Steiner trees in the stochastic mean-field model of distance

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Joint work with Ayalvadi Ganesh
and Balint Toth

Climbing Redwoods

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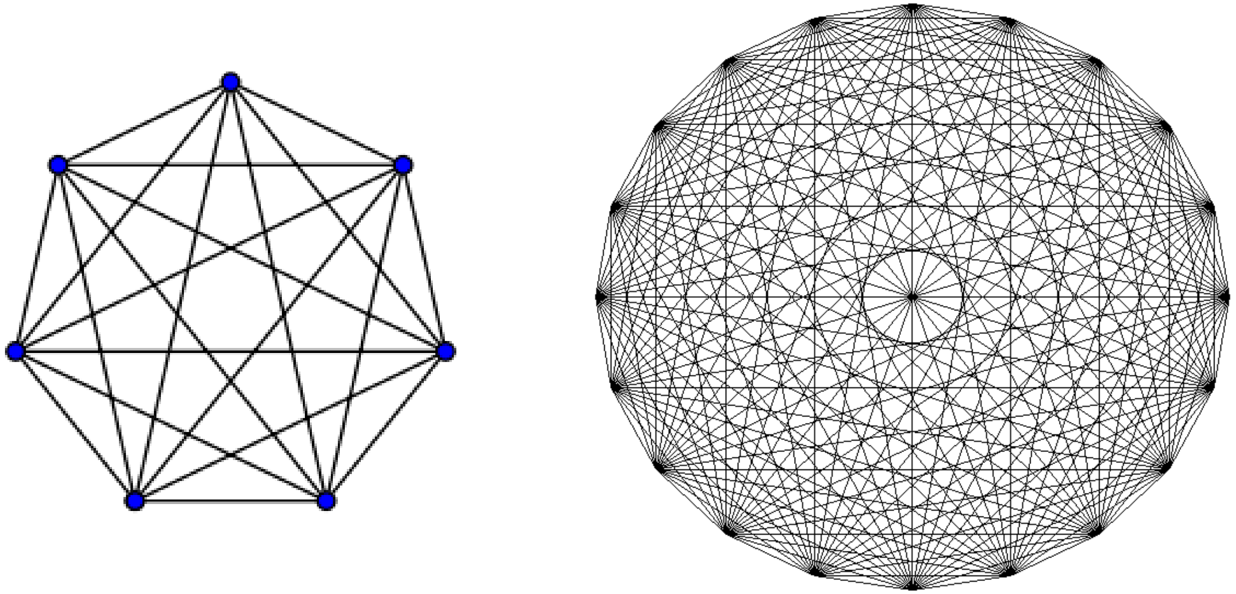
Random edge-weighted graphs

- Complete graph on n nodes, $G = K_n$
- Random edge weights, iid $Exp(1)$



Distance between nodes: Epidemic

- First passage percolation
- Single node infected initially
- Edge weight = time for infection to cross that edge
- Length of shortest path between nodes *u* and *v* is the same as the time for infection started at *u* to reach *v* (or vice versa)

Analysis of first-passage percolation

- T_k : first time that k nodes are infected
- Number of edges between infected and uninfected nodes : $k(n-k)$
- Time to infect one more node is minimum of $k(n-k)$ independent $Exp(1)$ r.v.s.

$$E[T_{k+1} - T_k] = \frac{1}{k(n-k)} = \frac{1}{n}\left(\frac{1}{k} + \frac{1}{n-k}\right)$$

Analysis (cont.)

- Time to infect all nodes is

$$T_n = (T_n - T_{n-1}) + \cdots + (T_2 - T_1) + T_1$$

- So $E[T_n] \sim 2\log(n)/n$
- Most nodes infected $\sim \log(n)/n$
- Diameter of graph $\sim 3\log(n)/n$

Steiner tree problem

- Fix k points on our graph G
- Find the minimum weight tree connecting the points
- For k typical nodes call this weight W_k
- Study asymptotics of this random variable, for k fixed and n tending to infinity

Previous results

- Bollobas, Gamarnik, Riordan and Sudakov (2004):

$$W_k \sim (k-1)\frac{\log(n)-\log(k)}{n}$$

- Here, the k nodes are chosen at random. Equivalently, the nodes are fixed first, and the edge weights assigned afterwards.

Previous results

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$$W_k \sim (k-1)\frac{\log(n)}{n}$$

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Previous results

- What if we first assign edge weights, then choose the k nodes that maximise the weight of the Steiner tree?
- Call this random variable M_k
- Janson (1999):

$$W_2 \sim \frac{\log(n)}{n}, \qquad M_2 \sim 3\frac{\log(n)}{n}$$

More precisely

$$\frac{W_2}{\log n/n} \xrightarrow{p} 1 \qquad as\ n \to \infty$$

$$\frac{M_2}{\log n/n} \xrightarrow{p} 3 \qquad as\ n \to \infty$$

$$\frac{W_k}{\log n/n} \xrightarrow{p} (k-1) \quad as\ n \to \infty$$

Distribution of Typical Distance

$$\frac{W_2}{\log n/n} \xrightarrow{p} 1 \ \ as \ n \to \infty$$

$$nW_2 - \log n \xrightarrow{d} \Lambda_1 + \Lambda_2 - \Lambda_3 \ \ as \ n \to \infty$$

Related work

- Random edge-weighted model very popular, has been used to study many combinatorial optimisation problems:

- Minimum spanning tree (Frieze, 1985):

$$W_n = M_n \sim \zeta(3) = \sum_{j=1}^{\infty} \frac{1}{j^3}$$

Related work

- Travelling salesman tour (Frieze, 2004):

$$\zeta(3) \leq W_{TSP} \leq 6$$

- Shortest path tree (van der Hofstad, Hooghiemstraa, van Mieghem, 2006):

$$W_{SPT} \sim \zeta(2) = \sum_{j=1}^{\infty} \frac{1}{j^2}$$

- Distribution of Diameter (Bhamidi, van der Hofstad, 2013)

Our results (1)

- The weight of the max-weight k-Steiner tree among all choices of k nodes is

$$M_k \sim (2k-1)\frac{\log(n)}{n}$$

Our results (1)

- Theorem (D., A. Ganesh)

$$\frac{M_k}{\log n/n} \xrightarrow{p} (2k-1) \;\; as \; n \to \infty$$

Intuition

- Typical distance between most pairs of nodes is $\log(n)/n$
- Some nodes are remote – at distance $\log(n)/n$ from their nearest neighbour
- But have ‘typical’ neighbours at this distance
- The typical neighbours are joined by a k-Steiner tree of weight $(k-1)\log(n)/n$
- Intuition can be turned into a lower bound

Some loose upper bounds

- Janson's result on graph diameter implies that
$$M_k \leq 3(k-1)\frac{\log(n)}{n}$$
- Consider infection started at a typical node: reaches all nodes by time $2\log(n)/n$
- Use resulting paths to connect given k nodes: yields
$$M_k \leq 2k\,\frac{\log(n)}{n}$$

Sketch of upper bound proof

- Want to show that, for any k nodes, the weight of the k-Steiner tree connecting them is bounded by $(2k-1)\frac{\log(n)}{n}$
- Use Chernoff bound on RV dominating weight of a typical k-Steiner tree, apply union bound

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$$
\begin{aligned}
& \mathbb{P}(W_k \geq (2k-1+\varepsilon)\log n/n) \\
\leq\ & \mathbb{P}(X(S) \geq (2k-1+\varepsilon)\log n/n) \\
\leq\ & \mathbb{E}(e^{Xnt-(2k-1+\varepsilon)t\log n}) \\
=\ & O(n^{-k-\varepsilon})
\end{aligned}
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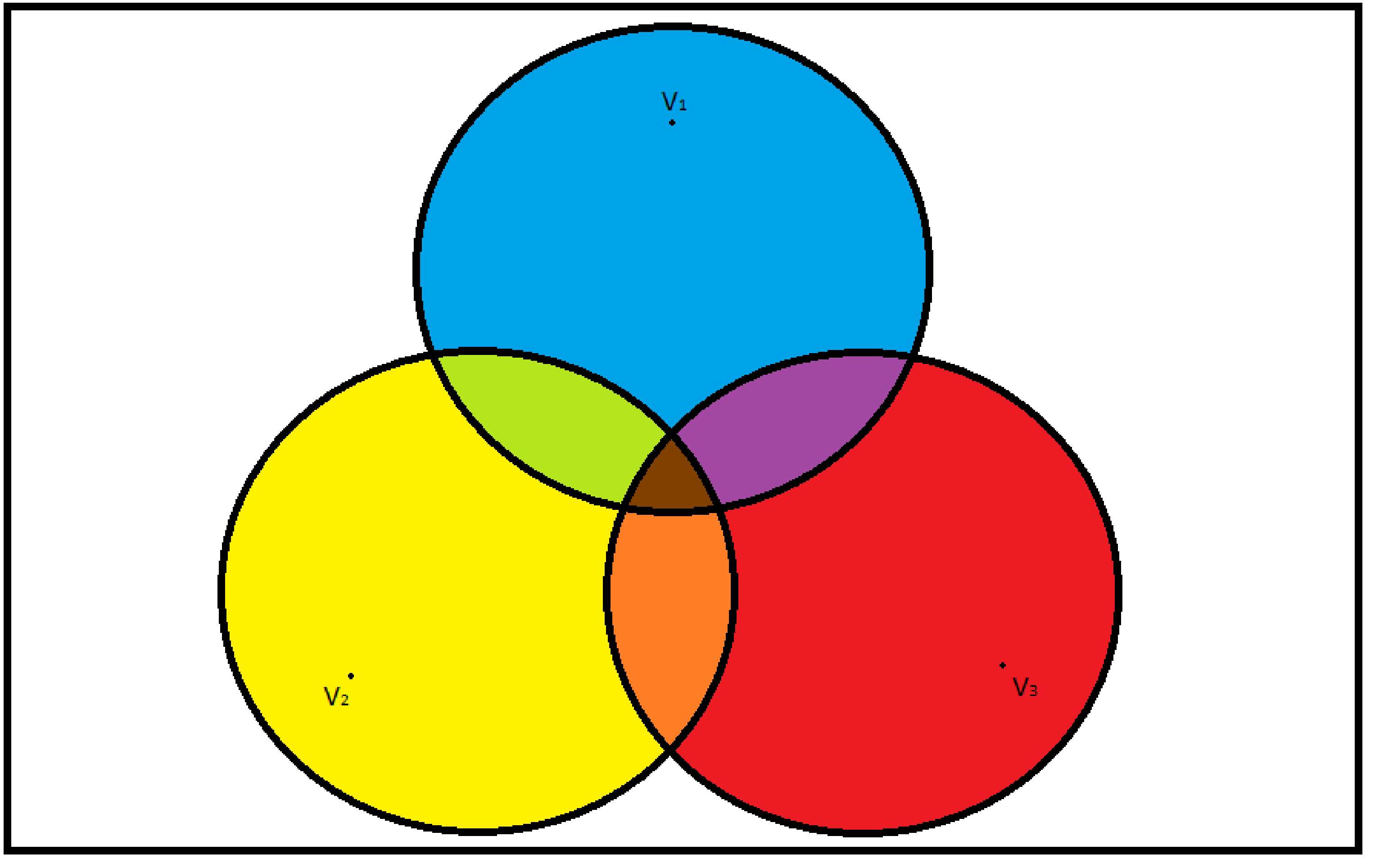
$$\begin{aligned}
& \mathbb{P}(M_k \geq (2k-1+\varepsilon)\log n/n) \\
\leq\ & \mathbb{P}\left(\bigcup_{|S|=k} X(S) \geq (2k-1+\varepsilon)\log n/n\right) \\
\leq\ & \binom{n}{k} O(n^{-k-\varepsilon}) \\
=\ & O(n^{-\varepsilon})
\end{aligned}$$

Upper bound: intuition

- Pick k nodes, and start infection simultaneously from each of them
- Grow infected sets until each has (a bit more than) $n^{(k-1)/k}$ nodes
- If subsets were independent, would all have a node in common

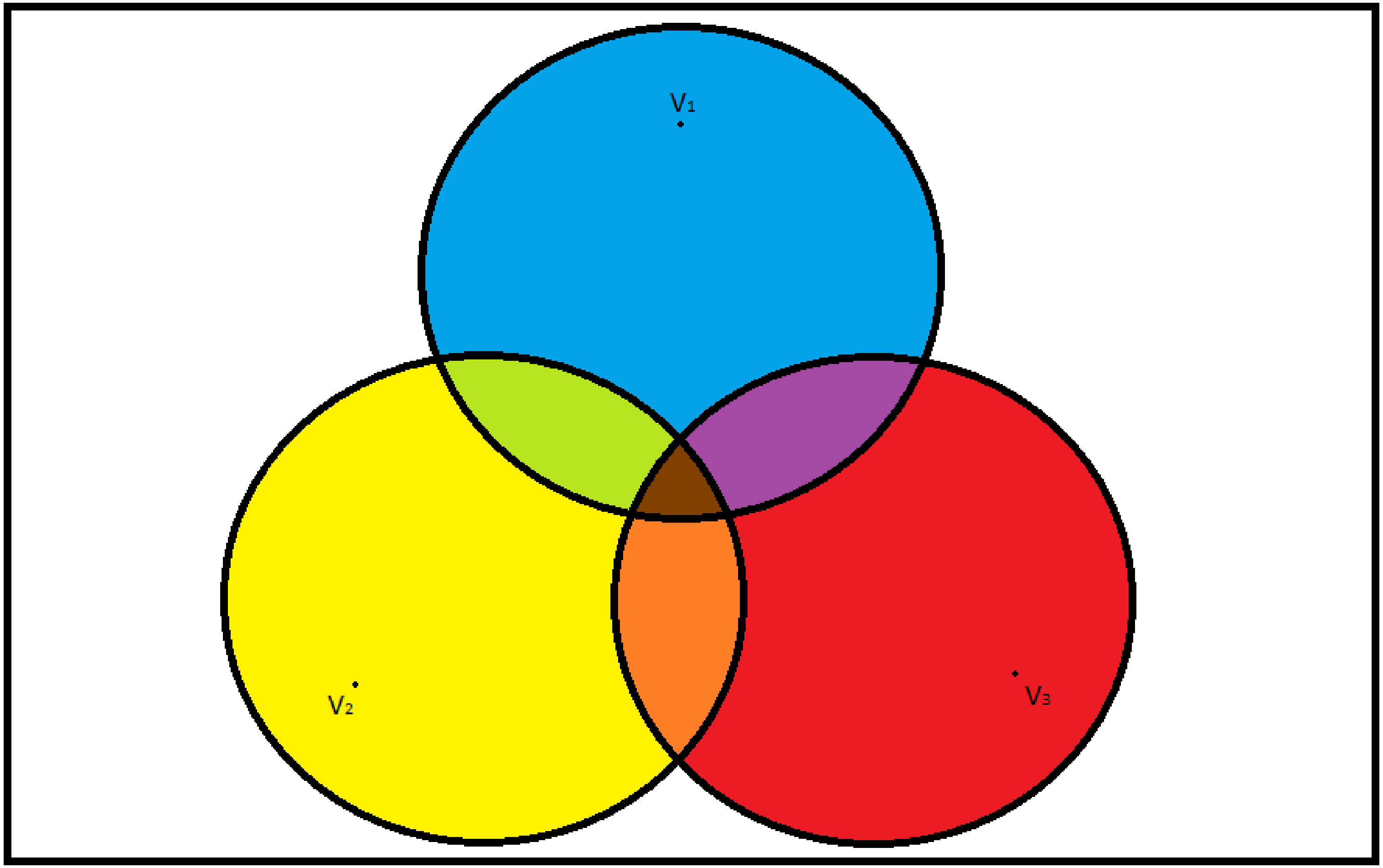
Upper bound: intuition



Upper bound: intuition

- Pick k nodes, and start infection simultaneously from each of them
- Grow infected sets until each has $Cn^{(k-1)/k}$ nodes
- If subsets were independent, would all have a node in common
- *But sets are not independent after first pair intersect*

