A product-sum lemma

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This is a small note on an important fact from calculus. It is good to know in general, will be used later on, and was not included in the Further Topics in Probability revision notes at the time of writing that one.

Lemma 1. *Let* $a_k \in (0,\,1)$ *with* $\lim_{k\to\infty} a_k = 0$. *Then*

$$\prod_{k=1}^{\infty}(1-a_k) = 0 \quad \Leftrightarrow \quad \sum_k a_k = \infty.$$

Proof. Convexity of the exponential function implies $1 - x \le \mathrm{e}^{-x}$ for any $x \in \mathbb{R}$. As terms in the product are non-negative,

$$0 \le \prod_{k=1}^{\infty}(1-a_k) \le \prod_{k=1}^{\infty}\mathrm{e}^{-a_k} = \mathrm{e}^{-\sum_{k=1}^{\infty} a_k}.$$

This proves $\Leftarrow$.

The function e^{-2x} is smooth with value 1 and derivative -2 at $x = 0$. Hence for all small enough $x > 0$, $1 - x \ge \mathrm{e}^{-2x}$. There is an index K that makes a_k small enough for this purpose for any $k \ge K$. Therefore

$$\prod_{k=K}^{\infty}(1-a_k) \ge \prod_{k=K}^{\infty}\mathrm{e}^{-2a_k} = \mathrm{e}^{-2\sum_{k=K}^{\infty} a_k}.$$

If $\prod_{k=1}^{\infty}(1-a_k) = 0$, then the left hand-side above is also zero, which proves $\Rightarrow$. $\square$