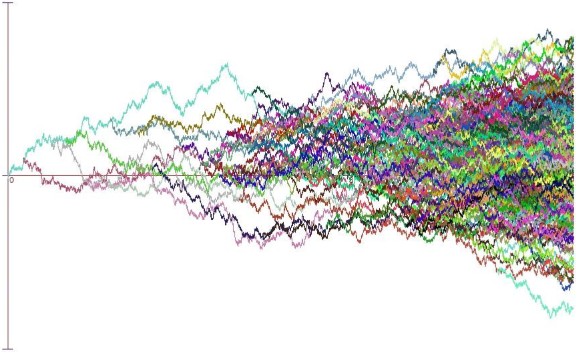
An invariance principle for branching diffusions in bounded domains

University of Bristol

Ellen Powell

11th November 2016

Branching Brownian Motion



$(X_1^t, \cdots, X_{N_t}^t) =$ system at time t.

Simulation by Matt Roberts.

FKPP Equation

$$u(t,x) := \mathbb{E}_x[\prod_{i=1}^{N_t} f(X_t^i)] \, ; \quad f : \mathbb{R}^d \to \mathbb{R}$$

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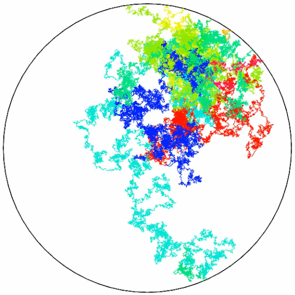
$\uparrow$ FKPP equation

Example: When $f = \mathbf{1}_{[0,\infty)}, d = 1 \rightsquigarrow u(t,x) = \mathbb{P}_0[R_t \leq x]$

Branching Brownian Motion in a Domain



- $D \subset \mathbb{R}^d$ bounded, C^1 domain.
- Start BBM at $x \in D$. Particles killed upon hitting the boundary.

Simulation by Henry Jackson.

Martingale

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MARTINGALE

β = branching rate $\qquad (\lambda, \varphi)$ = first eigenpair of $-\frac{1}{2}\Delta$ on D.

MARTINGALE

$$M_t := e^{(\lambda-\beta)t} \sum_{i=1}^{N_t} \varphi(X_t^i)$$

Extinction (BBM with binary branching)

Phase Transition (Sevast'yanov 1958, Watanabe 1965)

There exists a critical value λ of the branching parameter β s.t.

- $\beta \leq \lambda \Rightarrow$ a.s. extinction
- $\beta > \lambda \Rightarrow$ survival with positive probability.

λ is the first eigenvalue of $-\frac{1}{2}\Delta$ on D.

Survival Probabilities

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- Decays exponentially in subcritical case.
- Critical case?

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Galton-Watson Case (Kolmogorov 1938, Miermont 2008)

Critical GW and multitype GW processes *with finite variance* have

$$\mathbb{P}(N_n > 0) \sim c/n$$

as $n \to \infty$.

Main Results (BBM with binary branching)

Theorem (Asmussen & Hering 1983, P. 2015)

In the critical case $\beta = \lambda$, for all $x \in D$ we have

$$\mathbb{P}_x\left(N_t > 0\right) \sim \frac{1}{t} \times \frac{\varphi(x)}{\lambda \int_D \varphi(y)^3\, dy}$$

as $t \to \infty$.

φ is the first eigenfunction of $-\frac{1}{2}\Delta$ on D, normalised to have unit L^2 norm.

Main Results (BBM with binary branching)

Theorem (Asmussen & Hering 1983, P. 2015)

For any measurable $E \subset D$, if N_t^E is the number of particles in E at time t, we have

$$\left(\frac{N_t^E}{t} \,\middle|\, N_t > 0 \right) \to Z$$

in distribution as $t \to \infty$, where Z is an exponential random variable with mean

$$\lambda \langle \varphi, \mathbf{1}_E \rangle_{L^2(D)} \int_D \varphi^3.$$

Main Results (BBM with binary branching)

Corollary

Let

$$\mu_t := \frac{1}{N_t} \sum_{i=1}^{N_t} \delta_{X_t^i}$$

be the uniform distribution on all particles alive at time t, given survival. Then, for each measurable $E \subset D$, we have that

$$\mu_t(E) \to \mu(E)$$

in distribution, and hence in probability, as $t \to \infty$, where

$$\mu(E) = \frac{\int_E \varphi(x)\, dx}{\int_D \varphi(x)\, dx}.$$

Notations

$A =$ offspring distribution - mean m.