Upper bound: intuition



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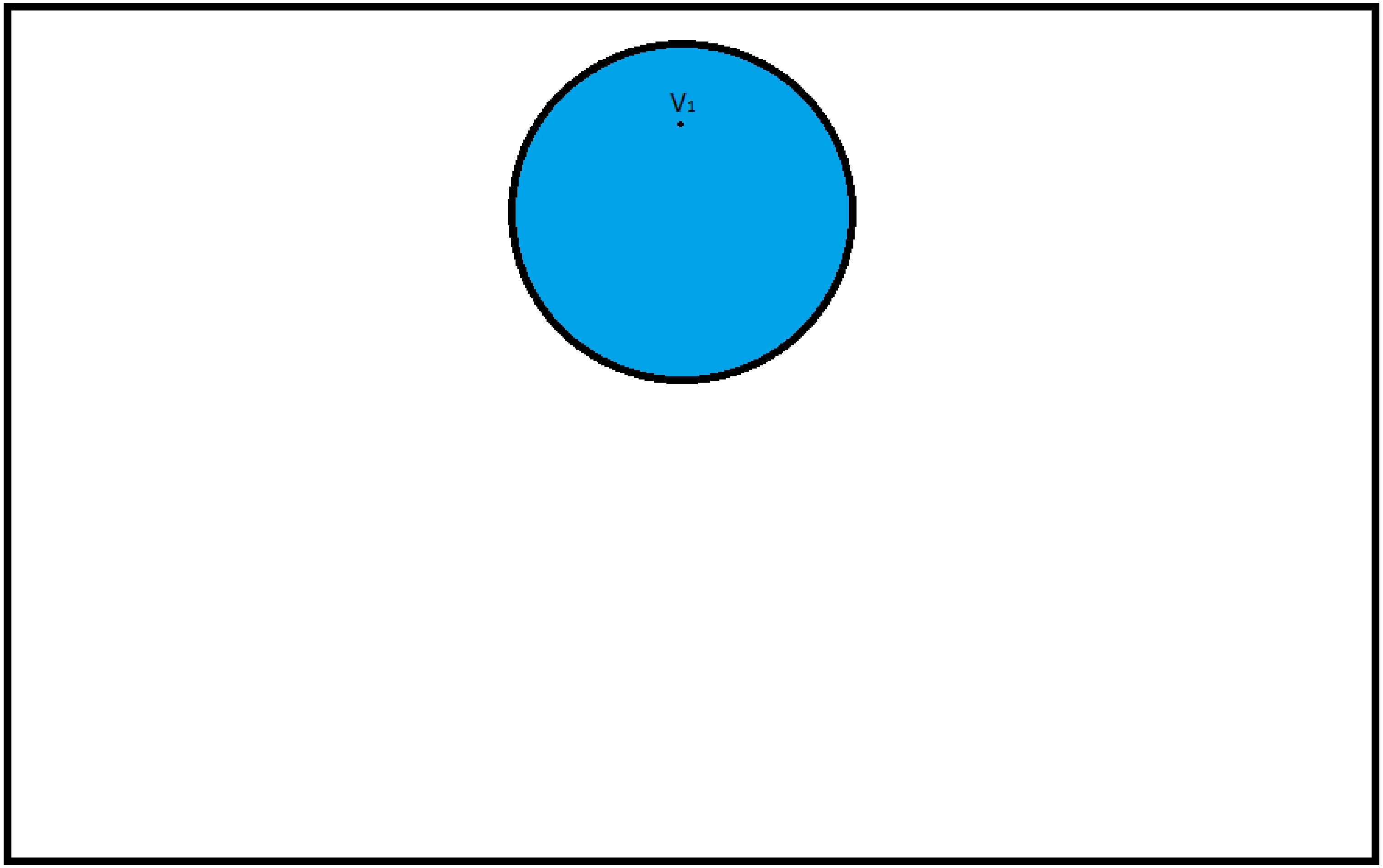

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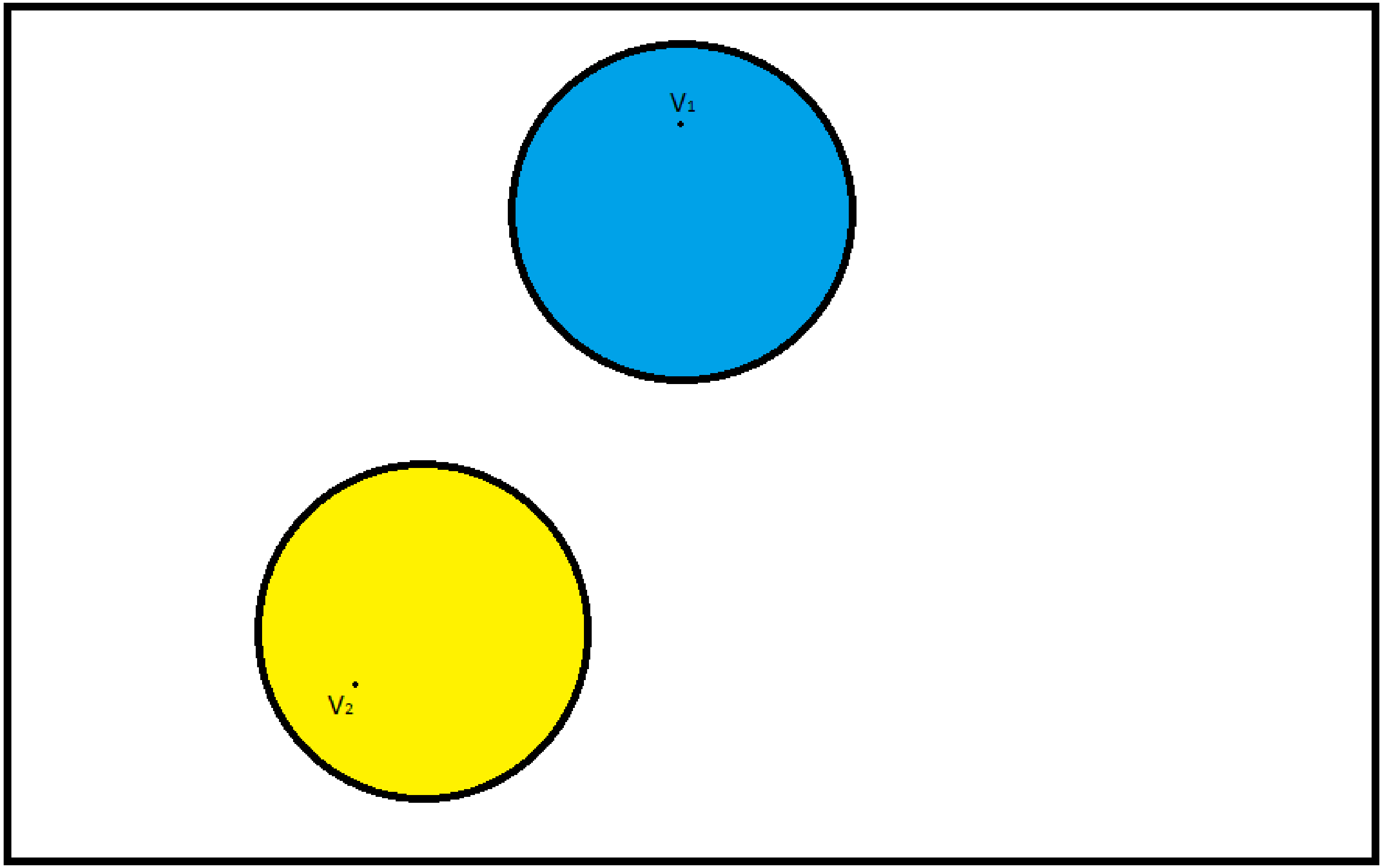

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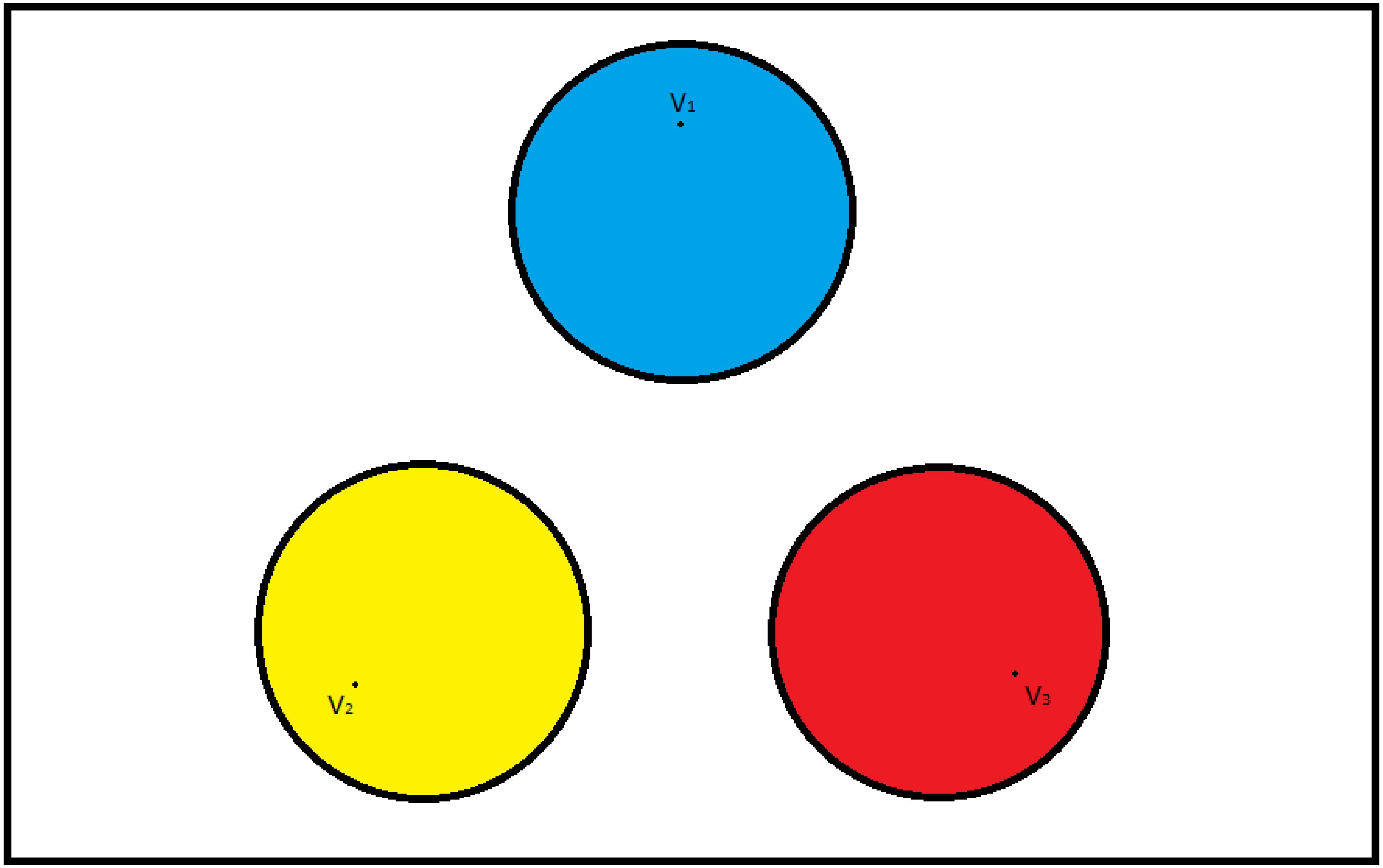
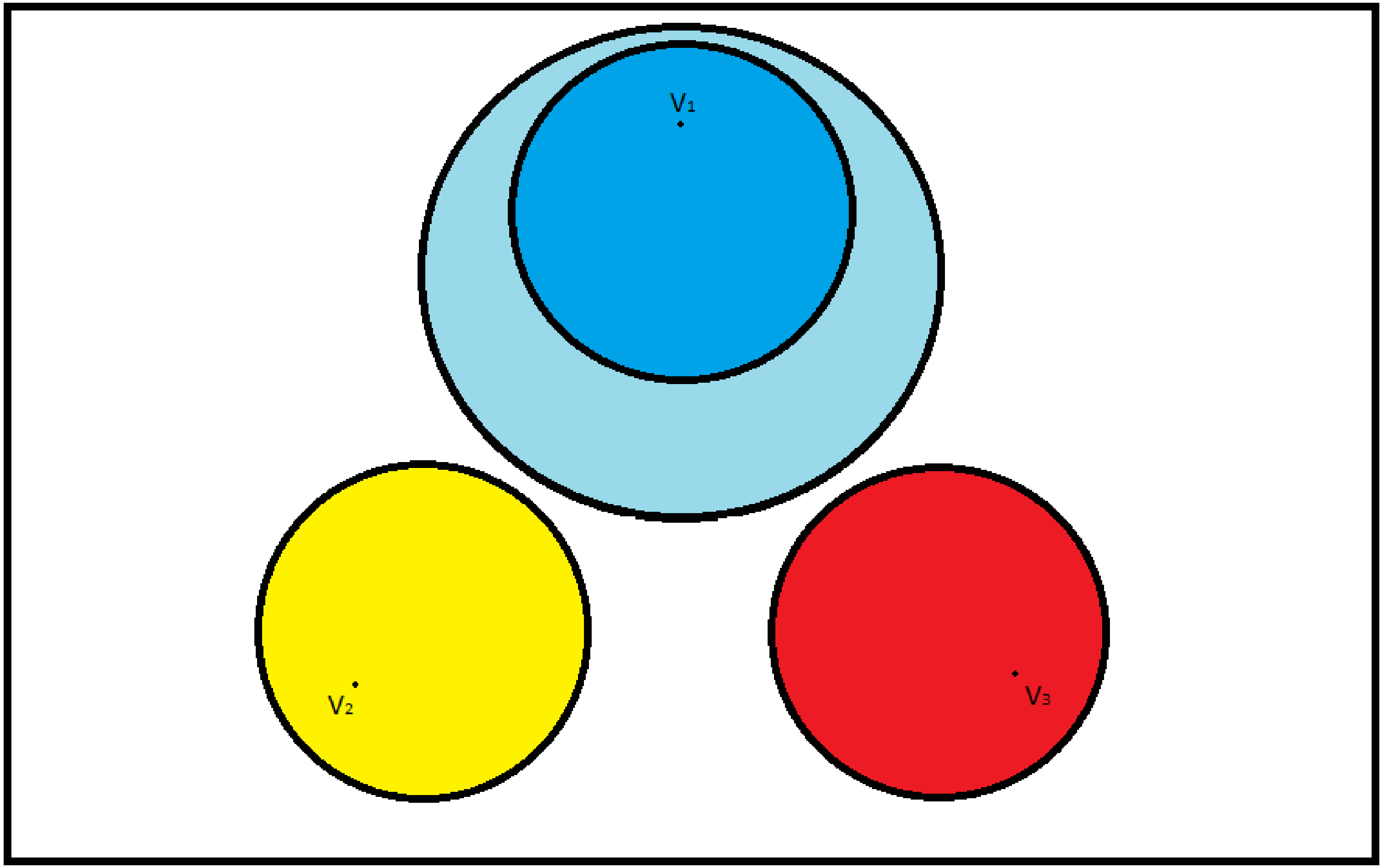
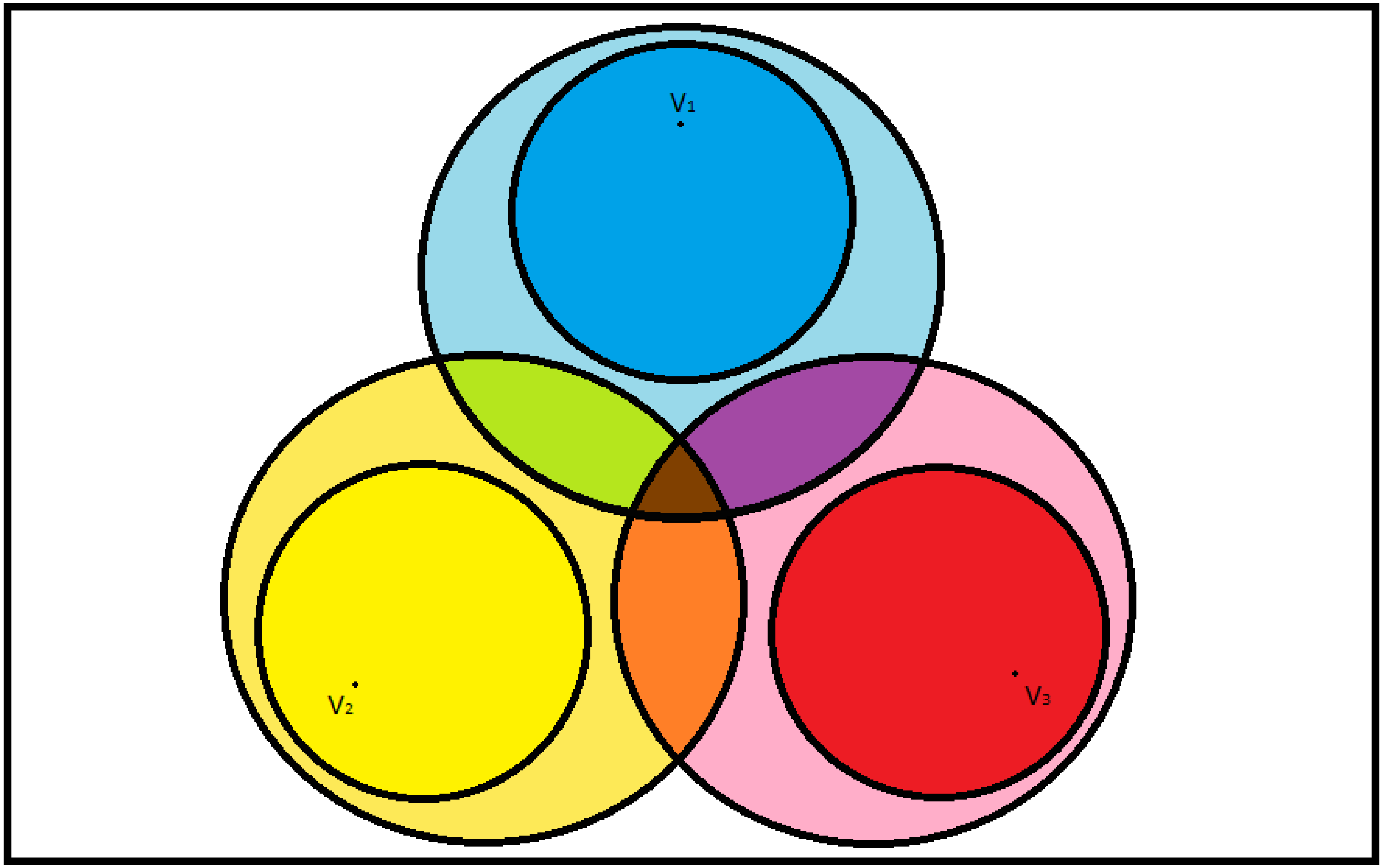


