When is a function nearly constant?

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Outline of the talk

References

- Based on: *A discrete log-Sobolev inequality under a Bakry-Émery type condition*
- `arxiv:1507.06268`.
- To appear in Annals IHP 2017 `doi:10.1214/16-AIHP778`

Approximately constant functions 1: derivatives

- Consider a differentiable function f, say from $\mathbb{R}$ to $\mathbb{R}$.
- What does it mean to say f is approximately constant?
- Can think in a variety of senses.
- One idea is "$f'(x) \simeq 0$".
- Given reference probability measure μ, we could think

$$\int f'(x)^2 d\mu(x) \quad \text{is "small"}.$$

Approximately constant functions 2: variances

- Given f and μ could also consider variance of f.
- Write $\lambda_{\mu,f} = \int f(x) d\mu(x)$
- Then define

$$\mathrm{Var}_\mu(f) = \int \left(f(x) - \lambda_{\mu,f}\right)^2 d\mu(x).$$

- Again, might require $\mathrm{Var}_\mu(f)$ "small".

Approximately constant functions 3: entropy

- Given positive f and μ also consider (relative) entropy of f.

$$\begin{aligned}
&\mathrm{Ent}_\mu(f) \\
&:= \int f(x) \log f(x) d\mu(x) - \left(\int f(x) d\mu(x) \right) \log \left(\int f(x) d\mu(x) \right) \\
&= \int f(x) \log f(x) d\mu(x) - \lambda_{\mu,f} \log \lambda_{\mu,f}.
\end{aligned}$$

- Jensen's inequality applied to $\theta(t) = t \log t$ (which is convex) means $\mathrm{Ent}_\mu(f) \geq 0$.
- If f constant, $f(x) \equiv c$ then $\mathrm{Ent}_\mu(f) = c \log c - c \log c = 0$.
- Hence $\mathrm{Ent}_\mu(f)$ "small" implies f close to constant?
- In fact work with $\mathrm{Ent}_\mu(f^2)$.

Comparison of conditions

- If f is constant, then
  1. $\int f'(x)^2 d\mu(x) = 0$,
  2. $\mathrm{Var}_\mu(f) = 0$
  3. $\mathrm{Ent}_\mu(f^2) = 0$.
- (Assuming μ supported everywhere) if any quantity is zero, then f is constant.
- Hence 1., 2. and 3. are equivalent, and each equivalent to f being constant.
- More interesting question: What is relationship between:
  - A. $\int f'(x)^2 d\mu(x)$ is small,
  - B. $\mathrm{Var}_\mu(f)$ is small
  - C. $\mathrm{Ent}_\mu(f^2)$ is small?

Poincaré inequalities

Definition

We say μ has Poincaré constant C if for all f

$$\mathrm{Var}_\mu(f) \leq C \int f'(x)^2 d\mu(x).$$

- Fix μ with mean zero and variance σ^2.
- Taking $f(x) = x$, $\mathrm{Var}_\mu(f) = \sigma^2$, RHS is $\int f'(x)^2 d\mu(x) = 1$.
- Hence μ has Poincaré constant at least σ^2.

Theorem (Folklore?)

If μ is Gaussian with variance σ^2 then μ has Poincaré constant σ^2.

- In fact, having PC equal to σ^2 characterizes Gaussian.
- Can prove theorem by expanding f in Hermite polynomials.
- Poincaré constant can be infinite.

Log-Sobolev inequalities

Definition

We say μ has log-Sobolev constant C if for all f

- $\mathrm{Ent}_\mu(f^2) \leq 2C \int f'(x)^2 d\mu(x).$
- Equivalently take $g = f^2$. Then $f'(x)^2 = g'(x)^2/(4g(x))$, so

$$\mathrm{Ent}_\mu(g) \leq \frac{C}{2} \int \frac{g'(x)^2}{g(x)} d\mu(x).$$

Theorem (Gross 1975)

μ Gaussian with variance σ^2 has log-Sobolev constant σ^2.

- In fact, follows from Entropy Power Inequality (Stam 1959).
- Log-Sobolev C implies Poincaré constant C (take $f = 1 + \epsilon h$).