$L =$ generator.

Assume: D is C^1, $\mathbb{E}[A^2] < \infty$ and L is uniformly elliptic and self-adjoint with smooth coefficients.

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BBM with binary branching: $A \equiv 2,\ L = \frac{1}{2}\Delta.$

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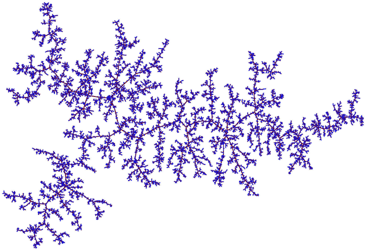
BBM with binary branching: $A \equiv 2,\; L = \dfrac{1}{2}\Delta.$

$\rightsquigarrow$ Everything still works!

Scaling Limit



Simulation by Igor Kortchemski.

- What does the conditioned tree look like?
- It converges (with appropriate rescaling) to the CRT.

FKPP equation

Key Tool: Let

$$u(x,t) = \mathbb{P}_x(N_t > 0).$$

Then u satisfies the FKPP equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \lambda(u - u^2)$$

in D with correct boundary/initial conditions.

Heuristics

Reminder: $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \lambda(u - u^2)$.

Write

$$u(x,t) = \sum_i a_i(t)\varphi_i(x); \quad a_i(t) = \int_D u(x,t)\varphi_i(x)dx.$$

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Then

$$\frac{da_i}{dt} = \int_D \left(\frac{1}{2}\Delta u + \lambda(u - u^2)\right)\varphi_i(x)\,dx.$$

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IBP $\Rightarrow$

$$\frac{da_i}{dt} = \int_D \left(-\lambda_i u + \lambda(u - u^2)\right)\varphi_i(x)\,dx.$$

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$$\frac{da_1}{dt} = -\lambda \int_D u^2(x,t)\varphi(x)\, dx$$

$$\frac{da_i}{dt} = (\lambda - \lambda_i)a_i(t) - \lambda \int_D u^2(x,t)\varphi_i(x)\, dx \text{ for } i \geq 2.$$

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Guess: $u(x,t) \sim a_1(t)\varphi(x)$.

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Guess: $u(x,t) \sim a_1(t)\varphi(x)$.

Sadly, doesn't quite work.

Heuristics

Reminder: $\frac{da_1}{dt} = -\lambda \int_D u^2(x,t)\varphi(x)\,dx.$

Suppose $u(x,t) \sim a_1(t)\varphi(x)$. Then

$$\frac{da_1}{dt} \sim -a_1^2 \times \lambda \int_D \varphi(x)^3\,dx.$$

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Suppose $u(x,t) \sim a_1(t)\varphi(x)$. Then

$$\frac{da_1}{dt} \sim -a_1^2 \times \lambda \int_D \varphi(x)^3\, dx.$$

Easy to prove:

$$a_1(t) \sim \frac{1}{t} \times \frac{1}{\lambda \int_D \varphi(y)^3}.$$

Martingales

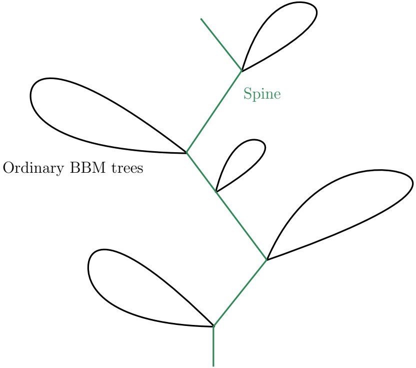
Goal: $u(x,t) \sim a_1(t)\varphi(x)$.

Remember:

$$M_t = \sum_{i=1}^{N_t} \varphi(X_t^i)$$

is a martingale.

Change of Measure - Critical Case



No extinction under $\mathbb{Q}_x$ (new measure).

- Change measure by $\frac{M_t}{\mathbb{E}_x[M_t]} = \frac{\sum_i \varphi(X_i^t)}{\varphi(x)}$.
- Spine particle is BM conditioned to remain in D.
- Branches at rate 2λ.
- Offspring are ordinary BBM processes.

Heuristics

Reminder: want $\mathbb{P}_x(N_t > 0) \sim a_1(t)\varphi(x)$.