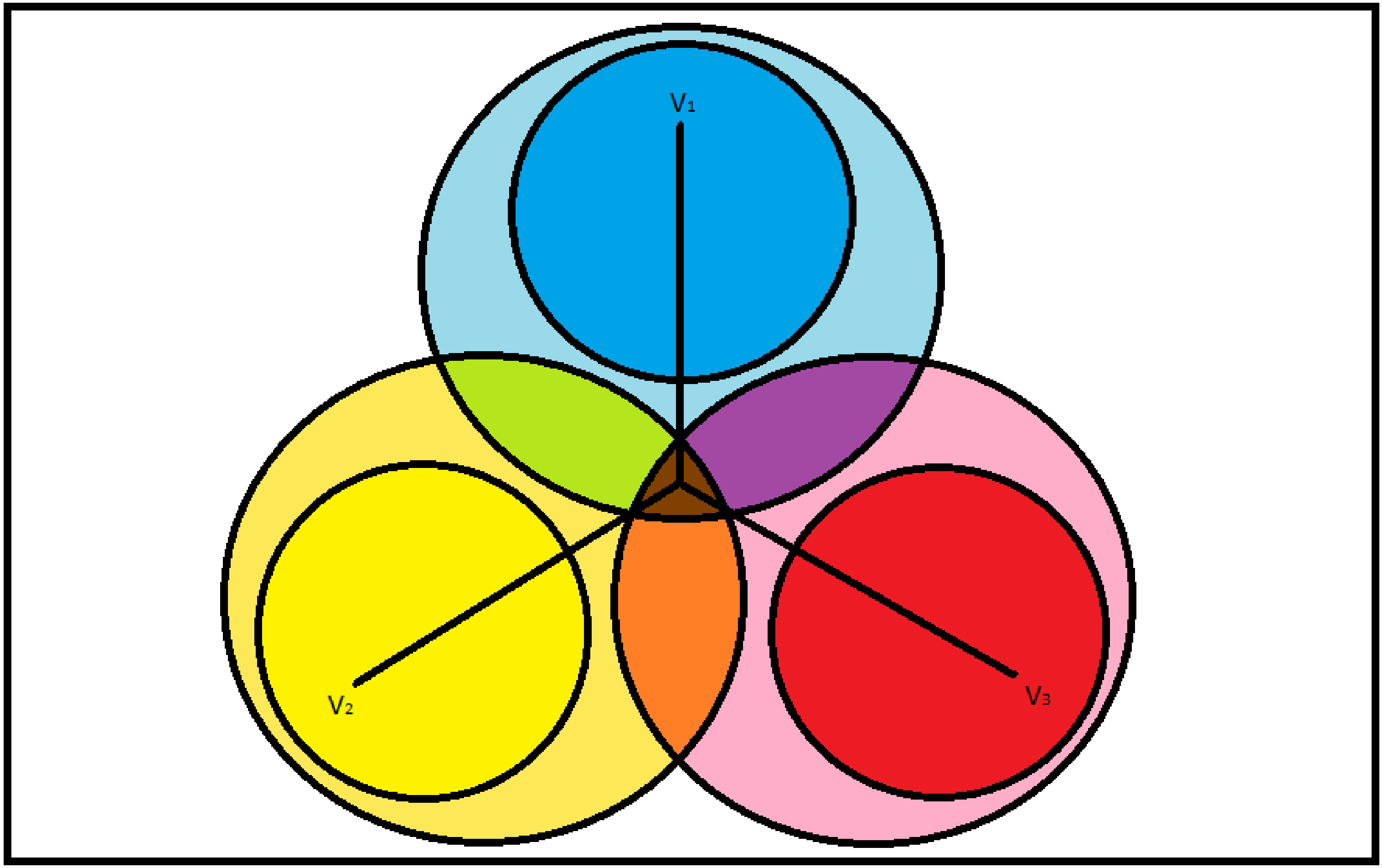
Upper bound: intuition



Upper bound: intuition



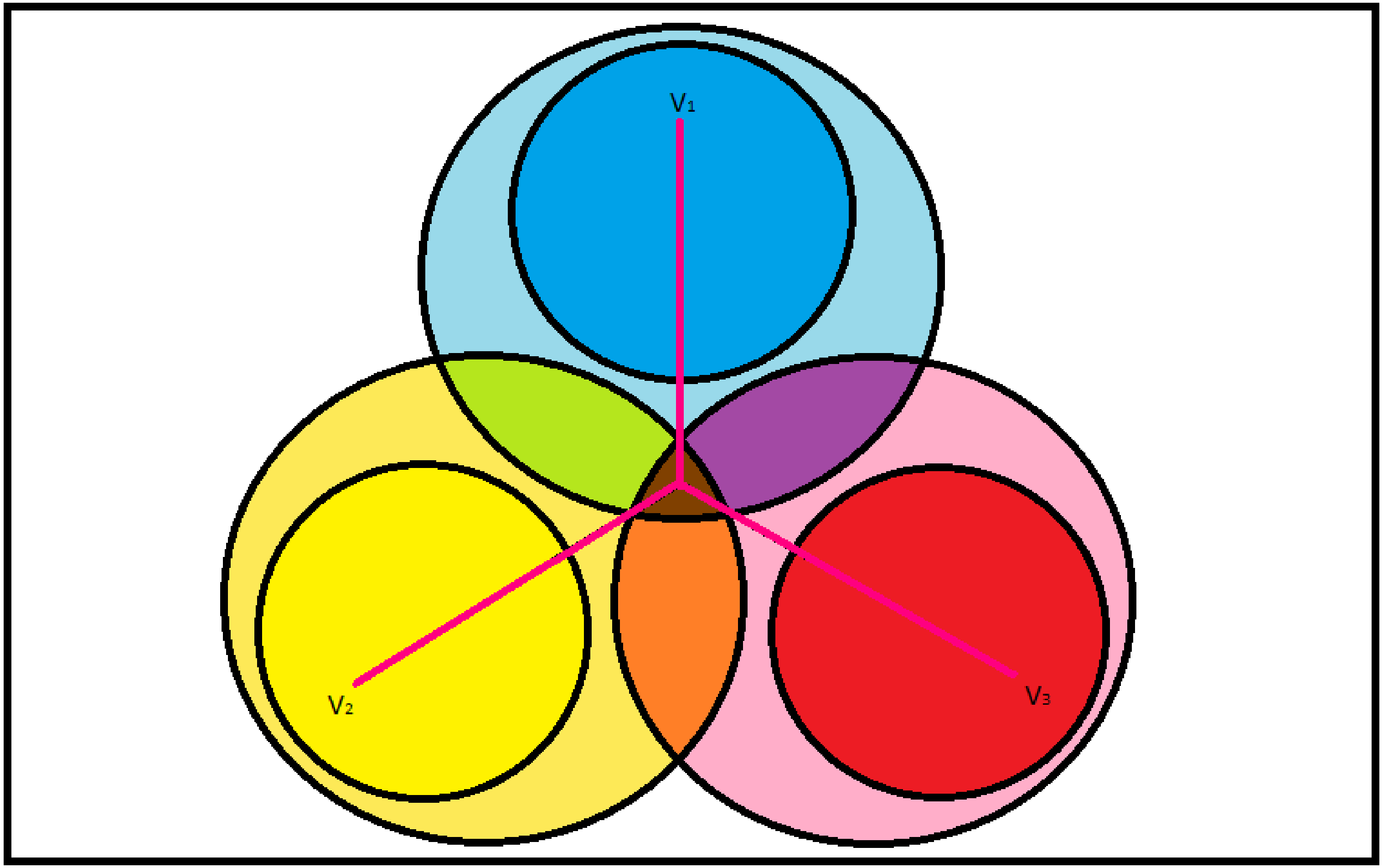
Upper bound: intuition



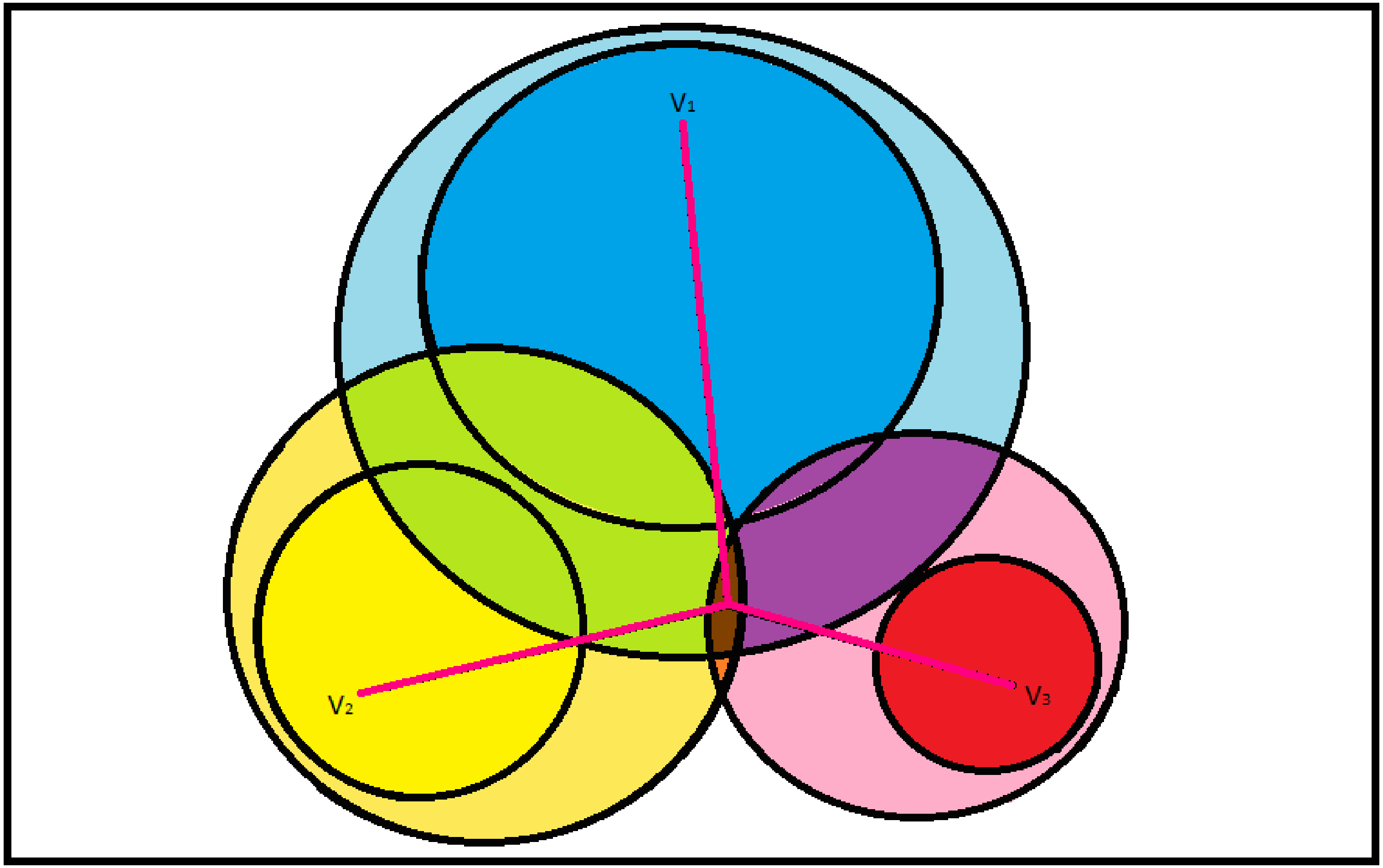
Asymptotic shapes

- Weight of k-Steiner tree is $(k-1)\log(n)/n$
- Same as sum of shortest path lengths from one of the k nodes to the other $k-1$
- Suggests tree is degenerate – has no internal nodes
- Growing infected sets from each node gives same total weight, but different shape
- Expected weight not sufficiently informative, need to look at fluctuations

