The symmetric simple exclusion with slow boundaries

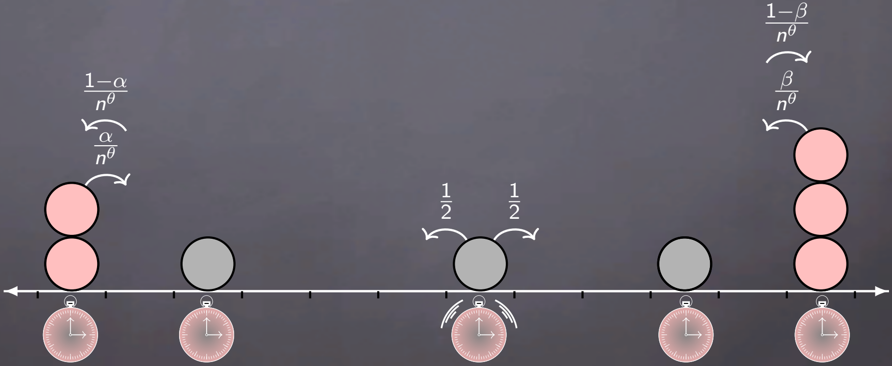
Patrícia Gonçalves
IST - Lisbon

Joint with Tertuliano Franco (UFBA, Brazil) and Adriana Neumann (UFRGS, Brazil)

Bristol
7th October 2016

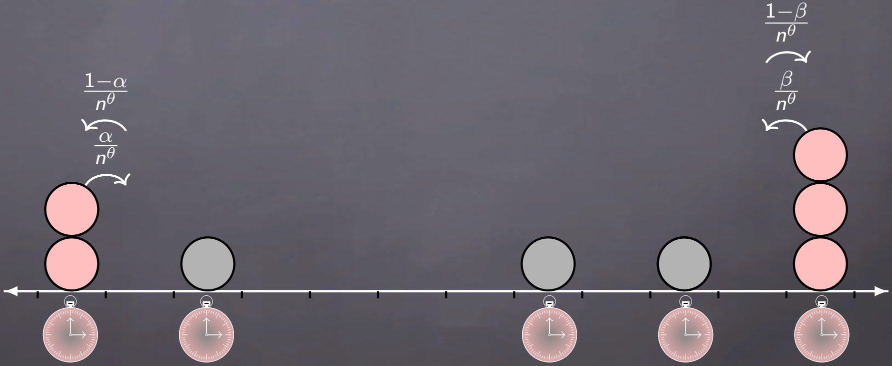
The dynamics

In the bulk:



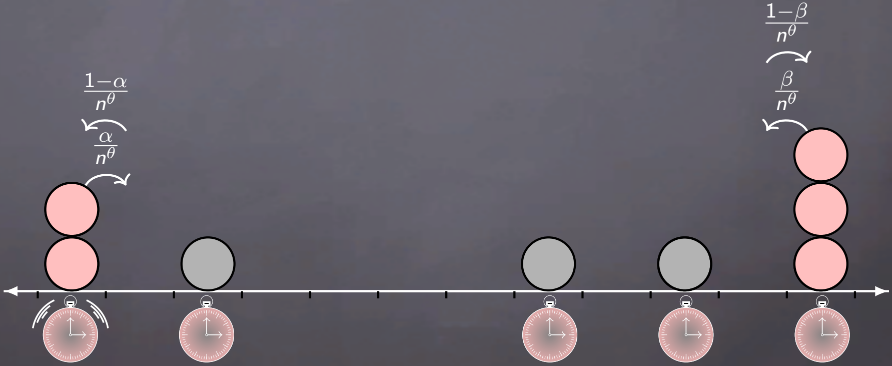
The dynamics

In the bulk:



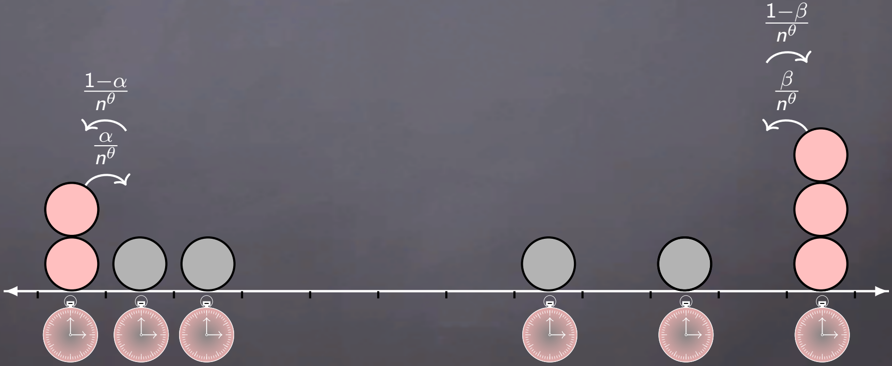
The dynamics

At the left boundary:



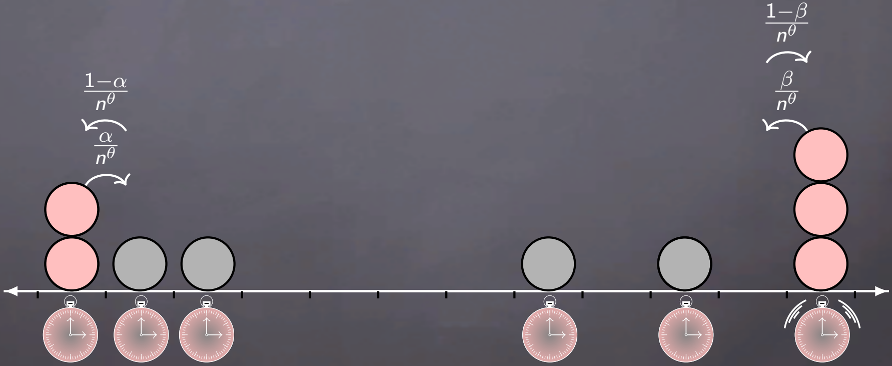
The dynamics

At the left boundary:



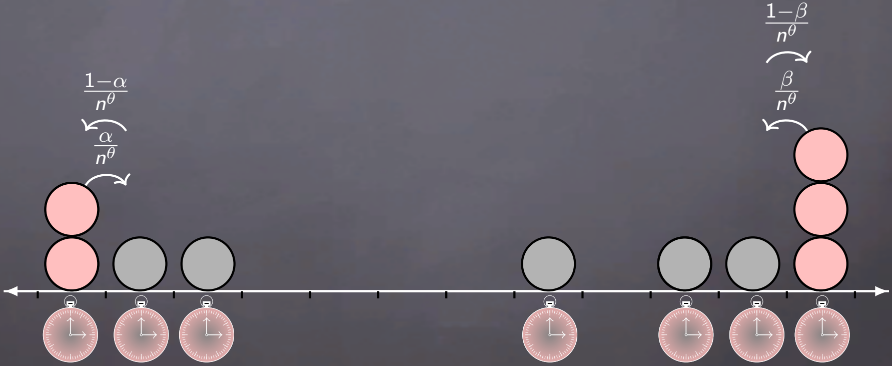
The dynamics

At the right boundary:



The dynamics

At the right boundary:



The dynamics (formally):

- For $n \geq 1$ let $\Sigma_n = \{1, \dots, n-1\}$.
- We denote the process by $\{\eta_t : t \geq 0\}$ which has state space $\Omega_n := \{0,1\}^{\Sigma_n}$.
- The infinitesimal generator $\mathcal{L}_n = \mathcal{L}_{n,0} + \mathcal{L}_{n,b}$ is given on $f : \Omega_n \to \mathbb{R}$, by

$$(\mathcal{L}_{n,0} f)(\eta) = \sum_{x=1}^{n-2} \Big(f(\eta^{x,x+1}) - f(\eta) \Big),$$

$$(\mathcal{L}_{n,b} f)(\eta) = \frac{1}{n^\theta} \sum_{x \in \{1, n-1\}} c_x(\eta) \Big(f(\sigma^x \eta) - f(\eta) \Big),$$

where for $x = 0, 1$, $c_x(\eta) = \Big[r_x(1 - \eta(x)) + (1 - r_x)\eta(x) \Big]$, $r_1 = \alpha$ and $r_{n-1} = \beta$.