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$$\begin{aligned}\frac{\mathbb{P}_x(N_t>0)}{\varphi(x)} &= \mathbb{E}_x\left[\frac{1}{M_t}\frac{M_t}{\varphi(x)}\mathbf{1}_{\{N_t>0\}}\right]\\ &= \mathbb{Q}_x\left[\frac{1}{M_t}\right]\end{aligned}$$

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This shouldn't depend on x for large t.

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So $\mathbb{P}_x(N_t > 0) \sim c(t)\varphi(x)$ as $t \to \infty$,

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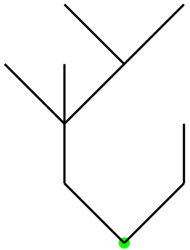
This shouldn't depend on x for large t.

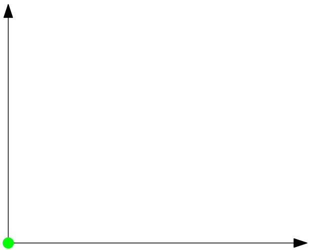
So $\mathbb{P}_x(N_t > 0) \sim c(t)\varphi(x)$ as $t \to \infty$, and $c(t) \sim a_1(t)$.

Conditioned results

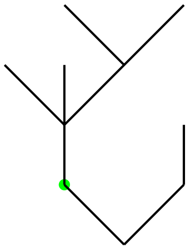
- Asymptotic for survival probability.
- Many-to-few Lemma.
- Method of Moments.

Scaling Limit - GW Trees





Height function:



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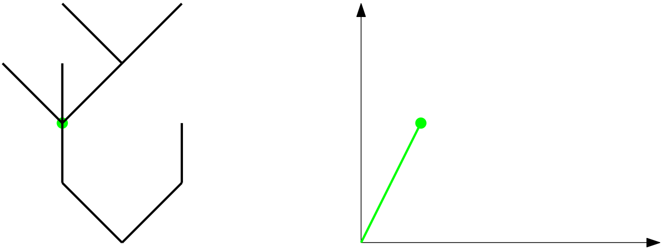

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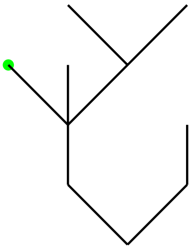

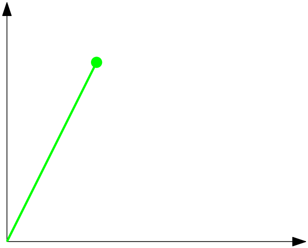

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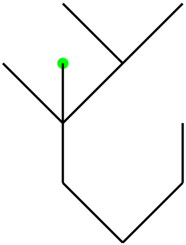

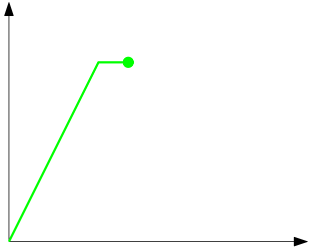

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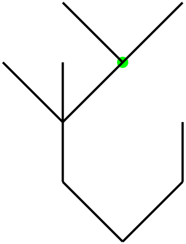

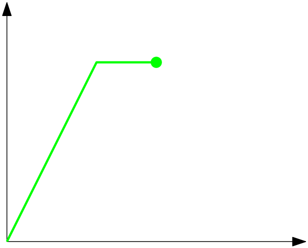

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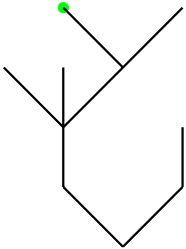

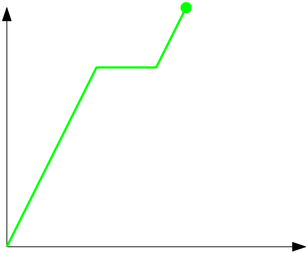

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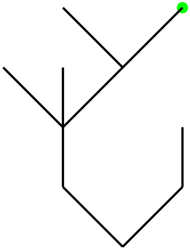

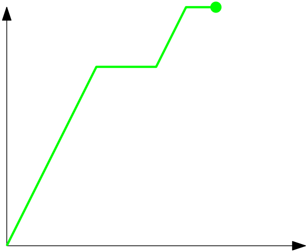

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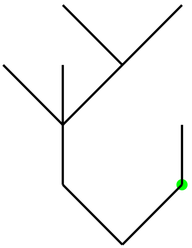