Asymptotic shapes



Asymptotic shapes



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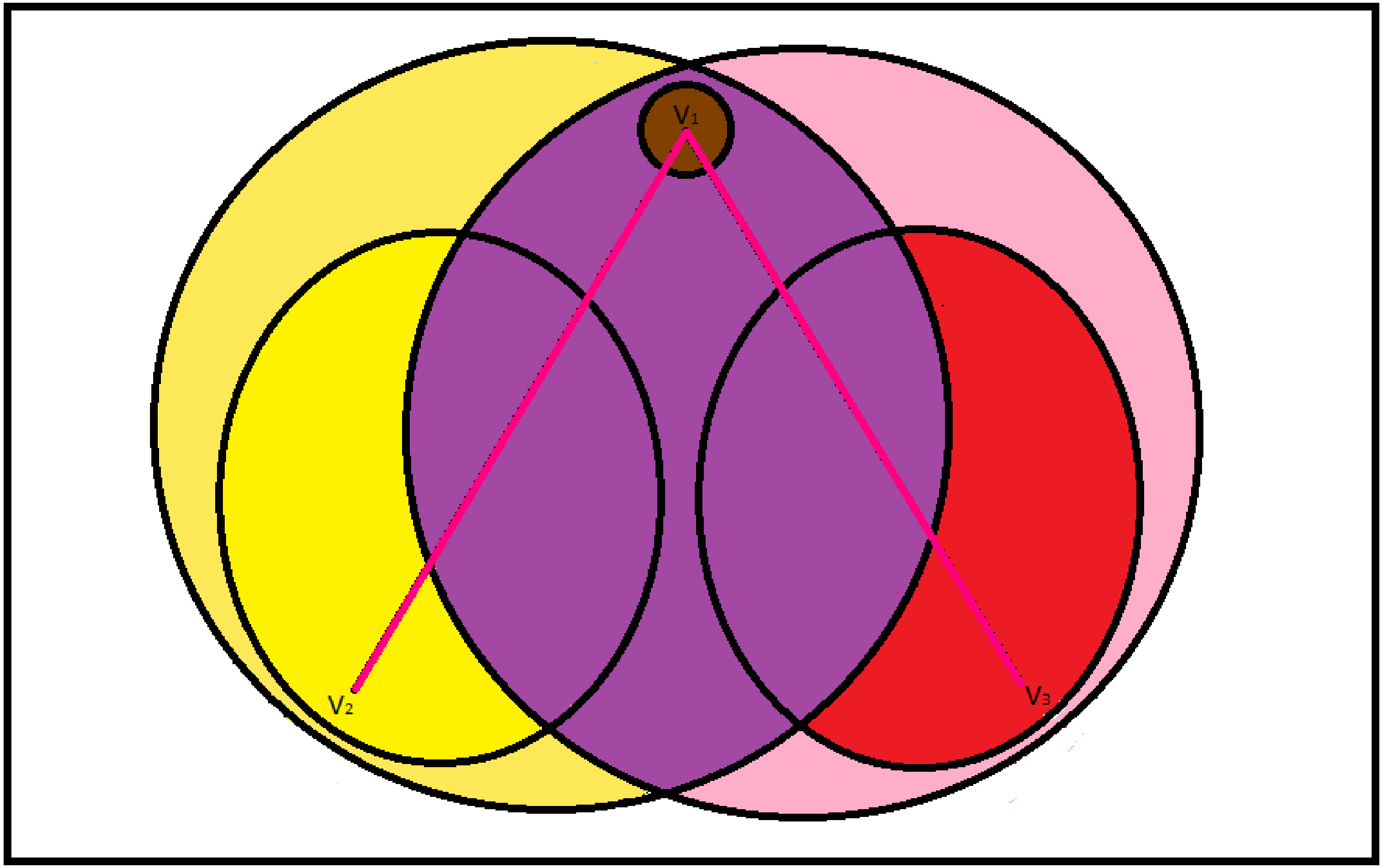
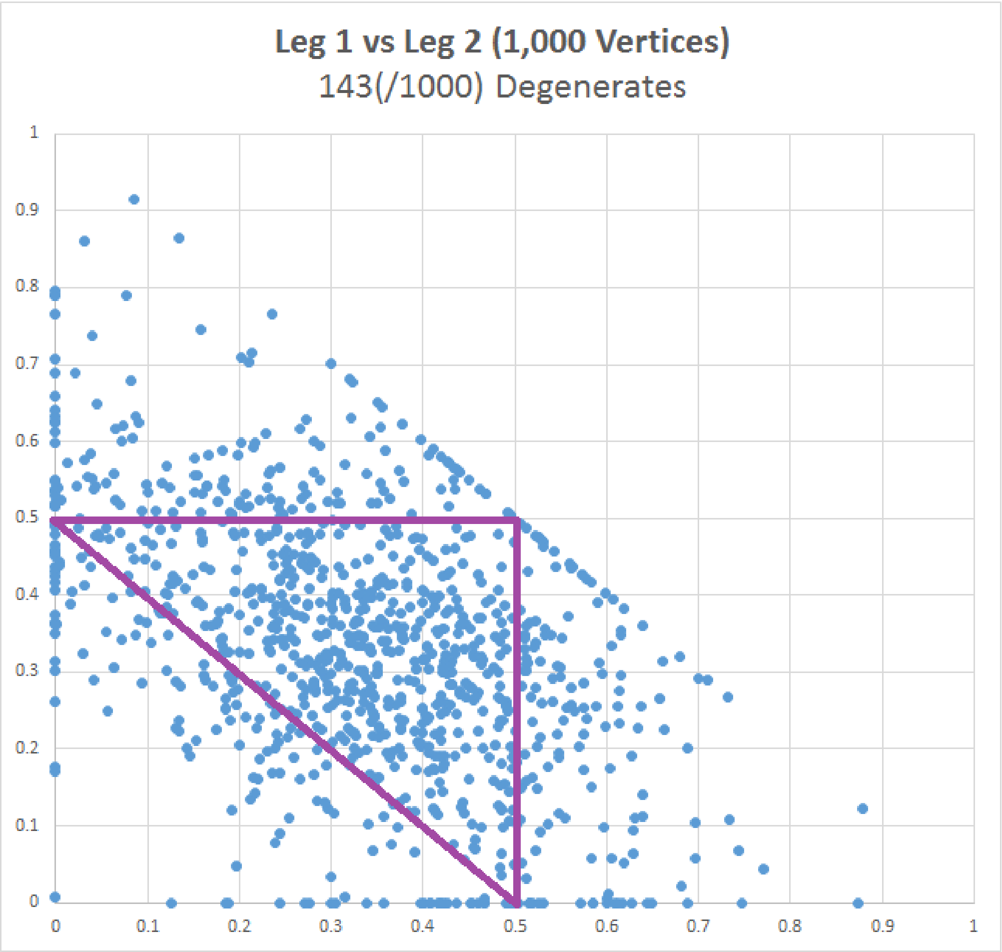
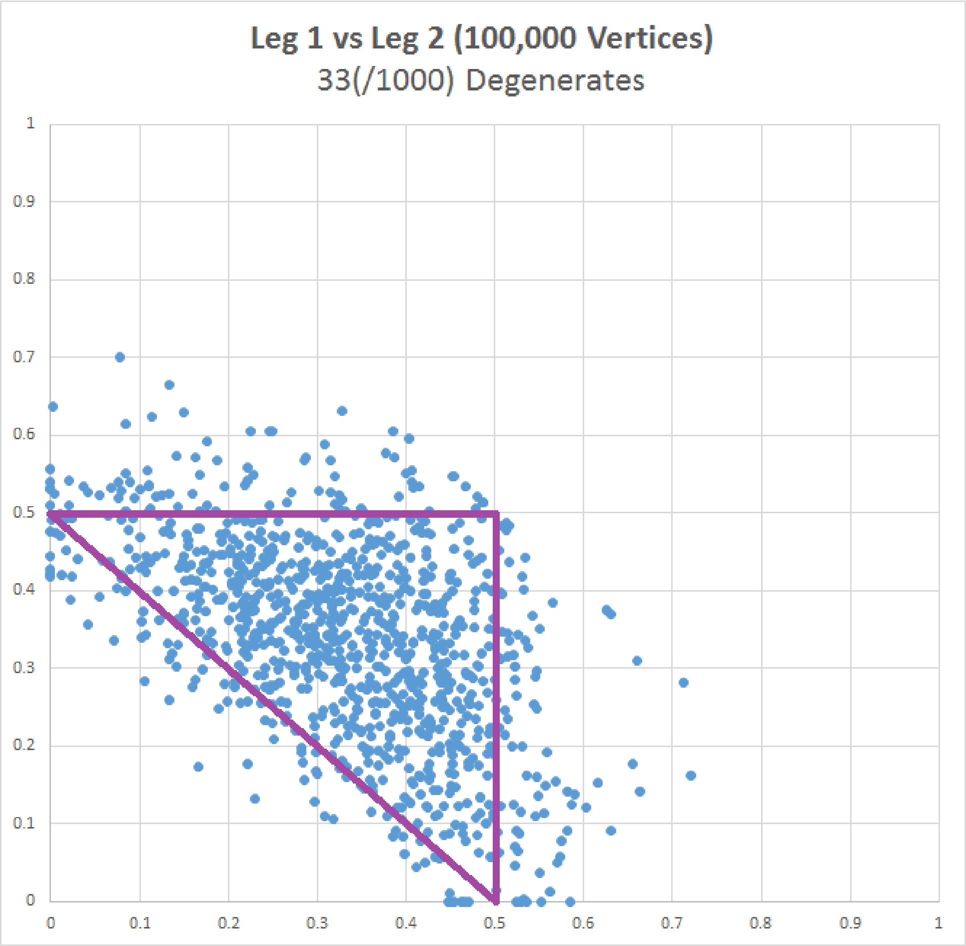


Simulations: 3-Steiner tree
2,250 Vertices



Simulations: 3-Steiner tree 100,000 Vertices



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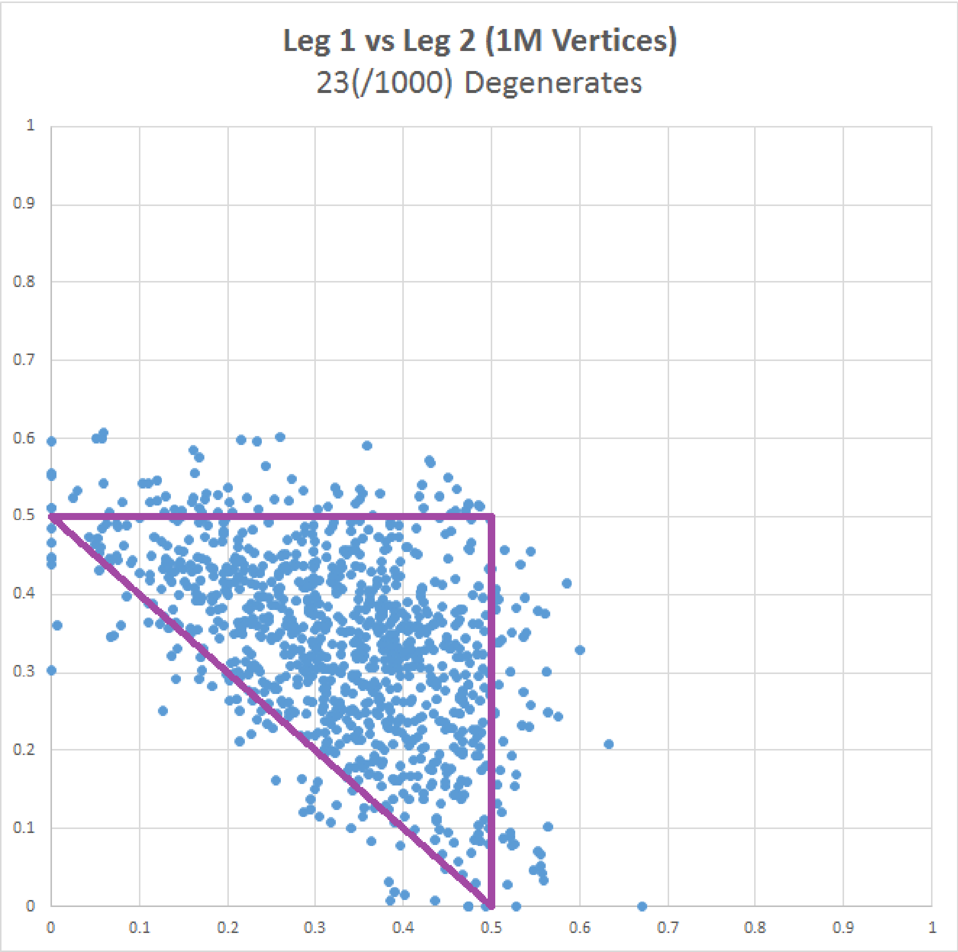
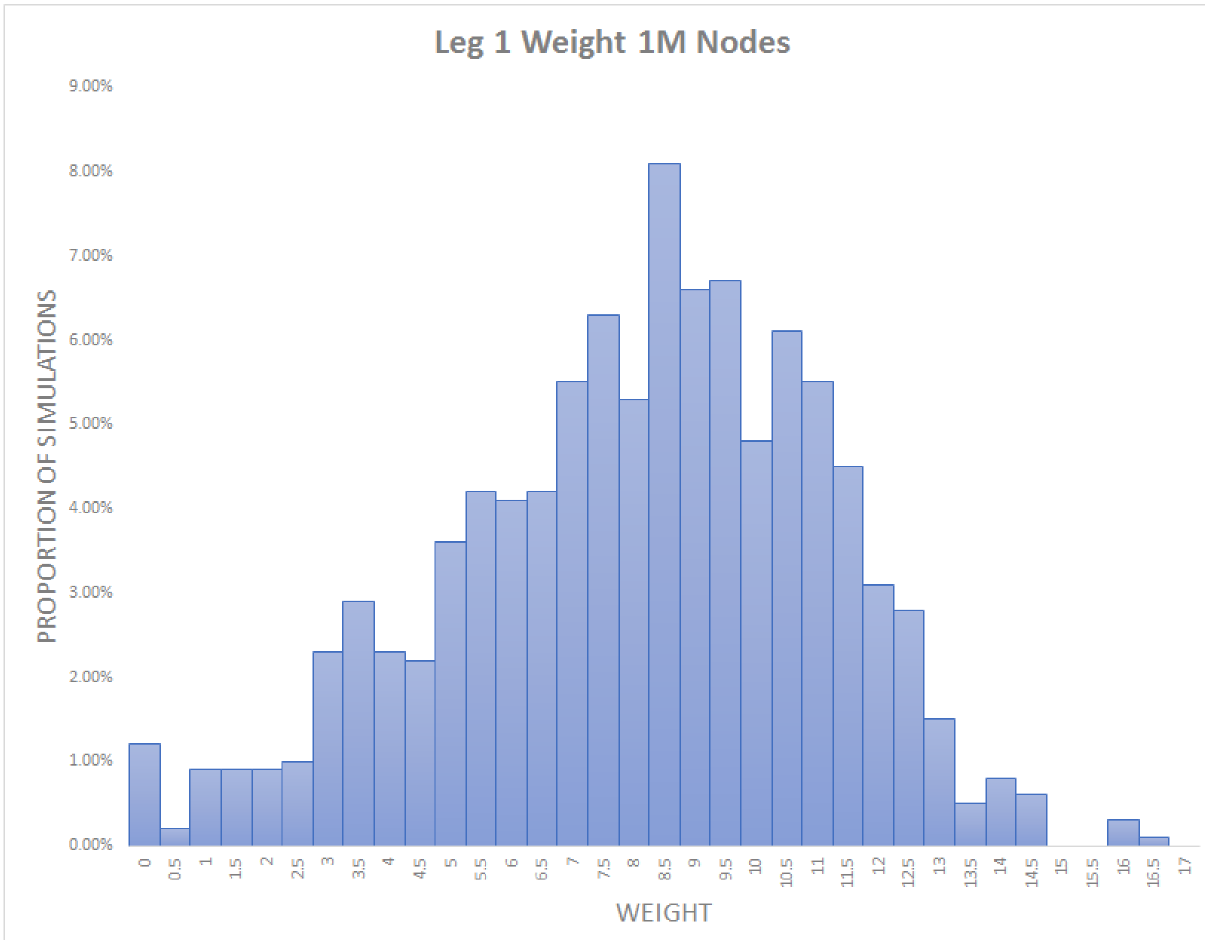
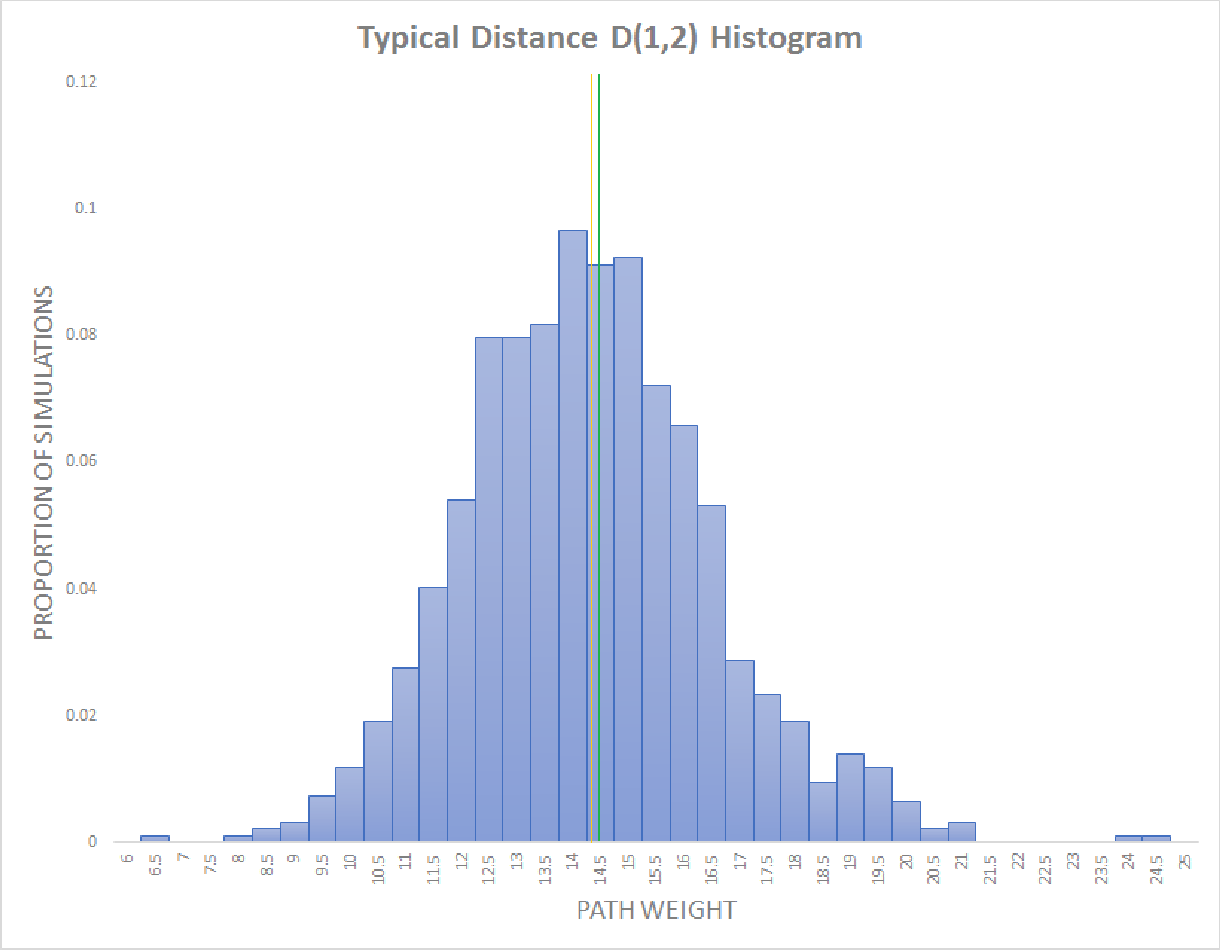