Invariant measures

- If $\alpha = \beta = \rho$ the Bernoulli product measures are invariant:

$$\nu_\rho\{\eta : \eta(x) = 1\} = \rho.$$

- If $\alpha \neq \beta$ the stationary measure μ_{ss} is not known.

Questions:

- **Hydrodynamics? Studied by Baldasso et al.**
- **Fluctuations? When $\theta = 0$, studied by Landim et al.**
  - **What about $\theta > 0$?**
  - **Here we give the answer for $\theta = 1$.**
    The other cases remain open.
- **What about non-nearest neighbor jumps?**
  What is known?

Hydrodynamic Limit:

• For $\eta \in \Omega_n$ let $\pi_t^n(\eta; du) = \frac{1}{n}\sum_{x=1}^{n-1} \eta_{tn^2}(x)\delta_{x/n}(du)$.

• Fix $\rho_0 : [0,1] \to [0,1]$ and μ_n such that for every $\delta > 0$ and every continuous function $f : [0,1] \to \mathbb{R}$,

$$\frac{1}{n}\sum_{x=1}^{n-1} f\left(\tfrac{x}{n}\right)\eta(x) \to_{n\to\infty} \int_0^1 f(u)\,\rho_0(u)du, \qquad (1)$$

wrt μ_n. Then for any $t > 0$, $\pi_t^n \to \rho(t,u)du$, as $n \to \infty$, where $\rho(t,u)$ evolves according to the heat equation

$$\partial_t \rho(t,u) = \partial_u^2 \rho(t,u)$$

with different type of boundary conditions depending on the value of θ.

Hydrodynamic Equations:

- $\theta < 1$: The heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u)\,, & \text{for } t>0\,,\ u\in(0,1)\,,\\ \rho(t,0)=\alpha \quad \rho(t,1)=\beta\,, & \text{for } t>0. \end{cases}$$

- $\theta = 1$: The heat equation with Robin boundary conditions

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u)\,, & \text{for } t>0\,,\ u\in(0,1)\,,\\ \partial_u\rho(t,0)=\rho(t,0)-\alpha\,, & \text{for } t>0\,,\\ \partial_u\rho(t,1)=\beta-\rho(t,1)\,, & \text{for } t>0. \end{cases}$$

- $\theta > 1$: The heat equation with Neumann boundary conditions

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho(t,u)\,, & \text{for } t>0\,,\ u\in(0,1)\,,\\ \partial_u\rho(t,0)=\partial_u\rho(t,1)=0\,, & \text{for } t>0. \end{cases}$$

Fluctuations: the case $\theta = 1$.

Definition (The space of test functions)

Let $\mathcal{S}$ denote the set of functions $f \in C^\infty([0,1])$ such that for any $k \in \mathbb{N} \cup \{0\}$ it holds that

$$\partial_u^{2k+1} f(0) = \partial_u^{2k} f(0) \quad \text{and} \quad \partial_u^{2k+1} f(1) = -\partial_u^{2k} f(1).$$

Definition (Density fluctuation field)

We define the density fluctuation field $\mathcal{Y}_\cdot^n$ as the time-trajectory of linear functionals acting on functions $f \in \mathcal{S}$ as

$$\mathcal{Y}_t^n(f) \;=\; \frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} f(\tfrac{x}{n}) \Big(\eta_{tn^2}(x) - \mathbb{E}_{\mu_n}[\eta_{tn^2}(x)] \Big) \,.$$

What are the conditions on the initial state μ_n?

- For each $n \in \mathbb{N}$, the measure μ_n is associated to a measurable profile $\rho_0 : [0,1] \to [0,1]$ in the sense of (1).
- There exists a constant $C_1 > 0$ not depending on n such that for $\rho_0^n(x) = \mathbb{E}_{\mu_n}[\eta_0(x)]$

$$\max_{x \in \Sigma_n} \left| \rho_0^n(x) - \rho_0(\tfrac{x}{n}) \right| \leq \frac{C_1}{n}.$$

- There exists a constant $C_2 > 0$ not depending on n such that for

$$\varphi_0^n(x,y) = \mathbb{E}_{\mu_n}[\eta(x)\eta(y)] - \rho_0^n(x)\rho_0^n(y)$$

it holds that

$$\max_{1 \leq x < y \leq n-1} \left| \varphi_0^n(x,y) \right| \leq \frac{C_2}{n}.$$

Non-equilibrium fluctuations: the case $\theta = 1$.