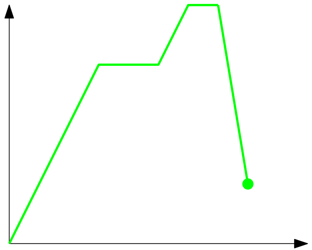

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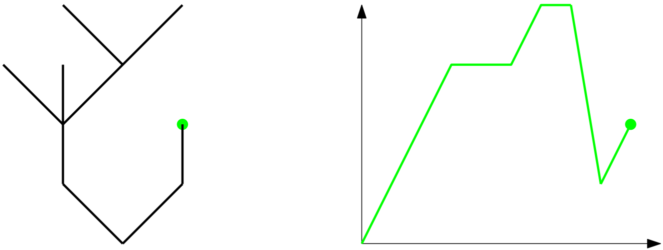

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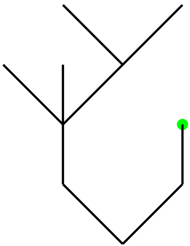
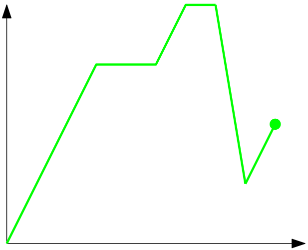
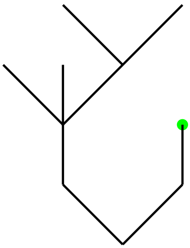
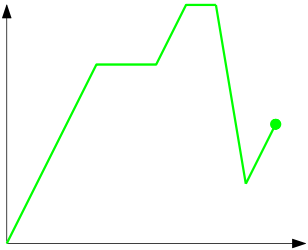




Height function: concatenate for an iid sequence of GW trees $\rightsquigarrow$ reflected Brownian motion.

Scaling Limit - GW Trees





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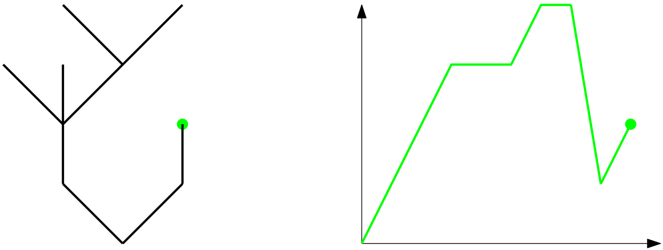

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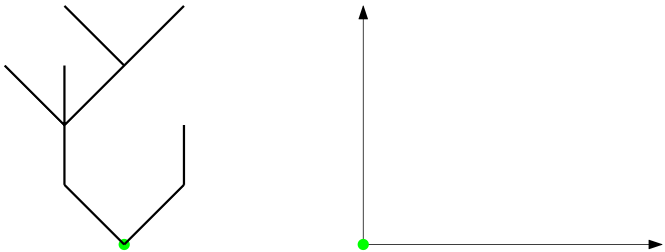
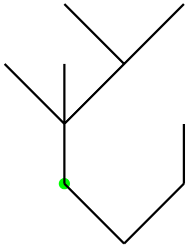


Lukasiewicz Path:

Scaling Limit - GW Trees



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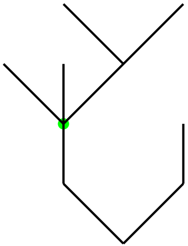

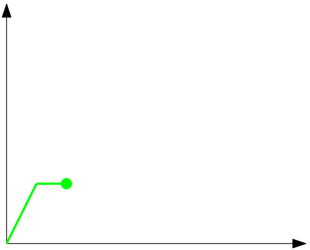

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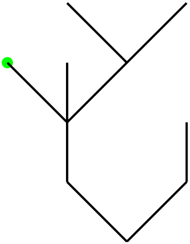

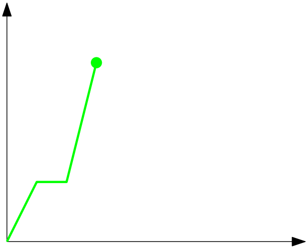

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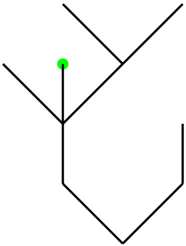