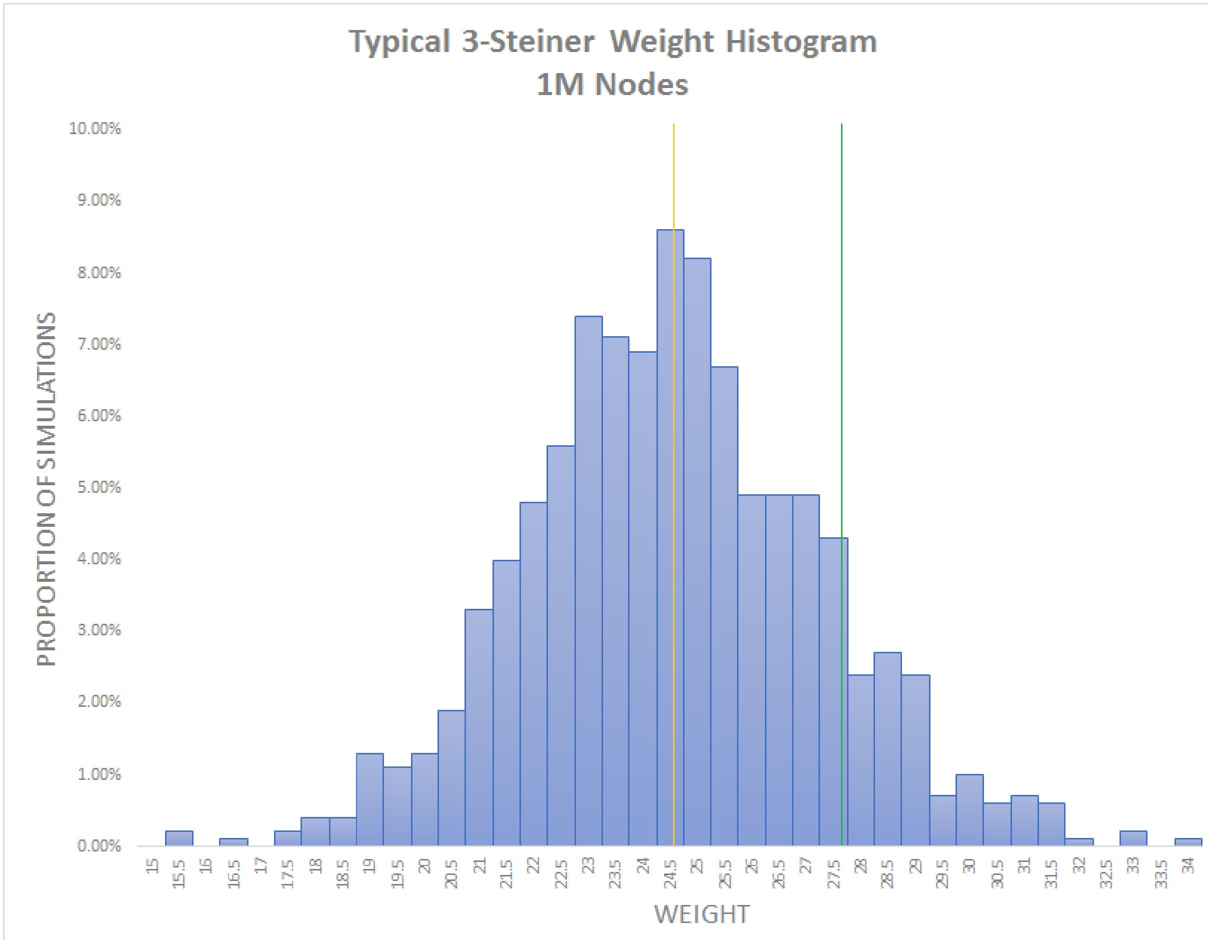
Leg 1 Distribution – 1M Vertices



Typical Distance Empirical Distribution cf. $\log(n) + \Lambda_1 + \Lambda_2 - \Lambda_{1,2}$



Typical 3-Steiner Tree Weight Empirical Distribution cf. 2log(n)



Average Tree Weight vs 2 log(n)

n	2log(n)	Average Tree Weight (1000 runs)	2log(n) - ATW
500	12.4	11.1	**1.35**
2250	15.4	13.7	**1.72**
100,000	23.0	20.5	**2.57**
1,000,000	27.6	24.5	**3.09**

Typical Steiner tree weight

- Conjecture:

$$nW_k - (k-1)\log n \xrightarrow{p} -\infty \ \ as\ n \to \infty$$

- Cf.

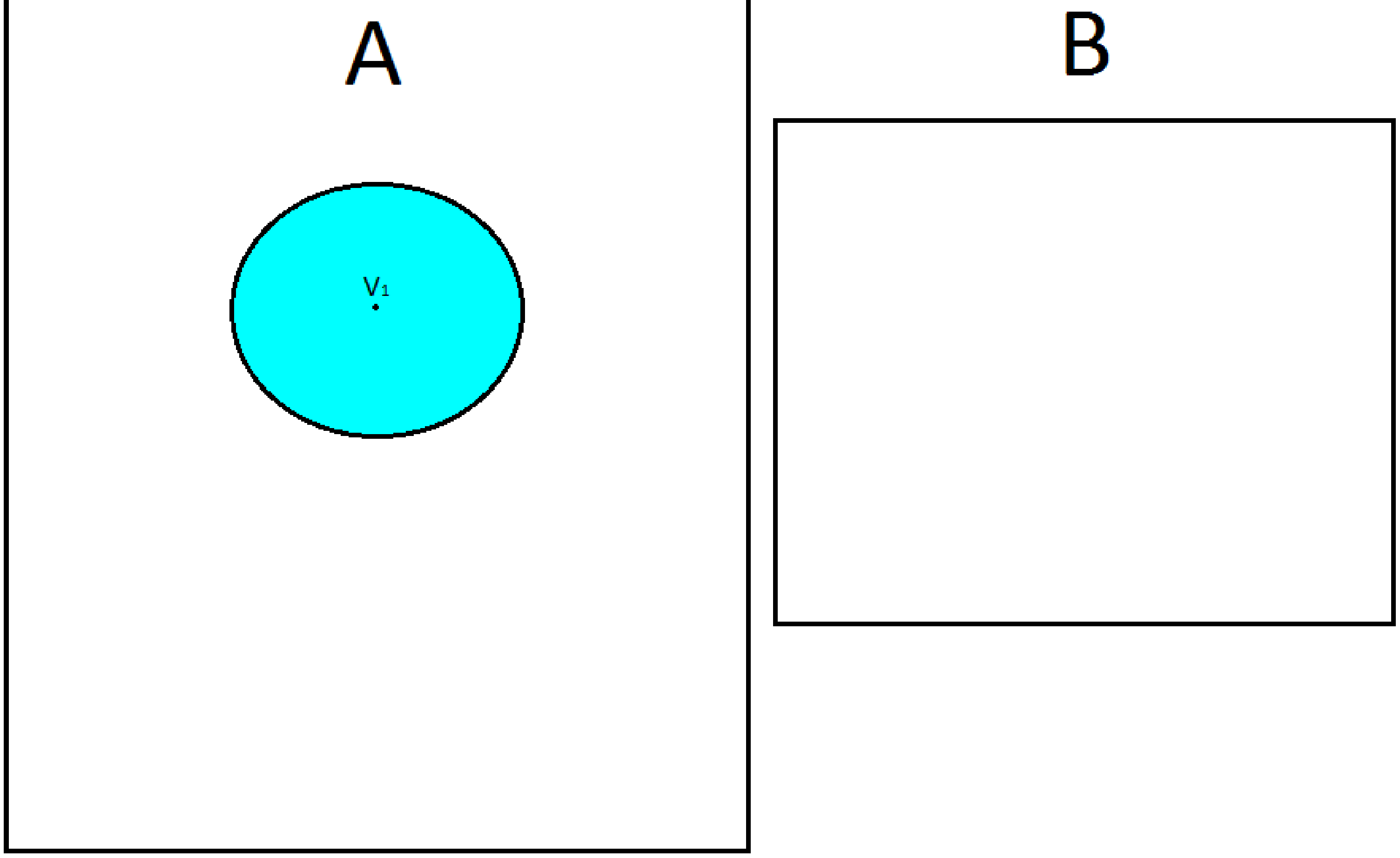
$$nW_2 - \log n \xrightarrow{d} \Lambda_1 + \Lambda_2 - \Lambda_3 \ \ as\ n \to \infty$$

Our results (2)

- Theorem (D., A. Ganesh):

$$nW_k - (k-1)\log n \xrightarrow{p} -\infty \quad as\ n \to \infty$$

Algorithm



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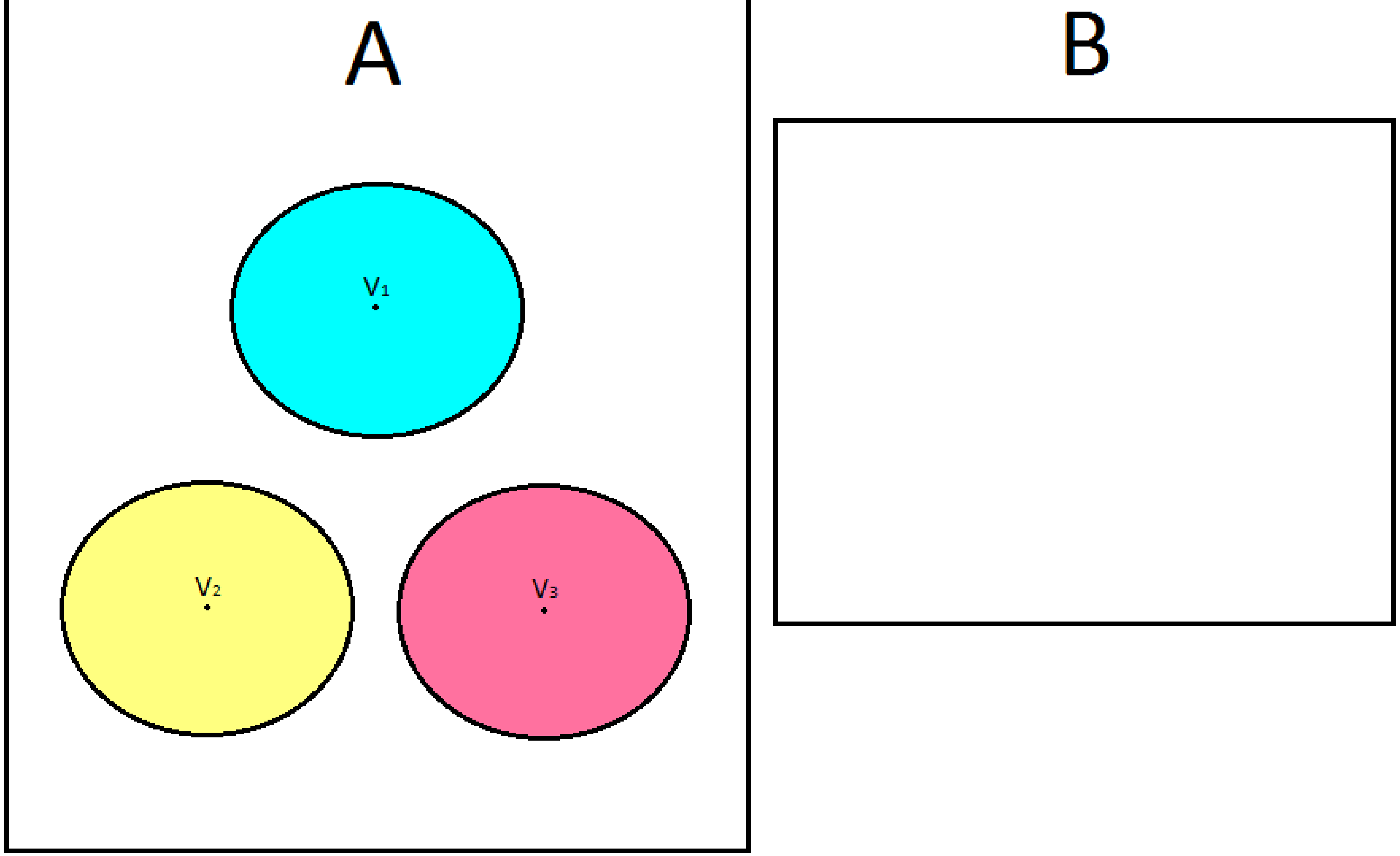

