\text{of edges}\}$).
Every cycle of σ_t is contained in an open connected component of G_t.

Subcritical phase:
Hamming graph: $|E| = n^2(n-1)$, $t = \beta n^2$, $p_t = 1 - e^{-t/|E|} \sim \frac{\beta}{n}$, the expected number of open edges at a vertex is 2β.
Adaptation of Erdős-Rényi arguments: for $\beta < 1/2$, all open connected components of G_t are $O(\log n)$.
Coupling: cycle lengths of σ_t are $O(\log n)$.

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Supercritical phase:

- Erdős-Rényi arguments $\implies$ unique component of size comparable to n^2.
- A priori, the giant component could be made up of many cycles of small length.
- Show that cycles of length $o(n^2)$ are more likely to merge than split $\implies$ giant component is covered by $O(1)$ many cycles

Complete graph:

Suppose that a cycle $\mathfrak{c}$ has length k.

$$\#\{\text{edges between vertices of } \mathfrak{c}\} = \binom{k}{2}$$

$$\#\{\text{edges from } \mathfrak{c} \text{ to } \{1, \ldots, n\} \backslash \mathfrak{c}\} = k(n-k)$$

Cycle is much more likely to merge then split when $\binom{k}{2} << k(n-k)$, or alternatively $k << n$ (graph volume is n).

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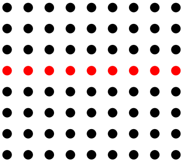
Hamming graph:
Big problem: cycle of length n (graph volume is n^2) which is equally likely to be merge as it is to split:



Isoperimetry

Let H denote the 2-dimensional Hamming graph. For $A \subset H$ let

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Heuristically what should $\iota(\mathfrak{c})$ of a cycle $\mathfrak{c} \subset \sigma_t$ look like?

- $v \mapsto \sigma_t(v)$ is the position of CSRW on H at time t,
- $(v, \sigma_t(v), \sigma_t \circ \sigma_t(v), \dots)$ looks like the trace of a CSRW
- CSRW mixes very quickly to the uniform measure so $\mathfrak{c}$ looks like a set of i.i.d. uniform points.

$$\iota(\mathfrak{c}) \approx 1 \vee \frac{|\mathfrak{c}|}{n} \log n.$$

The isoperimetry lemma

Let

$$\mathrm{orb}_t^k(v) := \{v, \sigma_t(v), \ldots, \underbrace{\sigma_t \circ \cdots \circ \sigma_t(v)}_{k \text{ times}}\}.$$

i.e.

$$(\overbrace{v, x_1, \ldots, x_k}^{\mathrm{orb}_t^k(v)}, \ldots)$$

Lemma

Suppose that for $k = o(n)$

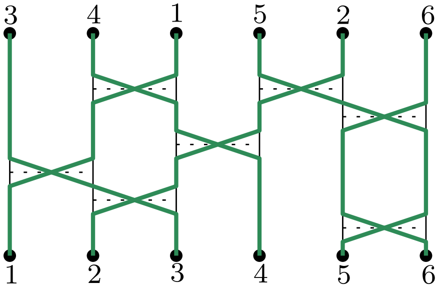
$$\liminf_{n\to\infty} \inf_{s\in[t-\Delta,t]} \mathbb{P}(|\mathrm{orb}_s^k(v)| = k) > 0$$

then

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s\in[t-\Delta,t]} \sup_{w} \iota(\mathrm{orb}_s^k(w)) \geq \log^2 n\right) = 0.$$

If cycles of length k exist, then they don't concentrate on any row or column.

Cyclic random walk



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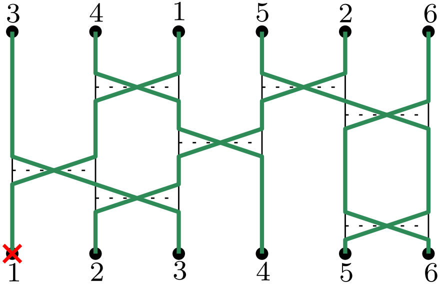

- Fix $t = \beta n^2$ and place a *bridge* when an edge rings prior to time t.
- $\mathcal{X} = (\mathcal{X}_u : s \geq 0)$ CRW with $\mathcal{X}_u \in \{1, \ldots, n\}^2 \times [0, t]$ with $\mathcal{X}_0 = (v, z)$
- $\mathcal{X}_u$ moves at unit speed up, switches to the other end of the cross, goes to the bottom when it reaches the top.