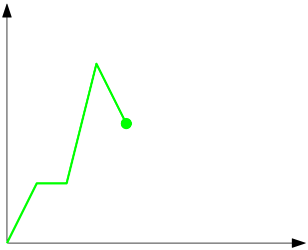

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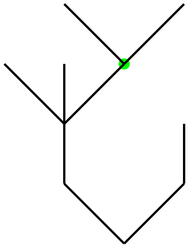

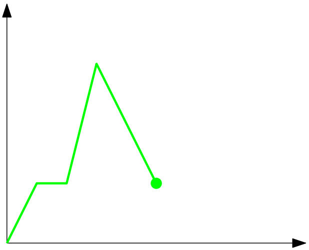

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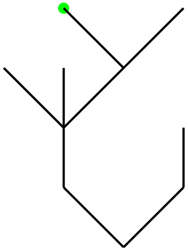

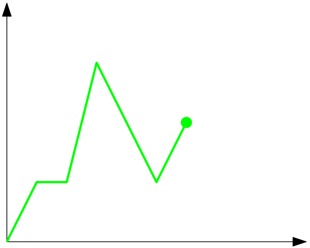

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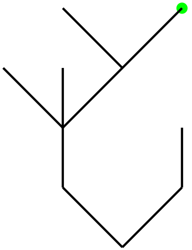
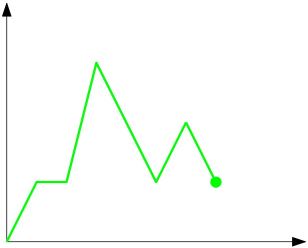
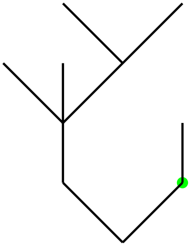
Scaling Limit - GW Trees

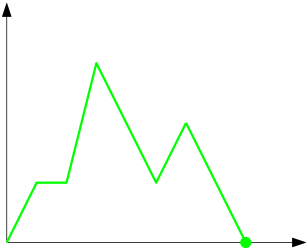




Lukasiewicz Path:

Scaling Limit - GW Trees





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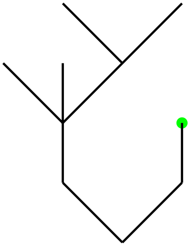
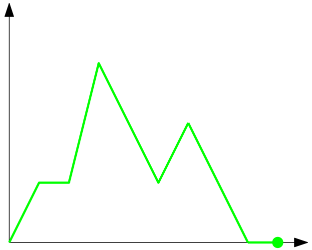
Scaling Limit - GW Trees





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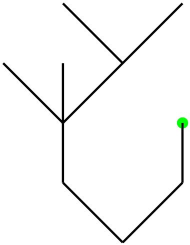
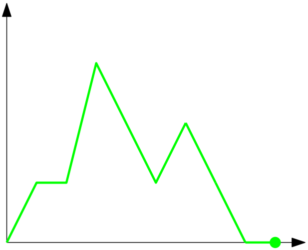
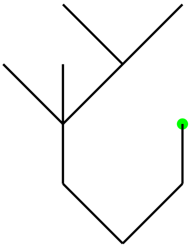
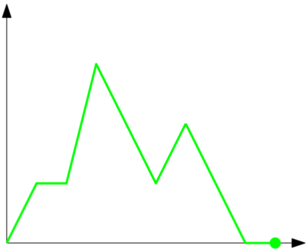




Lukasiewicz Path: RW

Scaling Limit - GW Trees





Lukasiewicz Path: RW $\rightsquigarrow$ get reflected BM.

Scaling Limit - GW trees

Can we compare?

Scaling Limit - GW trees

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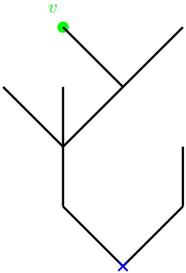

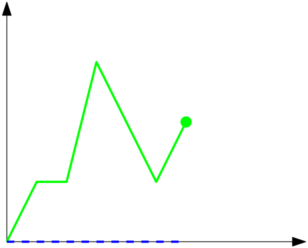

Scaling Limit - GW trees

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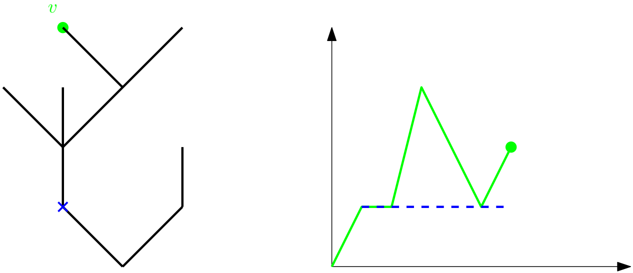