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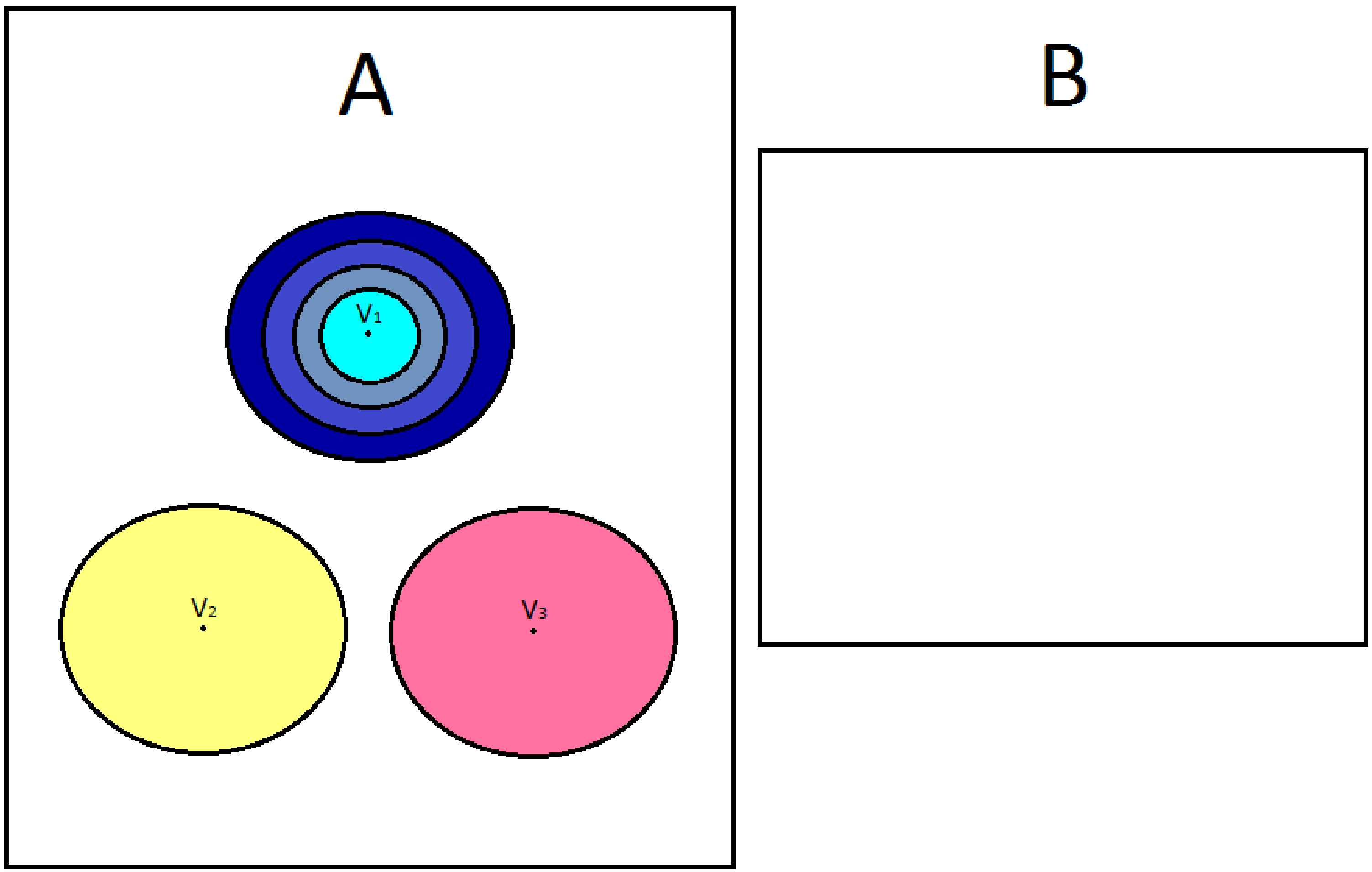

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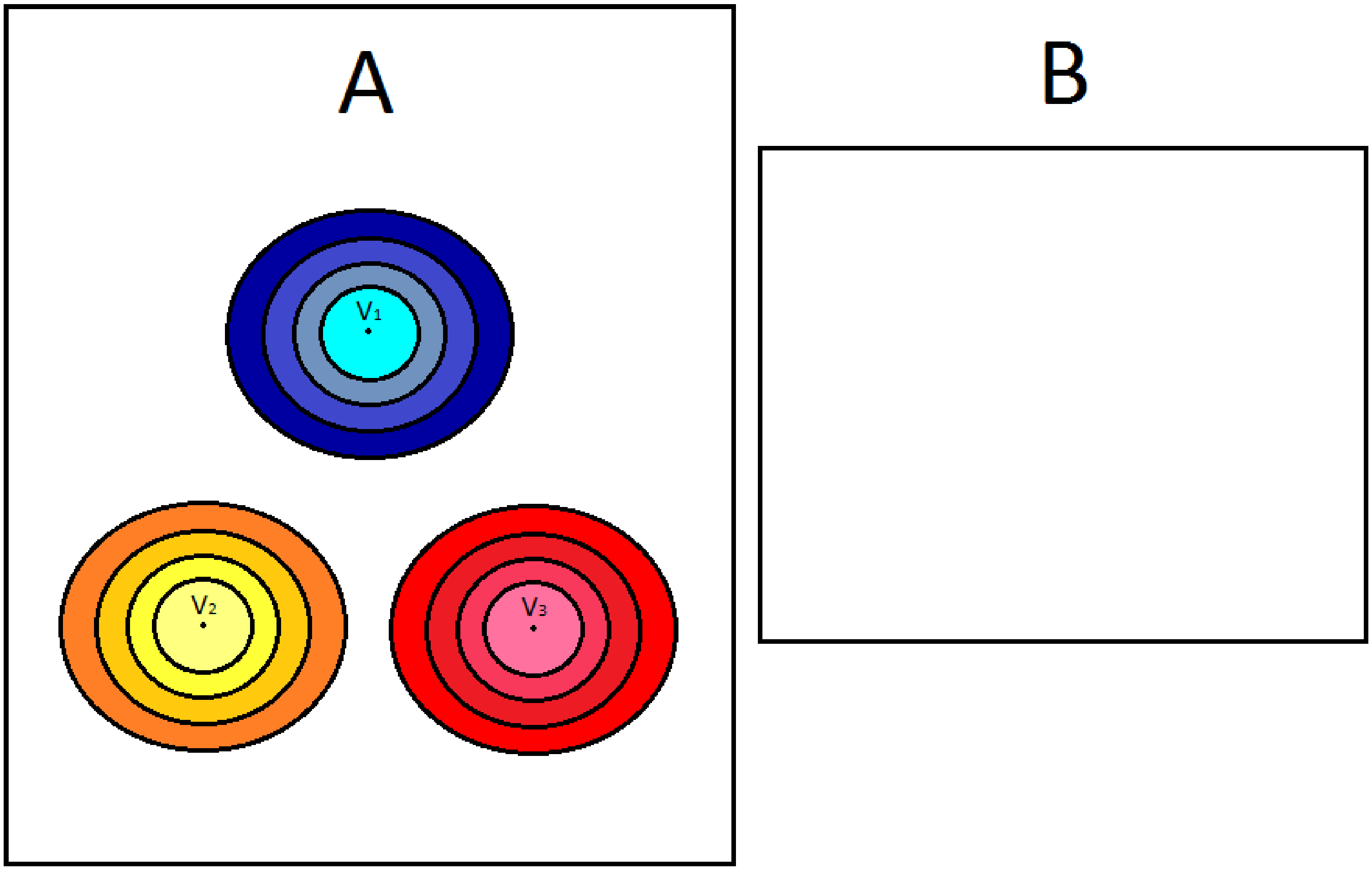

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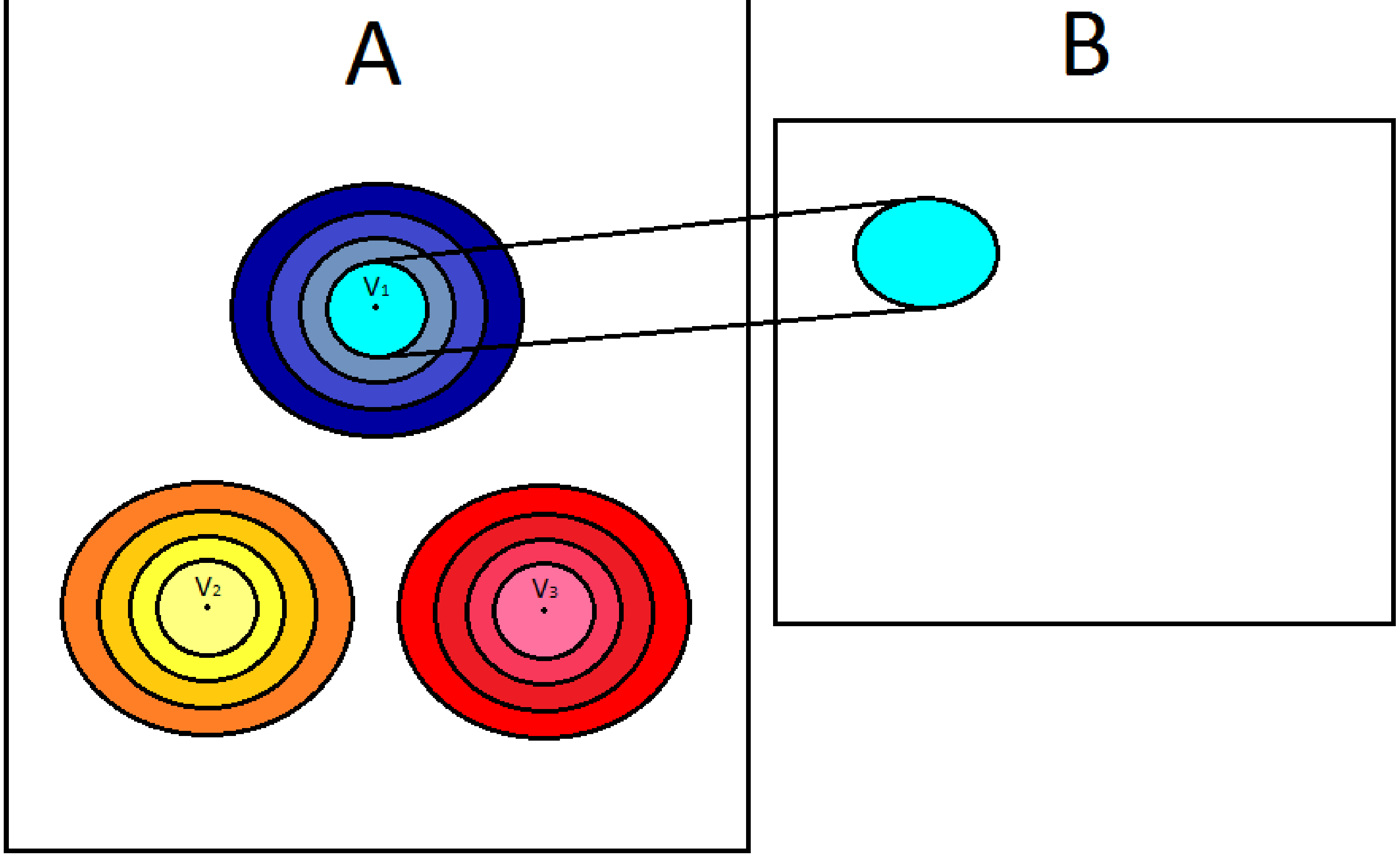

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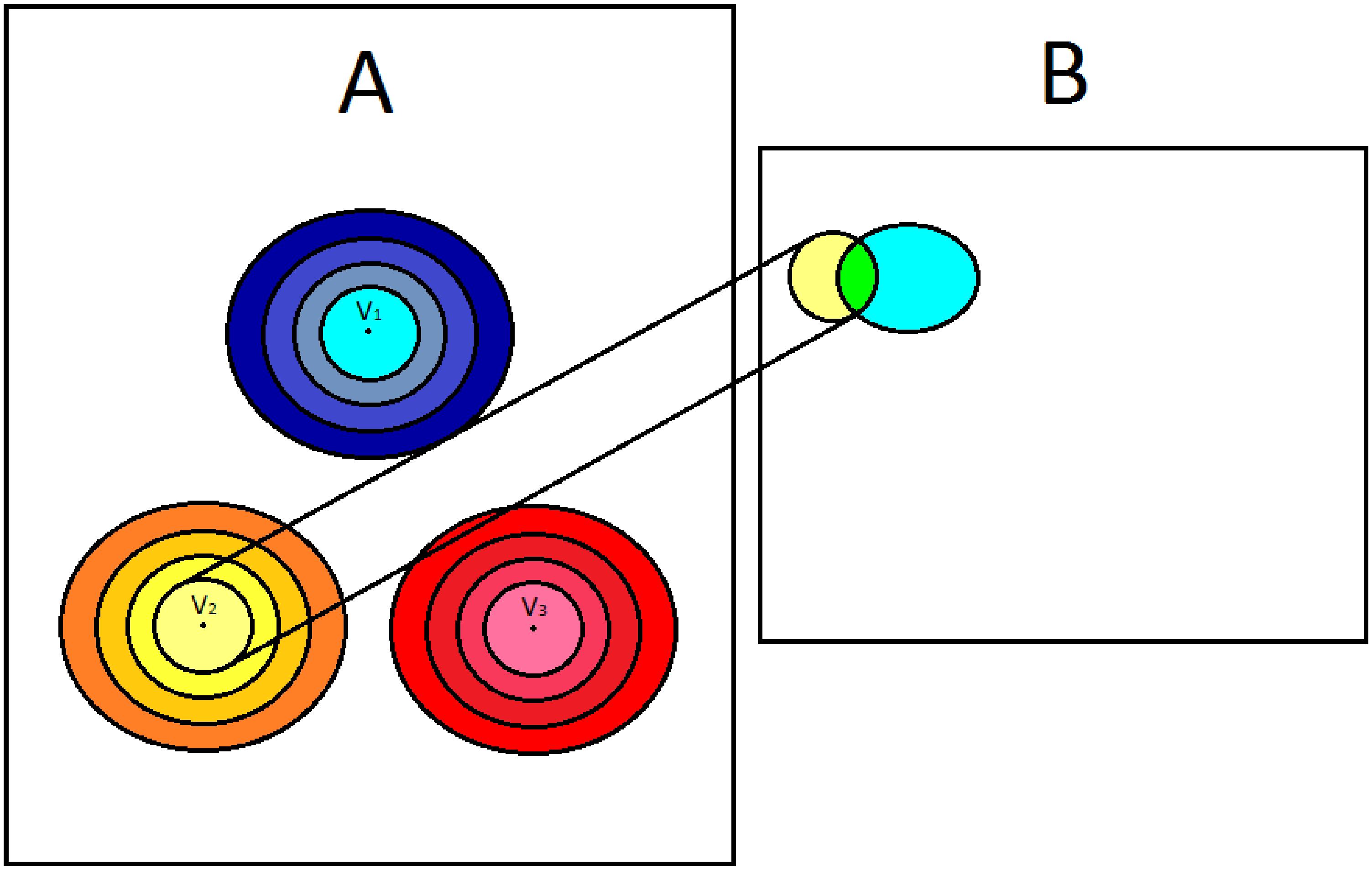