Continuous Bakry-Émery theory

- Value of Poincaré and log-Sobolev constant for Gaussians can be generalized.
- Key: (second-order differential) operator L, self-adjoint wrt μ.
- $$\text{i.e.} \quad \int Lf(x)g(x)d\mu(x) = \int f(x)Lg(x)d\mu(x).$$
- Allows the creation of so-called carré du champ operator Γ_1.

Definition

For any functions f and g, write

$$\Gamma_1(f,g) = \frac{1}{2}\left[L(fg) - fLg - gLf\right].$$

Motivating example

Example

- Suppose $d\mu(x) = dx$ (OK, not a prob. measure).
- Take $Lf(x) = f''(x)$. Self-adjoint by integration by parts

$$\int f''(x)g(x)dx = -\int f'(x)g'(x)dx = \int f(x)g''(x)dx.$$

- Then $\Gamma_1(f,g) = \frac{1}{2}\left[L(fg) - fLg - gLf\right]$ means

$$\Gamma_1(f,g) = \frac{1}{2}\left((f''g + 2f'g' + fg'') - fg'' - gf''\right) = f'g'.$$

- NB $\int \Gamma_1(f,g)(x)dx = -\int f(x)Lg(x)dx = -\int g(x)Lf(x)dx$.

Iterated operator Γ_2

Definition

For any functions f and g, write

$$\Gamma_2(f,g) = \frac{1}{2}\left[L\left(\Gamma_1(f,g)\right) - \Gamma_1(f,Lg) - \Gamma_1(g,Lf)\right].$$

Example

For motivating example $Lf = f''$:

$$\begin{aligned}\Gamma_2(f,g) &= \frac{1}{2}\left((f'g')'' - f'g''' - g'f'''\right) \\ &= f''g''\end{aligned}$$

Central definition

Condition (Bakry-Émery condition)

Say the Bakry-Émery condition holds with constant c if for all functions f:

$$\Gamma_2(f, f) \geq c\Gamma_1(f, f).$$

- Deep. Relates to curvature of underlying space?

Further key example

Example

- For function $U(x)$, take $d\mu(x) = \frac{1}{Z}\exp(-U(x))dx$.
- Write
$$Lf(x) = f''(x) - U'(x)f'(x).$$
- Simple to verify L is self-adjoint with respect to μ.
- Further $\Gamma_1(f,g) = f'g'$, and
$$\Gamma_2(f,g) = f''(x)g''(x) + U''(x)f'(x)g'(x).$$
- If $U''(x) \geq c$ then $\Gamma_2(f,f) \geq c\Gamma_1(f,f)$, i.e Bakry-Émery condition holds with constant c.
- Think of $U(x) = cx^2/2$ and μ Gaussian with variance $1/c$.
- In general $U''(x) \geq c$ is 'log-concave plus'

Main results of Bakry-Emery theory

Theorem

If the Bakry-Émery condition holds with constant c then

1. *the Poincaré inequality holds with constant $1/c$; i.e.*

$$\mathrm{Var}_\mu(f) \leq \frac{1}{c} \int \Gamma_1(f,f)(x) d\mu(x).$$

2. *the logarithmic Sobolev inequality holds with constant $1/c$; i.e.*

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{2c} \int \frac{\Gamma_1(f,f)(x)}{f(x)} d\mu(x).$$

- If μ is Gaussian with variance σ^2 then as above take $c = 1/\sigma^2$, and recover folklore result and Gross log-Sobolev.
- Proof via considering Markov process with generator L.

Discrete Bakry-Émery theory

- Give version of BE theory for mass functions V (supported on whole of $\mathbb{Z}_+$).

Definition

Fix probability mass function V, write L_V and L_V^* for the adjoint (with respect to counting measure) by:

$$
\begin{aligned}
L_V f(x) &:= (f(x+1) - f(x)) - \frac{V(x-1)}{V(x)} (f(x) - f(x-1)), \\
L_V^* f(x) &:= f(x-1) - \left(1 + \frac{V(x-1)}{V(x)}\right) f(x) + \frac{V(x)}{V(x+1)} f(x+1).
\end{aligned}
$$

Use convention $V(-1) = 0$.