For each $n \geq 1$, let Q_n be the probability measure on $\mathcal{D}([0,T],\mathcal{S}')$ induced by $\mathcal{Y}^n_\cdot$ and μ_n.

Theorem (Non-equilibrium fluctuations)

The sequence of measures $\{Q_n\}_{n\in\mathbb{N}}$ is tight on $\mathcal{D}([0,T],\mathcal{S}')$ and all limit points Q are p.m. concentrated on paths $\mathcal{Y}_\cdot$ satisfying

$$\mathcal{Y}_t(f) \;=\; \mathcal{Y}_0(T_t f) + W_t(f)\,,$$

for any $f \in \mathcal{S}$. Above $T_t : \mathcal{S} \to \mathcal{S}$ is the semigroup associated to the hydrodynamic equation with $\alpha = \beta = 0$, and $W_t(f)$ is a mean zero Gaussian variable of variance $\int_0^t \|\nabla T_{t-r} f\|^2_{L^2(\rho_r)} dr$, where $\rho(t,u)$ is the solution of the hydrodynamic equation, and $\chi(u) = u(1-u)$. Moreover, $\mathcal{Y}_0$ and W_t are uncorrelated in the sense that $\mathbb{E}_Q\Big[\mathcal{Y}_0(f)\, W_t(g)\Big] \;=\; 0$ for all $f, g \in \mathcal{S}$.

Above, for $r > 0$

$$\begin{aligned}\langle f, g\rangle_{L^2(\rho_r)} =& [\alpha - (1-2\alpha)\rho(r,0)]\, f(0)g(0)\\ &+ [\beta - (1-2\beta)\rho(r,1)]\, f(1)g(1)\\ &+ \int_0^1 2\chi(\rho(r,u))\, f(u)g(u)\, du.\end{aligned}$$

Definition

Let $\Delta : \mathcal{S} \to \mathcal{S}$ be the Laplacian operator which is defined on $f \in \mathcal{S}$ as

$$\Delta f(u) = \begin{cases} \partial_u^2 f(u)\,, & \text{if } u \in (0,1)\,,\\ \partial_u^2 f(0^+)\,, & \text{if } u = 0\,,\\ \partial_u^2 f(1^-)\,, & \text{if } u = 1\,.\end{cases}$$

Above, $\partial_u^2 f(a^\pm)$ denotes the side limits at the point a. The definition of the operator $\nabla : \mathcal{S} \to C^\infty[0,1]$ is analogous.

Theorem (Ornstein-Uhlenbeck limit)

Assume that the sequence of initial density fields $\{\mathcal{Y}_0^n\}_{n\in\mathbb{N}}$ *converges, as* $n \to \infty$*, to a mean-zero Gaussian field* $\mathcal{Y}_0$ *with covariance given on* $f, g \in \mathcal{S}$ *by*

$$\lim_{n\to\infty} \mathbb{E}_{\mu_n}\Big[\mathcal{Y}_0^n(f)\mathcal{Y}_0^n(g)\Big] = \mathbb{E}\Big[\mathcal{Y}_0(f)\mathcal{Y}_0(g)\Big] := \sigma(f,g).$$

Then, the sequence $\{Q_n\}_{n\in\mathbb{N}}$ *converges, as* $n \to \infty$*, to a generalized Ornstein-Uhlenbeck process, which is the formal solution of the equation:*

$$\partial_t \mathcal{Y}_t = \Delta \mathcal{Y}_t dt + \sqrt{2\chi(\rho_t)}\nabla W_t,$$

where $\mathcal{W}_t$ *is a space-time white noise of unit variance. As a consequence, the covariance of the limit field* $\mathcal{Y}_t$ *is given on* $f, g \in \mathcal{S}$ *by*

$$E\left[\mathcal{Y}_t(f)\mathcal{Y}_s(g)\right] = \sigma(T_t f, T_s g) + \int_0^s \langle \nabla T_{t-r} f, \nabla T_{s-r} g\rangle_{L^2(\rho_r)} dr.$$