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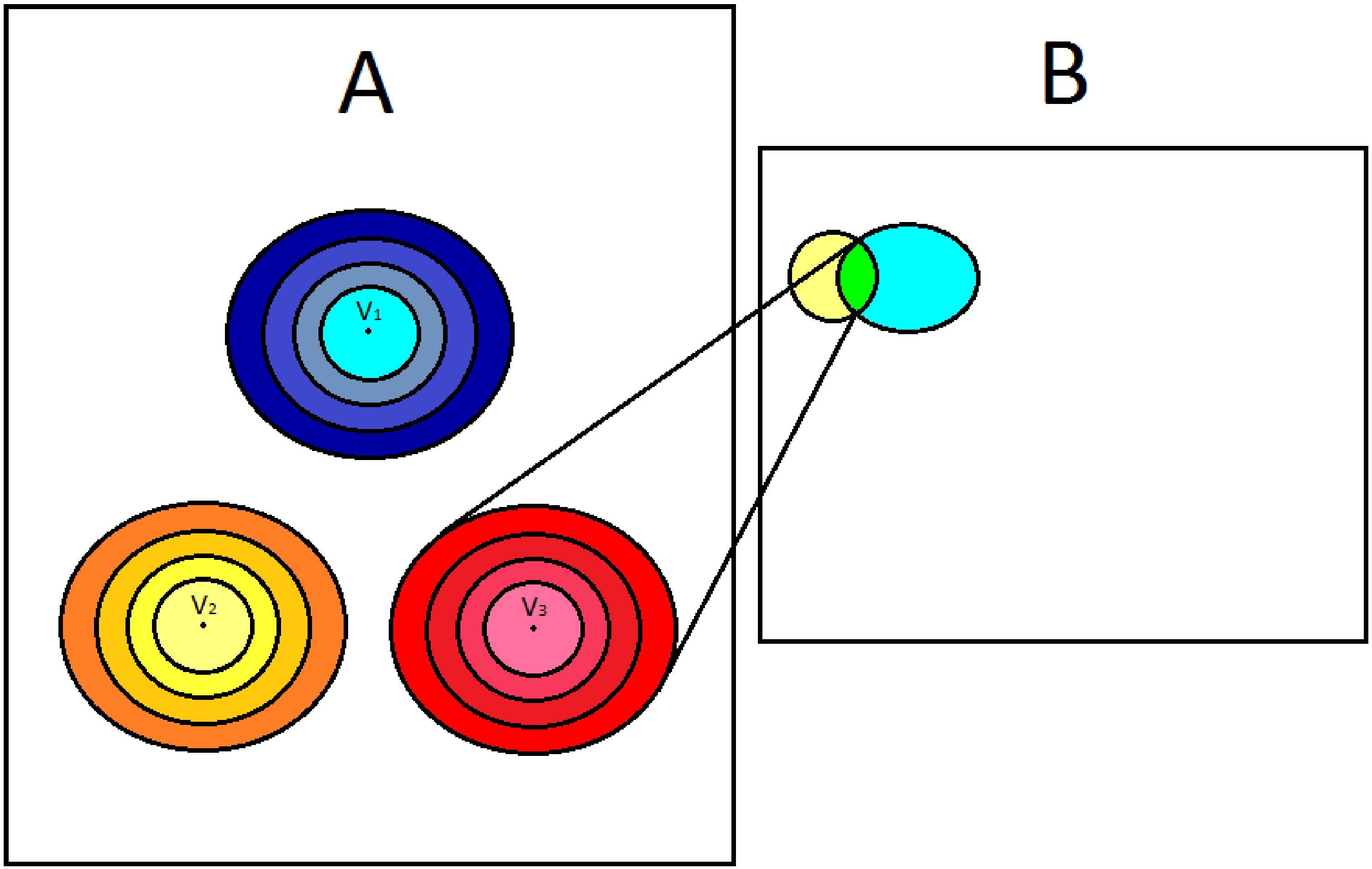

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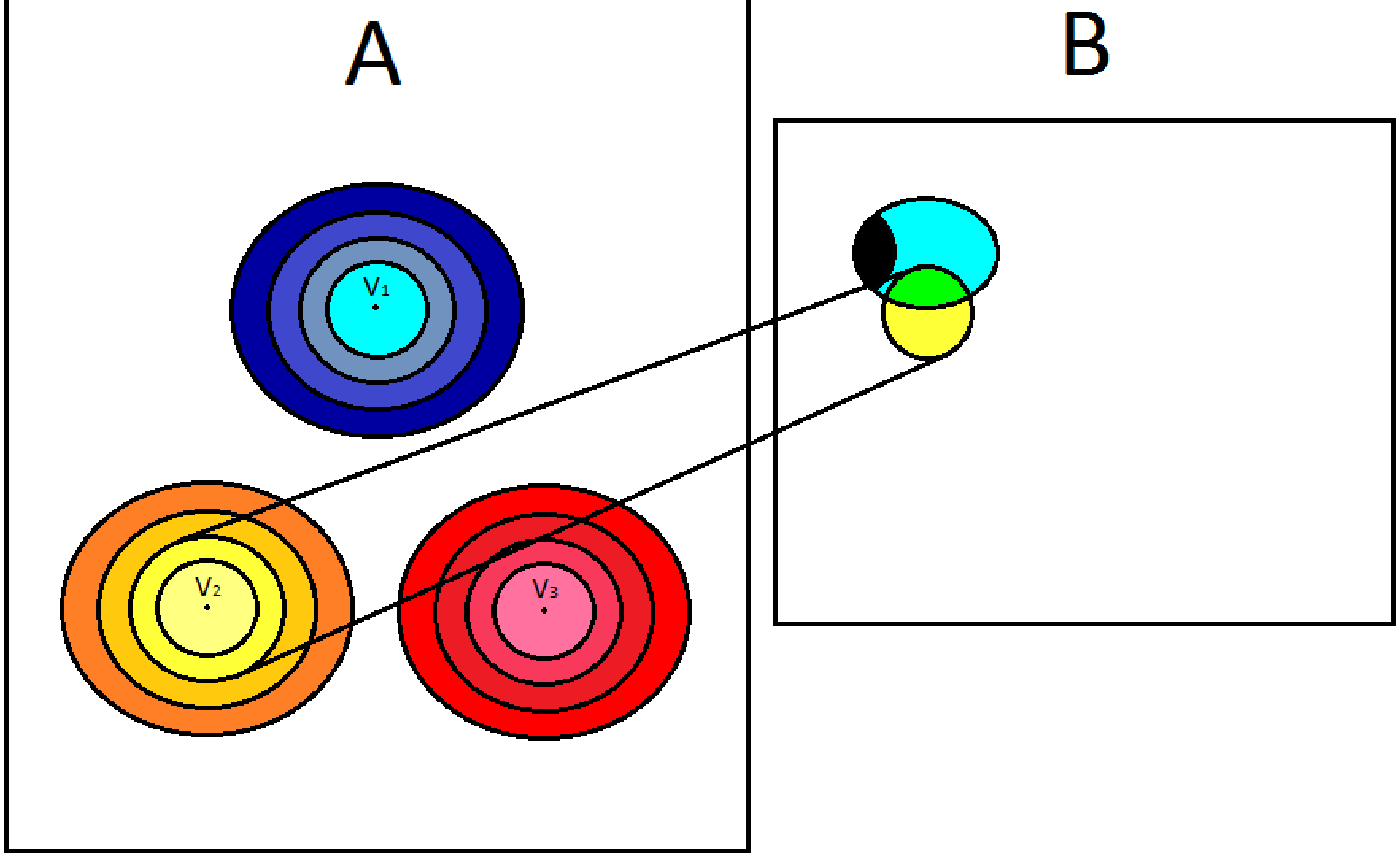

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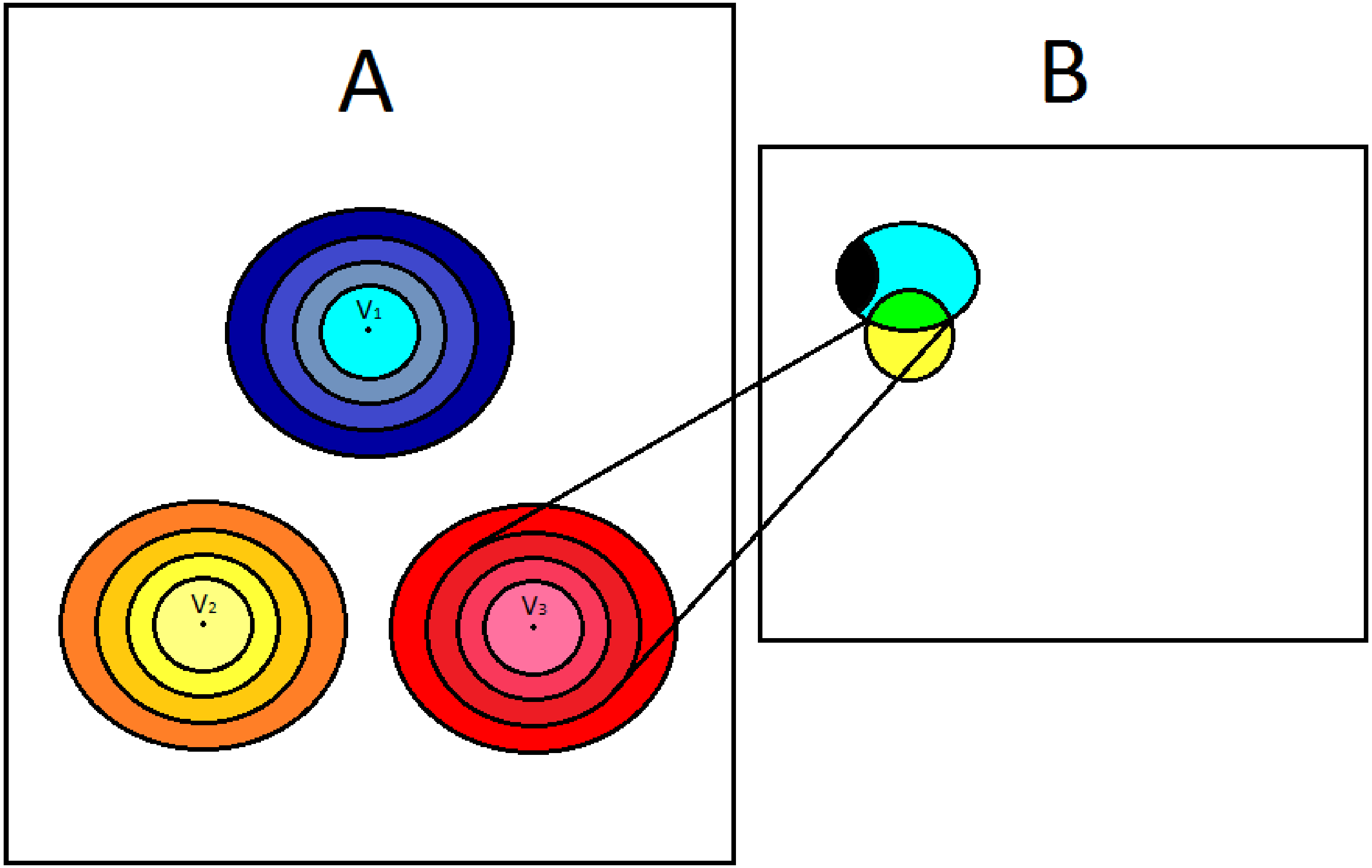

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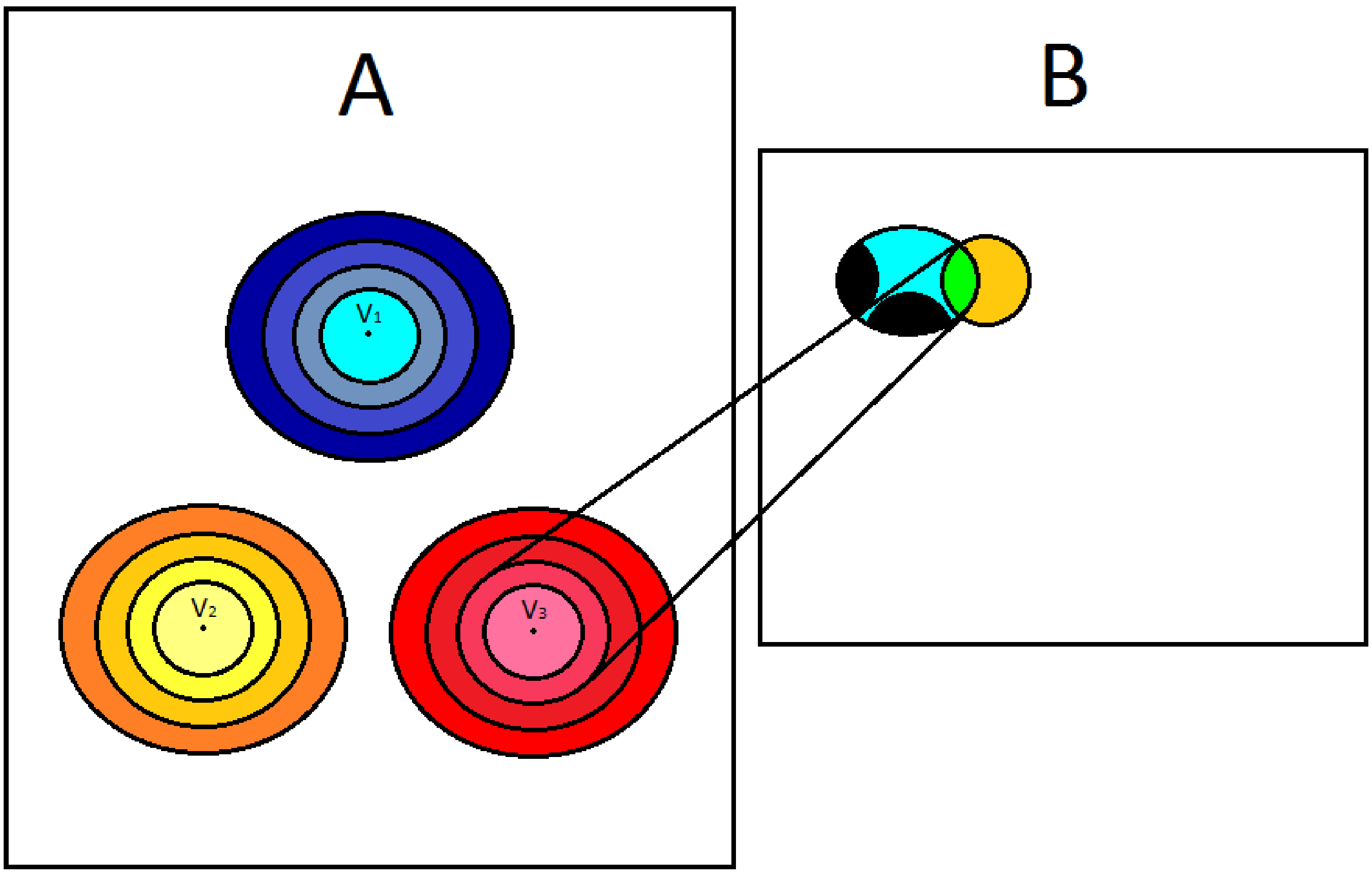