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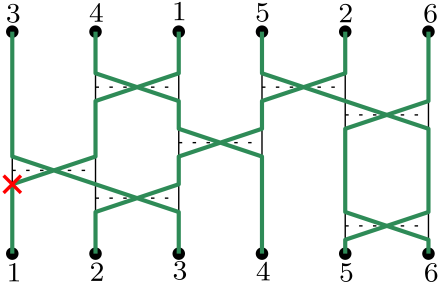

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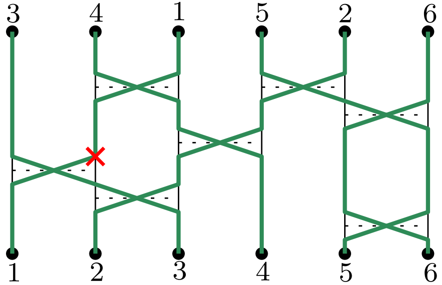

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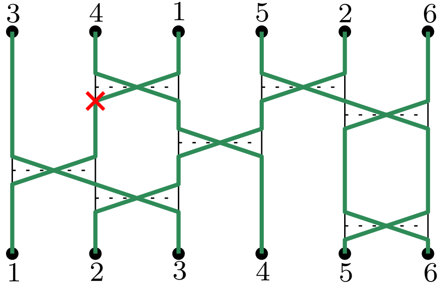

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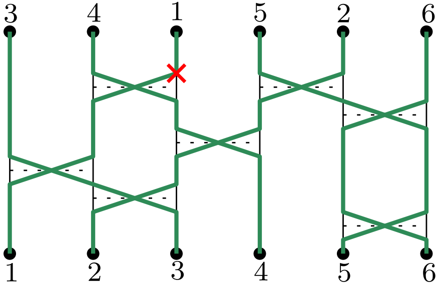

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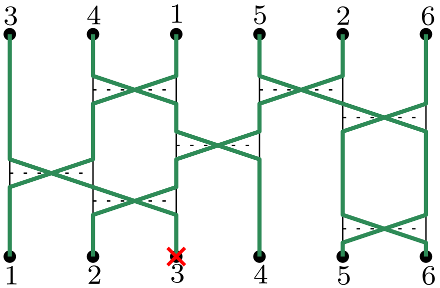

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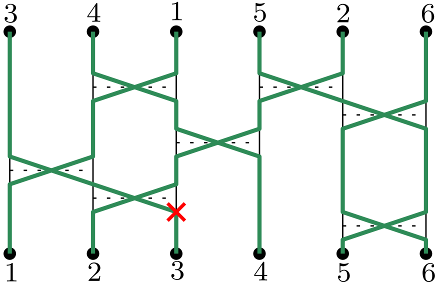

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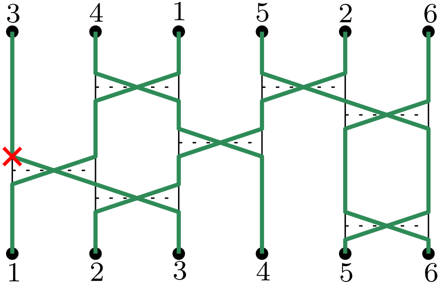

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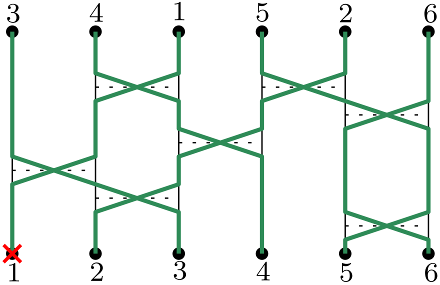

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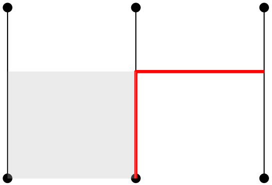
Properties

- $\mathcal{X}$ is periodic.
- $\mathcal{X}$ is measurable w.r.t. $(\sigma_{t'} : t' \leq t)$.
- $\mathcal{X}$ is non-Markovian:

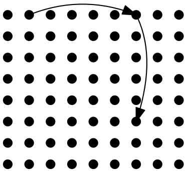


- The cycle containing v is given by $\{X_u|_{[n]^2}$ s.t. $X_u|_{[0,t]} = t\}$
- $\iota(\{\text{first } k \text{ vertices visited by } \mathcal{X}\}) \approx \iota(\mathrm{orb}_t^k(v))$

Why doesn't the CRW concentrate on rows/columns?

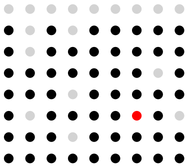
Control the number of vertices it visits on the first row:

- At each pair of steps, there is a bounded probability that we do an *L-shaped jump* from the first row:



- Suppose L-shaped jump happens at time T, then $\mathcal{X}_T = (v, z)$ is roughly uniform.

- Condition on $\mathcal{A} = \sigma\{\mathcal{X}_u : u \leq T\}$ and let $A = \{\text{vertices visited before time } T\}$



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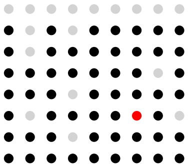

- The remaining looks like the original graph so $\mathbb{P}(\mathcal{X} \text{ stays away from first row next } k \text{ steps} | \mathcal{A}) \approx \mathbb{P}(\mathcal{X} \text{ visits at least } k \text{ vertices}) \geq \text{const.}$

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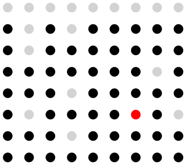

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- Positive probability of L-shaped jump + escape at each step on the first row $\implies$ can't spend more than $O(1)$ steps on the first row.