Corollary (Local Gibbs state)

Fix a Lipschitz profile $\rho_0 : [0,1] \to [0,1]$ *and suppose to start the process from a* <u>*Bernoulli product*</u> *measure given by* $\mu_n\{\eta : \eta(x) = 1\} = \rho_0(\frac{x}{n})$*. Then, the previous theorem remains in force and the covariance in this case is given on* $f, g \in \mathcal{S}$ *by*

$$E\left[\mathcal{Y}_t(f)\mathcal{Y}_s(g)\right] = \int_0^1 \chi(\rho_0(u))\, f(u)g(u)\, du + \int_0^s \langle \nabla T_{t-r} f, \nabla T_{s-r} g \rangle_{L^2(\rho_r)} dr,$$

where $\rho(t,u)$ *is the solution of the hydrodynamic equation with initial condition given by* $\rho_0(\cdot)$*.*

Stationary fluctuations: the case $\theta = 1$.

Theorem (Stationary fluctuations)

Suppose to start the process from μ_{ss} with $\alpha \neq \beta$. Then, $\mathcal{Y}^n$ converges to the centered Gaussian field $\mathcal{Y}$ with covariance given on $f, g \in \mathcal{S}$ by:

$$
\begin{aligned}
E_{\mu_{ss}}[\mathcal{Y}(f)\mathcal{Y}(g)] \; = \; & \int_0^1 \chi(\overline{\rho}(u)) f(u) g(u)\, du \\
& - \Big(\frac{\beta-\alpha}{3}\Big)^2 \int_0^1 [(-\Delta)^{-1} f(u)] g(u)\, du \\
& + \frac{2(2\beta+\alpha)(2\beta-1)}{3} \int_0^\infty T_t f(1)\, T_t g(1)\, dt \\
& + \frac{2(\beta+2\alpha)(2\alpha-1)}{3} \int_0^\infty T_t f(0)\, T_t g(0)\, dt\,,
\end{aligned}
$$

with $\overline{\rho}(u) = \big(\frac{\beta-\alpha}{3}\big)\, u + \frac{\beta+2\alpha}{3}$, which is the stationary solution of the hydrodynamic equation.

How do we prove the results?

Two things to do:

- Tightness;
- Characterization of limit points.

Let us focus on the second point.

Associated martingale.

Let $\phi : [0,1] \to \mathbb{R}$ be a test function and note that

$$M_t^n(\phi) := \mathcal{Y}_t^n(\phi) - \mathcal{Y}_0(\phi) - \int_0^t n^2 \mathcal{L}_n \mathcal{Y}_s^n(\phi)\, ds$$

is a martingale where

$$\begin{aligned} n^2 \mathcal{L}_n \mathcal{Y}_s^n(\phi) &= \frac{1}{\sqrt{n}} \sum_{x=1}^{n-1} \Delta_n \phi(\tfrac{x}{n}) \Big(\eta_{sn^2}(x) - \rho_s^n(x) \Big)\, ds \\ &+ \sqrt{n} \left[\nabla_n^+ \phi(0) - \phi(\tfrac{1}{n}) \right] \Big(\eta_{sn^2}(1) - \rho_s^n(1) \Big) \\ &+ \sqrt{n} \left[\phi(\tfrac{n-1}{n}) + \nabla_n^- \phi(1) \right] \Big(\eta_{sn^2}(n-1) - \rho_s^n(n-1) \Big). \end{aligned}$$

Note that the second term at the right hand side of the previous expression is $\mathcal{Y}_s^n(\Delta_n \phi)$. Above, we have used the notation

$$\nabla_n^+ \phi(x) = n \left[\phi(\tfrac{x+1}{n}) - \phi(\tfrac{x}{n}) \right] \quad \text{and} \quad \nabla_n^- \phi(x) = n \left[\phi(\tfrac{x}{n}) - \phi(\tfrac{x-1}{n}) \right].$$

The empirical profile.