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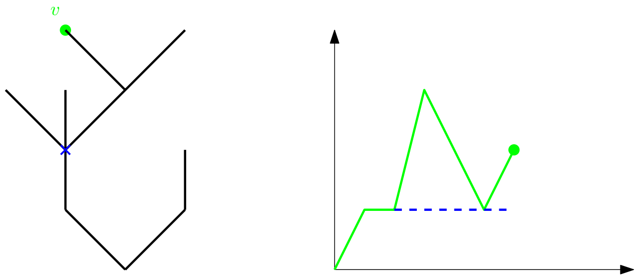

Scaling Limit - GW trees

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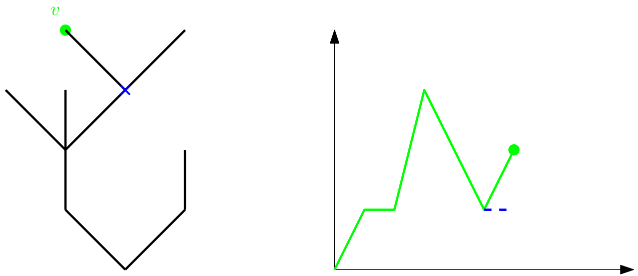
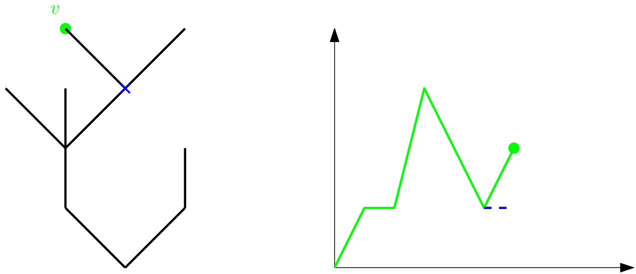


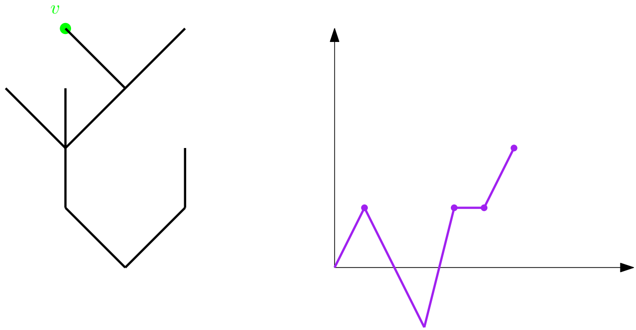
Scaling Limit - GW trees

Can we compare? $h(v)$ = no. of times Lukasiewicz path reaches a new "future infima"



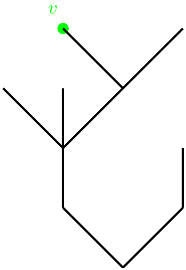
Scaling Limit - GW trees

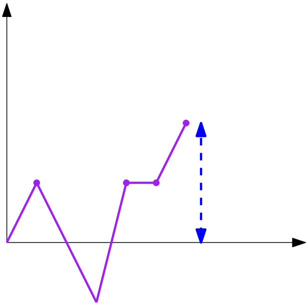
Reverse the path: $h(v)$=no. of new maxima.



Scaling Limit - GW trees

Reverse the path: $h(v)$=no. of new maxima. $l(v)$=total height reached.





Scaling Limit - GW trees

Note: Reversed path is a RW with same law.

SCALING LIMIT - GW TREES

Note: Reversed path is a RW with same law. $\Rightarrow$

$$(h(v_n), l(v_n)) \stackrel{\text{law}}{=} \sum_{i=1}^{K_n} (S_{\tau_i} - S_{\tau_{i-1}})$$

where $(0 = \tau_0, \tau_1, ..., \tau_{K_n})$ times $< n$ when SRW S hits new maxima.