Understanding action of L_V

- Introduced by Caputo, Dai Pra and Posta in this context.
- Corresponds to evolution of birth-and-death Markov chain:
    1. upward jumps at rate 1,
    2. downward jumps from x at rate $V(x-1)/V(x)$.
- If $V = \Pi_\lambda$ (Poisson) this is $M/M/\infty$ queue.
- Note V is invariant distribution.
- Probability mass functions $p_t := p \exp(tQ)$ satisfy

$$\frac{\partial}{\partial t} p_t(x) = L_V^* p_t(x).$$

- Or, functions $f_t = \exp(tQ) f$ satisfy

$$\frac{\partial}{\partial t} f_t(x) = L_V f_t(x).$$

Discrete log-concavity 'log-concave plus'

Definition

Given probability mass function V, write

$$\mathcal{E}^{(V)}(x) := \frac{V(x)^2 - V(x-1)V(x+1)}{V(x)V(x+1)} = \frac{V(x)}{V(x+1)} - \frac{V(x-1)}{V(x)}.$$

Condition (c-log-concavity)

If $\mathcal{E}^{(V)}(x) \geq c$ for all $x \in Z_+$, we say V is c-log-concave.

Example

- If $V = \Pi_\lambda$ then $\mathcal{E}^{(V)}(x) \equiv \lambda$.
- If V is ULC (see Pemantle, Liggett), it is c-log-concave.
- Sum of Bernoulli p_i has $c = \left(\sum_j p_j/(1-p_j)\right)^{-1}$.

Discrete Γ_1 and Γ_2

Definition

For any functions f and g, write

$$\Gamma_1^{(V)}(f,g) = \frac{1}{2}\left[L_V(fg) - fL_V g - gL_V f\right]$$

$$\Gamma_2^{(V)}(f,g) = \frac{1}{2}\left[L_V\left(\Gamma_1^{(V)}(f,g)\right) - \Gamma_1^{(V)}(f, L_V g) - \Gamma_1^{(V)}(g, L_V f)\right]$$

Condition (Integrated BE(c))

V satisfies the integrated BE(c) condition if for all functions f:

$$\sum_{x=0}^{\infty} V(x)\Gamma_2^{(V)}(f,f)(x) \geq c\sum_{x=0}^{\infty} V(x)\Gamma_1^{(V)}(f,f)(x).$$

Brute force calculation

Proposition

For any f and g, writing $Lf(x) = f(x+1) - 2f(x) + f(x-1)$ and $\Delta f(x) = f(x+1) - f(x)$ we deduce:

$$\sum_{x=0}^{\infty} V(x)\Gamma_1^{(V)}(f,g)(x) = \sum_{x=0}^{\infty} V(x)(\Delta f(x))(\Delta g(x)),$$

$$\sum_{x=0}^{\infty} V(x)\Gamma_2^{(V)}(f,g)(x) = \sum_{x=0}^{\infty} V(x)\mathcal{E}^{(V)}(x)(\Delta f(x))(\Delta g(x)) + \sum_{x=0}^{\infty} V(x)Lf(x+1)Lg(x+1).$$

Hence, if V is c-log-concave (if $\mathcal{E}^{(V)}(x) \geq c$ for all x) then the integrated $BE(c)$ condition holds.

Main result of my paper

Theorem (New modified log-Sobolev inequality)

Fix c-log-concave V. For any positive f:

$$\mathrm{Ent}_V(f) \leq \frac{1}{c} \sum_{x=0}^{\infty} V(x) f(x+1) \left(\log \left(\frac{f(x+1)}{f(x)} \right) - 1 + \frac{f(x)}{f(x+1)} \right).$$

- Since $\log 1/u - 1 + u \geq 0$ for all $u > 0$, RHS is positive.
- Strengthens and generalizes previous log-Sobolev inequalities of 1. Wu 2. Caputo et al 3. Bobkov and Ledoux.
- e.g. linearizing log and taking $V = \Pi_\lambda$ recover

$$\mathrm{Ent}_V(f) \leq \lambda \sum_{x=0}^{\infty} V(x) \frac{\Delta f(x)^2}{f(x)}.$$

- Is sharp: equality holds for $V = \Pi_\lambda$, $f(x) = \exp(ax + b)$.