Return to percolation coupling

Each time an edge e rings, declare it to be open. (G_t is bond percolation with $p_t \approx \beta/n$)
Every cycle of σ_t is contained in an open connected component of G_t.

Lemma

For $\alpha \in (0,1/2)$ and $\beta > \beta' > 1/2$, there exists a $\delta \in (0,1)$ such that with probability approaching 1,

$$\inf_{s \in [\beta' n^2, \beta n^2]} \#\{\textit{vertices in cycles of length} \geq n^\alpha \textit{ at time } s\} \geq \delta n^2$$

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- Let $s \in [\beta' n^2, \beta n^2]$, then G_s has a unique giant component of size $O(1) n^2$
- Consider a vertex $v \in \{\text{giant component of } G_s\}$ such that $|\mathrm{orb}_s^\infty(v)| \leq n^\alpha$.
  This vertex must have been in a cycle prior to time s which was involved in a split where one of the resulting pieces has length $\leq n^\alpha$

- Probability a uniformly chosen edge $e = (u, w)$ makes such a split is at most $n^{\alpha-1}$:

$$(\dots, \overbrace{x_1, \dots, x_{n^\alpha}}^{w \text{ must fall here}}, u, \overbrace{y_1, \dots, y_{n^\alpha}}^{\text{or here}}, \dots)$$

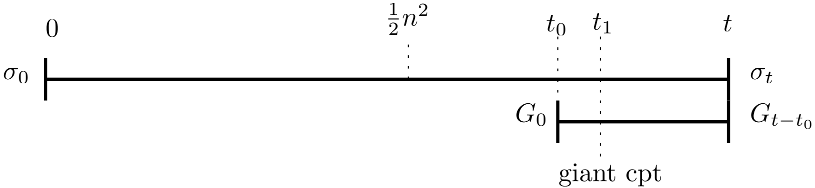
- Thus the total number of vertices in the giant cpt and in cycles of length $\leq n^\alpha$ is at most

$$\underbrace{2n^\alpha}_{\#\text{ of vertices in cycle}} \times \underbrace{\beta n^2}_{\text{time interval}} \times n^{\alpha-1} = O(n^{1+2\alpha}) = o(n^2)$$

- Giant component has size $O(1)n^2$

Inducting

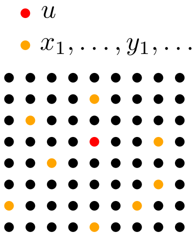
Set $\beta > 1/2$, $t = \beta n^2$, $t_0 = t - 2n^{2-\alpha} \log n$, $t_1 = t - n^{2-\alpha} \log n$.
Let $\tilde{G}_0$ be a graph with the same connected cpts as σ_{t_0}. Add an edge to $\tilde{G}$ whenever an edge is selected for swap after time t_0.



G_0 has a lot of vertices in n^α cpts + sprinkling $\implies$ $\tilde{G}_s$ has a giant cpt when $s \geq t_1$
Every cycle of σ_{t_0+s} is contained in an open connected component of $\tilde{G}_s$.

- Consider a vertex $v \in \{\text{giant component of } \tilde{G}_s\}$ such that $|\mathrm{orb}^{\infty}_{s+t_0}(v)| \leq n^{\gamma}$.
  This vertex must have been in a cycle prior at time $s' \in [t_0, s+t_0]$ which was involved in a split where one of the resulting pieces has length $\leq n^{\gamma}$
- Probability a uniformly chosen edge $e = (u, w)$ makes such a split

$$(\ldots, \overbrace{x_1, \ldots, x_{n^\gamma}}^{w \text{ must fall here}}, u, \overbrace{y_1, \ldots, y_{n^\gamma}}^{\text{or here}}, \ldots)$$



is $\iota(\mathrm{orb}^{2n^\gamma}_{s'}(x_1))/2n$

- $$\iota(\mathrm{orb}_{s'}^{2n^\gamma}(x_1)) \le \underbrace{\max_w \iota(\mathrm{orb}_{s'}^{n^\alpha}(w))}_{\iota \text{ of a slice}} \times \underbrace{2n^{\gamma-\alpha}}_{\#slices} \le 2n^{\gamma-\alpha}\log^2 n$$

  by isoperimetry lemma
- Thus the total number of vertices in the giant cpt and in cycles of length $\le n^\gamma$ is at most

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Powers go $\gamma \mapsto (1/2)(1 + \gamma + \min\{\gamma, 1\})$ which has fixed point at $\gamma = 2$.

Thank you!