Scaling Limit - GW trees

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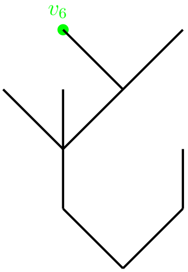

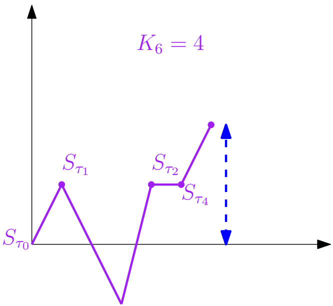

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LLN: $\frac{\text{Height}}{\text{Lukasiewisc path}} \to$ const.

Scaling Limit - BBM

Tree/height function in continuous time:

Scaling Limit - BBM

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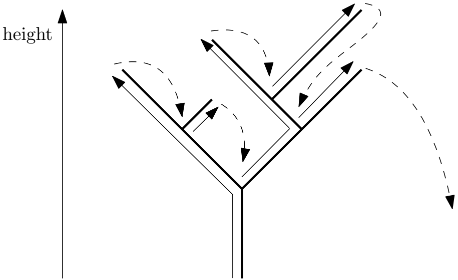

Scaling Limit - BBM

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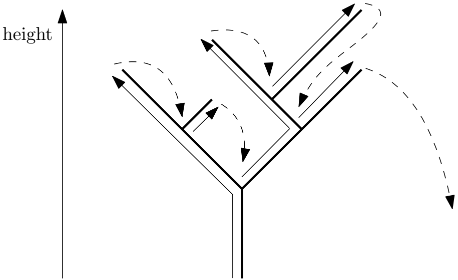
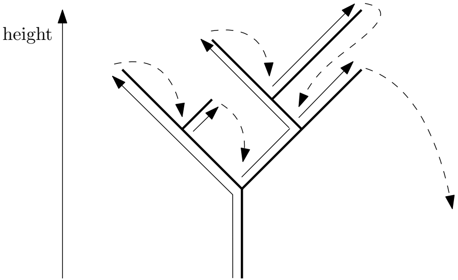


How to generalise?

Scaling Limit - BBM

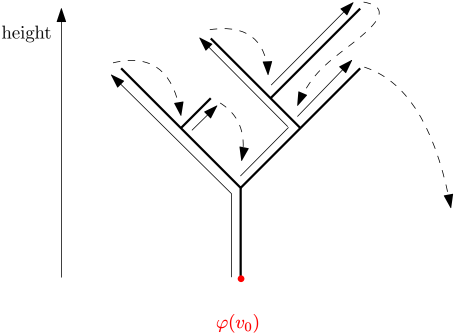
Tree/height function in continuous time:



How to generalise? Use Martingale.

Scaling Limit - BBM

v_t=vertex visited at time t.



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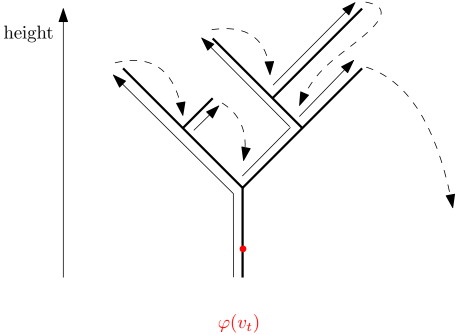

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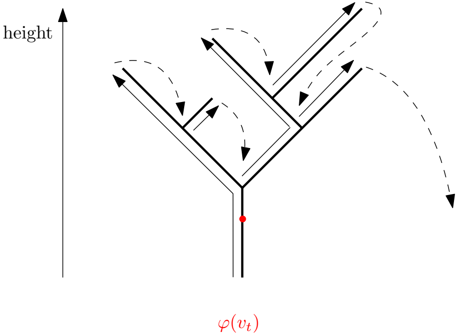

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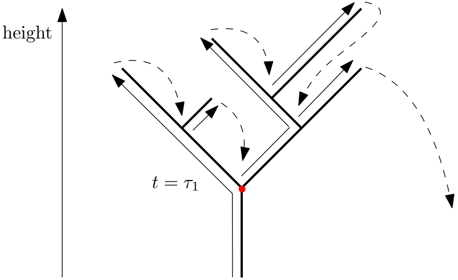

$\varphi(v_t) + \varphi(v_{\tau_1})$

Scaling Limit - BBM

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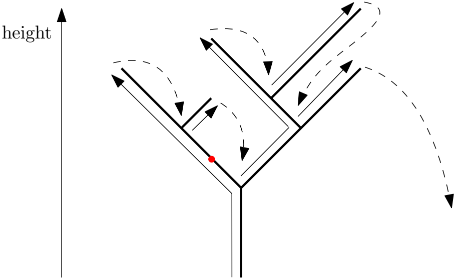