Fix an initial measure μ_n in Ω_n. For $x \in \Sigma_n$ and $t \geq 0$, let

$$\rho_t^n(x) \;=\; \mathbb{E}_{\mu_n}[\eta_{tn^2}(x)]\,.$$

We extend this definition to the boundary by setting

$$\rho_t^n(0) \;=\; \alpha \text{ and } \rho_t^n(n) \;=\; \beta\,, \text{ for all } t \geq 0\,.$$

A simple computation shows that $\rho_t^n(\cdot)$ is a solution of the discrete equation given by

$$\begin{cases} \partial_t \rho_t^n(x) \;=\; (n^2 \mathcal{B}_n \rho_t^n)(x)\,, \;\; x \in \Sigma_n\,, \;\; t \geq 0\,, \\ \rho_t^n(0) \;= \alpha\,, \;\; t \geq 0\,, \\ \rho_t^n(n) \;=\; \beta\,, \;\; t \geq 0\,, \end{cases}$$

where the operator $\mathcal{B}_n$ acts on functions $f : \Sigma_n \cup \{0, n\} \to \mathbb{R}$ as

$$(\mathcal{B}_n f)(x) \;=\; \sum_{y=0}^{n} \xi_{x,y}^n (f(y) - f(x))\,, \quad \text{for } x \in \Sigma_n\,,$$

where

$$\xi^n_{x,y} = \begin{cases} 1\,, & \text{if } |y-x| = 1 \text{ and } x, y \in \Sigma_n\,, \\ \frac{1}{n}\,, & \text{if } x = 1, y = 0 \text{ and } x = n-1\,, y = n\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Proposition

Let $\rho^n_t(\cdot)$ be as above. Then, there exists $C > 0$ which does not depend on n such that

$$\left|\rho^n_t(x+1) - \rho^n_t(x)\right| \leq \frac{C}{n}\,,$$

for all $x \in \{1, \ldots, n-2\}$, uniformly in $t \geq 0$.

The correlation estimate.

Definition (Two-point correlation function)

For each $x, y \in \Sigma_n$, $x < y$, and $t \in [0, T]$, we define the two-point correlation function as

$$\varphi_t^n(x, y) \;=\; \mathbb{E}_{\mu_n}[\eta_{tn^2}(x)\eta_{tn^2}(y)] - \rho_t^n(x)\rho_t^n(y)\,,$$

where ρ_t^n is as above. Moreover, for $x = 0$ or $y = n$, we set $\varphi_t^n(x, y) = 0$,

Proposition

There exists $C > 0$ such that

$$\sup_{t \geq 0} \max_{(x,y) \in V_n} |\varphi_t^n(x, y)| \;\leq\; \frac{C}{n}\,,$$

where $V_n \;=\; \{(x, y)\,;\, x, y \in \mathbb{N},\, 0 < x < y < n\}$.

If jumps are arbitrarily big? (Joint with C. Bernardin and B. Oviedo (University of Nice))

Let $\gamma > 2$ and $p(\cdot)$ be a translation invariant transition probability given at $z \in \mathbb{Z}$ by

$$p(z) = \begin{cases} \frac{c_\gamma}{|z|^{\gamma+1}}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

where $c_\gamma = \frac{2}{\zeta(\gamma+1)}$. Since $p(\cdot)$ is symmetric it is mean zero, that is:

$$\sum_{z \in \mathbb{Z}} z p(z) = 0$$

and since $\gamma > 2$ we define its variance by

$$\sigma_\gamma^2 = \sum_{z \in \mathbb{Z}} z^2 p(z) < \infty.$$

The infinitesimal generator:

$\mathcal{L}_n = \mathcal{L}_{n,0} + \mathcal{L}_{n,r} + \mathcal{L}_{n,\ell}$ where

$$(\mathcal{L}_{n,0} f)(\eta) = \frac{1}{2} \sum_{x,y \in \Sigma_n} p(x-y)[f(\sigma^{x,y}\eta) - f(\eta)],$$

$$(\mathcal{L}_{n,\ell} f)(\eta) = \frac{1}{n^\theta} \sum_{\substack{x \in \Sigma_n \\ y \le 0}} p(x-y) c_x(\eta;\alpha)[f(\sigma^x\eta) - f(\eta)],$$

$$(\mathcal{L}_{n,r} f)(\eta) = \frac{1}{n^\theta} \sum_{\substack{x \in \Sigma_n \\ y \ge n}} p(x-y) c_x(\eta;\beta)[f(\sigma^x\eta) - f(\eta)]$$

where

$$c_x(\eta) := [(1-\eta_x) r_x + (1-r_x)\eta_x] \, .$$

Hydrodynamic Equations:

- $\theta < 1$: The heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho(t,u) = \frac{\sigma^2}{2}\partial_u^2 \rho(t,u)\,, & \text{for } t > 0\,,\ u \in (0,1)\,, \\ \rho(t,0) = \alpha \quad \rho(t,1) = \beta\,, & \text{for } t > 0. \end{cases}$$

- $\theta = 1$: The heat equation with Robin boundary conditions

$$\begin{cases} \partial_t \rho(t,u) = \frac{\sigma^2}{2}\partial_u^2 \rho(t,u)\,, & \text{for } t > 0\,,\ u \in (0,1)\,, \\ \partial_u \rho(t,0) = \frac{2m}{\sigma^2}(\rho(t,0) - \alpha)\,, & \text{for } t > 0\,, \\ \partial_u \rho(t,1) = \frac{2m}{\sigma^2}(\beta - \rho(t,1))\,, & \text{for } t > 0. \end{cases}$$

Above $m = \sum_{y \geq 1} y p(y)$.

- $\theta > 1$: The heat equation with Neumann boundary conditions

$$\begin{cases} \partial_t \rho(t,u) = \frac{\sigma^2}{2}\partial_u^2 \rho(t,u)\,, & \text{for } t > 0\,,\ u \in (0,1)\,, \\ \partial_u \rho(t,0) = \partial_u \rho(t,1) = 0\,, & \text{for } t > 0. \end{cases}$$

What about $\gamma \in (1,2)$?

This is in progress. So far we know that for $\gamma \in (1,2)$ and $\theta = 0$, we get the fractional heat equation with Dirichlet boundary conditions:

$$\begin{cases} \partial_t \rho(t,u) = -(-\Delta)^{\gamma/2} \rho(t,u)\,, & \text{for } t>0\,,\ u \in (0,1)\,, \\ \rho(t,0) = \alpha \quad \rho(t,1) = \beta\,, & \text{for } t > 0, \end{cases}$$

where the fractional Laplacian $(-\Delta)^{\gamma/2}$ of exponent $\gamma/2$ is defined on the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|f(u)|}{(1+|u|)^{1+\gamma}} du < \infty$$

by

$$(-\Delta)^{\gamma/2} f(u) = c_\gamma \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \mathbf{1}_{|y-u| \geq \varepsilon} \frac{f(u) - f(y)}{|y-u|^{1+\gamma}} dy.$$

References.

1. Baldasso, R. and Menezes, O. and Neumann, A., Souza, R.: *Exclusion process with slow boundary*, at arxiv.1407.7918.

2. Franco, G., Neumann: *Hydrodynamical behavior of symmetric exclusion with slow bonds*, Annales Institute Henri Poincaré: Prob. Stats., 49 n. 2, 402–427, (2013).

3. Franco, G., Neumann: *Phase transition in equilibrium fluctuations of symmetric slowed exclusion*, Stoc. Proc. Appl., 123, no. 12, 4156–4185, (2013).

4. Franco, T., G., Neumann, A. (2016): *Non-equilibrium and stationary fluctuations for a slowed boundary symmetric exclusion process*, at arxiv.org.

5. Landim, C., Milanes, A., Olla, S.: *Stationary and nonequilibrium fluctuations in boundary driven exclusion processes*, Markov Processes Related Fields, 14, 165–184 (2008).

THANK YOU VERY MUCH FOR THE ATTENTION!