Sketch of proof

- Study $\Theta(t) := \sum_x V(x) f_t(x) \log f_t(x)$ for $f_t = \exp(tQ)f$.
- As in Caputo et al.

$$\Theta'(t) = -\sum_{x=0}^{\infty} V(x) \left(f_t(x+1) - f_t(x) \right) \log \left(\frac{f_t(x+1)}{f_t(x)} \right).$$

- Similarly take:

$$\psi(t) = \sum_{x=0}^{\infty} V(x) \left(f_t(x+1) \log \left(\frac{f_t(x+1)}{f_t(x)} \right) - f_t(x+1) + f_t(x) \right).$$

- Deduce

$$\psi'(t) \leq -\sum_{x=0}^{\infty} V(x) \mathcal{E}^{(V)}(x) (f_t(x+1) - f_t(x)) \log \left(\frac{f_t(x+1)}{f_t(x)} \right).$$

Sketch of proof (cont.)

- Deduce $(-\Theta'(t)) \leq \frac{1}{c}(-\psi'(t))$.
- Hence

$$\begin{aligned} \mathrm{Ent}_V(f) &= \Theta(0) - \Theta(\infty) \\ &= \int_0^\infty -\Theta'(t) dt \\ &\leq \frac{1}{c} \int_0^\infty (-\psi'(t)) dt = \frac{1}{c}\psi(0) \end{aligned}$$

and the result follows.

Poincaré inequality

Theorem (Poincaré inequality)

- *Fix probability mass function V.*
- *For any f, writing $\Delta f(x) = f(x+1) - f(x)$:*

$$\mathrm{Var}_V(f) \leq \frac{1}{c} \sum_{x=0}^{\infty} V(x) \Delta f(x)^2,$$

if and only if V satisfies the integrated BE(c) condition.

- *Hence certainly true for c-log-concave V.*

Application 1: Concentration of measure

Proposition

- *Fix probability mass function V satisfying new modified log-Sobolev inequality with constant c.*
- *Then, writing $h(s) = (1+s)\log(1+s) - s$, for any function g with $\sup_x |g(x+1) - g(x)| \leq 1$:*

$$V\left(\{g \geq \mu_V g + t\}\right) \leq \exp\left(-\frac{h(ct)}{c}\right).$$

- Hence certainly true for c-log-concave V.
- Sharp: if $V = \Pi_\lambda$ and $g(x) = x$ then equality holds.

Application 2: Decay of entropy

- Consider probability distributions such that:

$$\begin{aligned} \frac{\partial}{\partial t} V_t(x) &= \alpha_t \left(V_t(x) - V_t(x-1) \right), \\ \frac{\partial}{\partial t} p_t(x) &= \alpha_t \left(\frac{V_t(x)}{V_t(x+1)} p_t(x+1) - \frac{V_t(x-1)}{V_t(x)} p_t(x) \right). \end{aligned}$$

- E.g. $V = \Pi_t$ and p_t is "p thinned by t".

Proposition

If $V_t(x)$ satisfies $\mathcal{E}^{(V_t)}(x) \geq c_t$ for all x then

$$D(p_t \| V_t) \leq D(p \| V) \exp\left(- \int_0^t \alpha_s c_s ds \right).$$

Application 3: hypercontractivity

Consider probability measures and functions such that:

$$\begin{aligned}
\frac{\partial}{\partial t} V_t(x) &= \alpha_t \left(V_t(x) - V_t(x-1) \right), \\
\frac{\partial}{\partial t} g_t(x) &= \alpha_t \frac{V_t(x-1)}{V_t(x)} \left(g_t(x) - g_t(x-1) \right).
\end{aligned}$$

Proposition

If V_t satisfies the new modified log-Sobolev inequality with constant c_t: writing $q(t) = p \exp\left(- \int_0^t \alpha_s c_s ds \right)$ then

$$\| \exp(g) \|_{V,p} \leq \| \exp(g_t) \|_{V_t, q(t)}.$$

(Note that $q(t) \leq p$).