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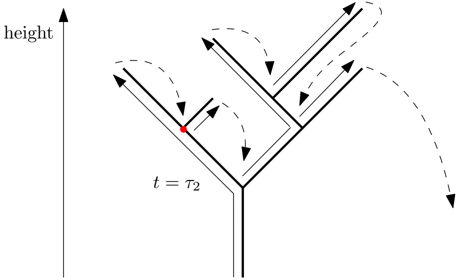

$\varphi(v_t) + \varphi(v_{\tau_1}) + \varphi(v_{\tau_2})$

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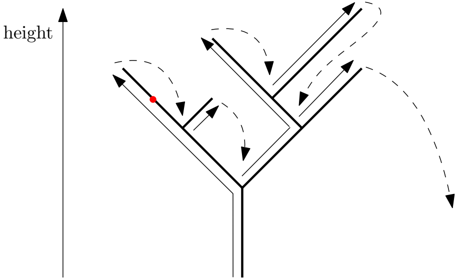

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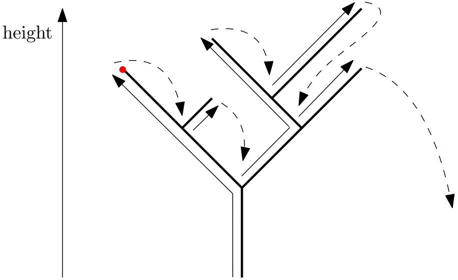

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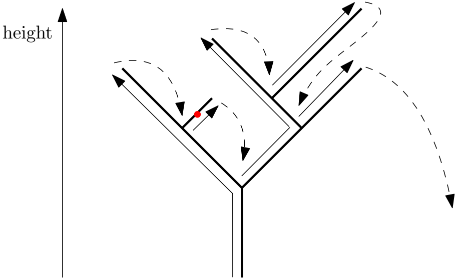

$\varphi(t) + \varphi(v_{\tau_1})$

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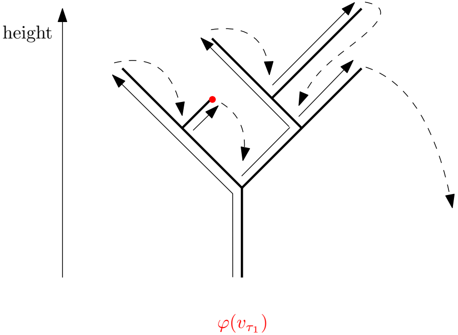

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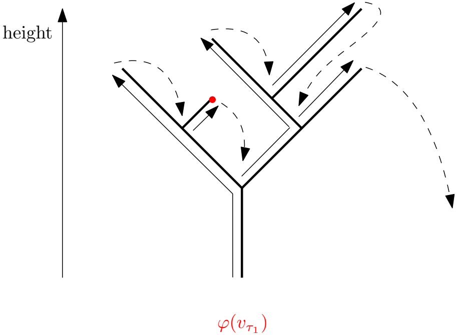

This is a martingale $\rightsquigarrow$ reflected BM

Connection with height function?

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- Lost reversibility.

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- Consider a random vertex in generation t, tree conditioned on $N_t > 0$.

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Scaling Limit - BBM

Connection with height function?

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- Consider a random vertex in generation t, tree conditioned on $N_t > 0$.
- Use change of measure to compare with spine.
- LLN $\Rightarrow$ result.

Scaling Limit - Result

Theorem (P. 2016)

$D \subset \mathbb{R}^d$ C^1 domain, L, A as before and $\varphi \in C^1(\overline{D})$. Then for any $y > 0$, and any starting point $x \in D$,

$$(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}) \underset{n \to \infty}{\longrightarrow} (\mathcal{T}_{e^y}, d_{e^y})$$

in distribution, with respect to the Gromov-Hausdorff distance, where

$$\alpha = \sqrt{\frac{4(m-1)}{\lambda \langle 1, \varphi \rangle \mathbb{E}[A^2 - A] \int_D \varphi(y)^3 \, dy}}.$$

Further Questions

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- What if D is not bounded?

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- What if D is not bounded?
- Can have very different behaviour.
- Can we classify the different types of critical behaviour?

Thanks for listening!