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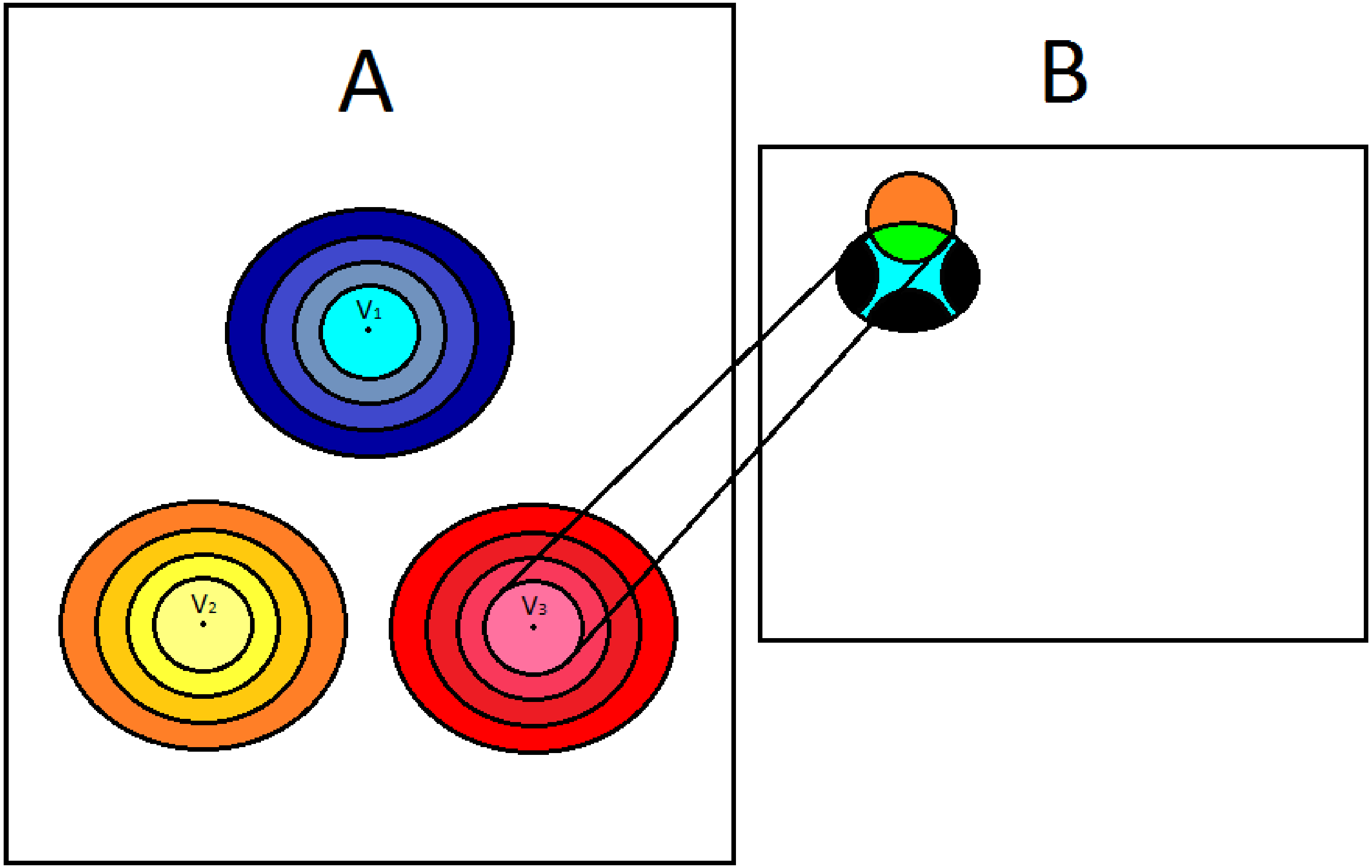

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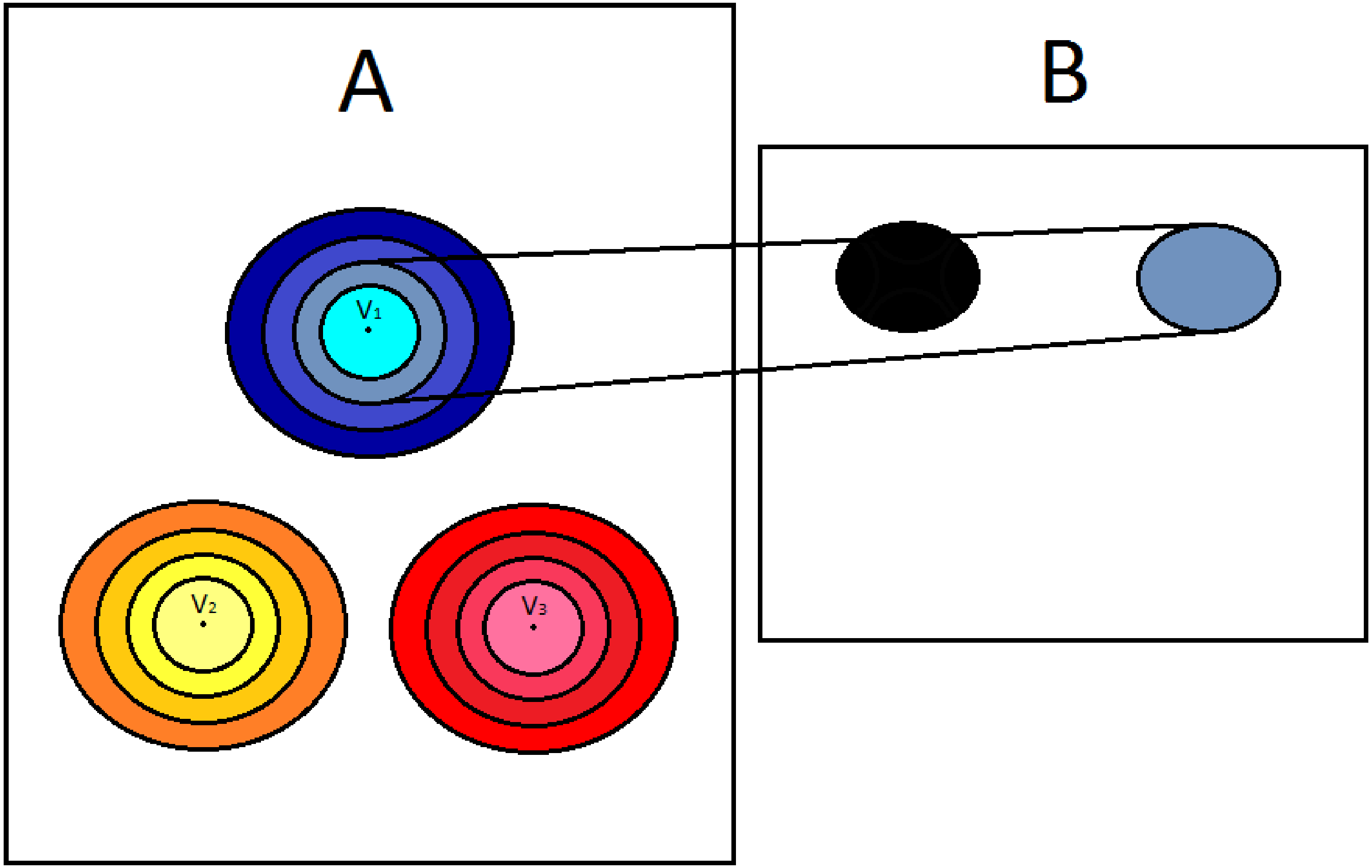

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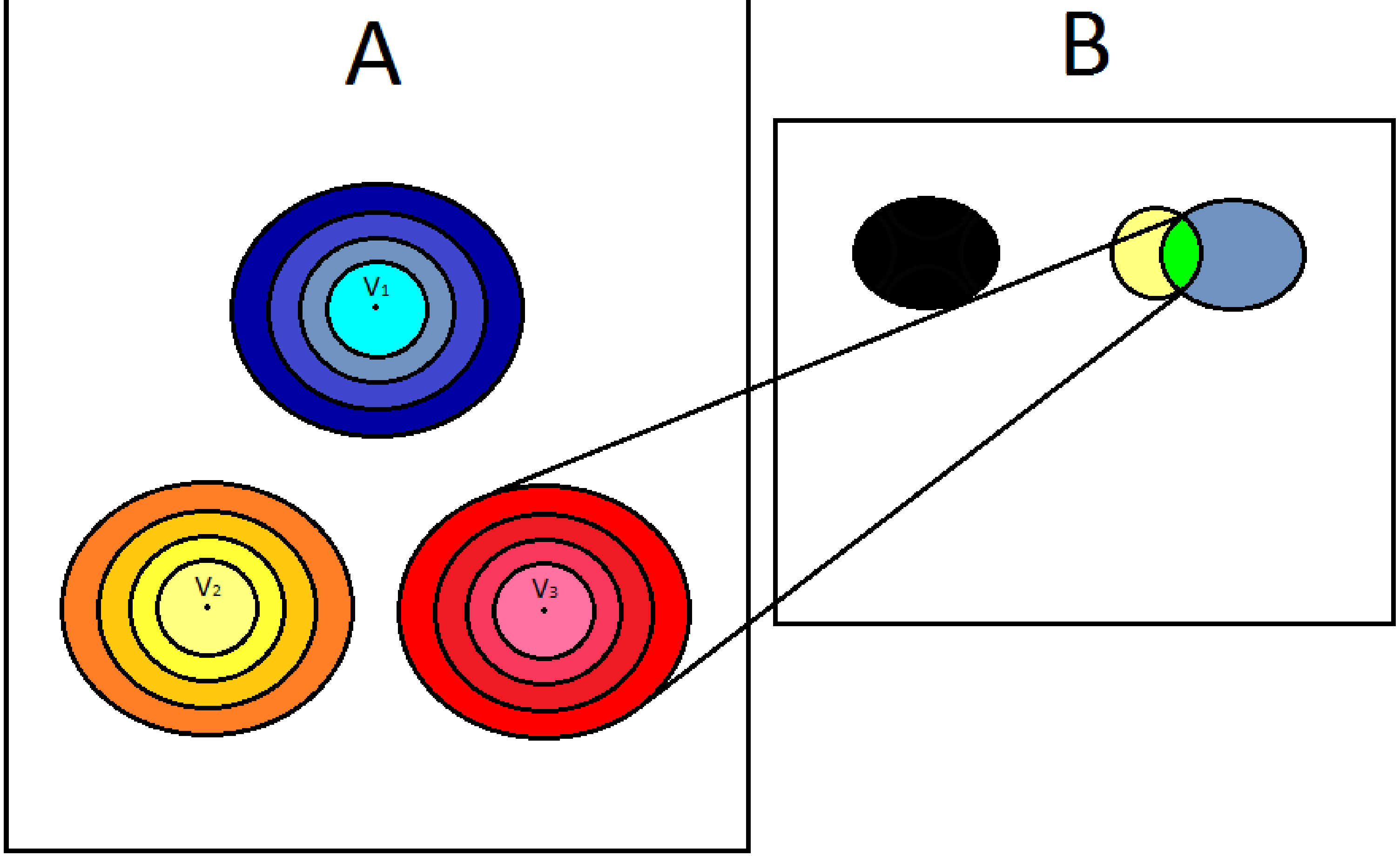

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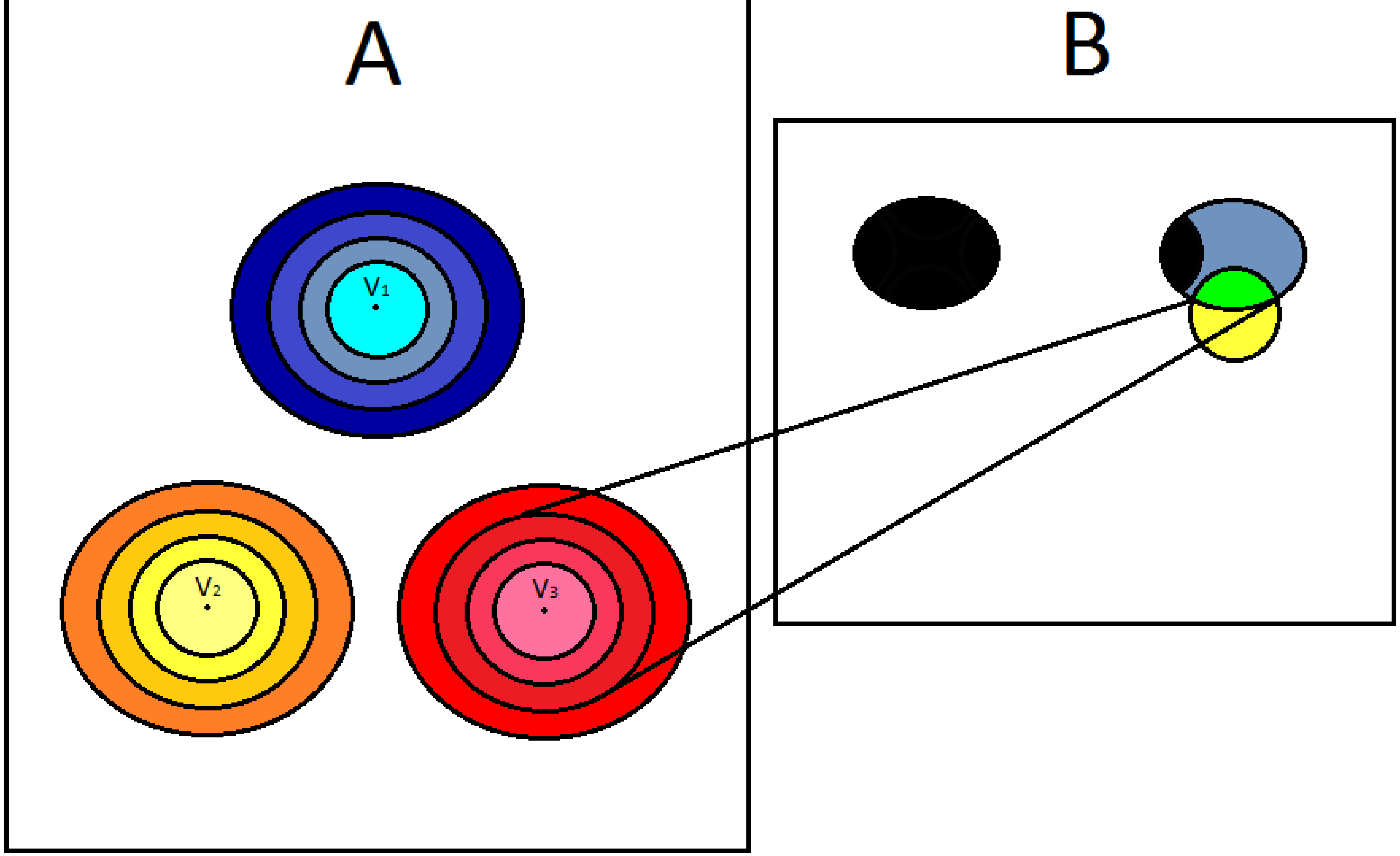

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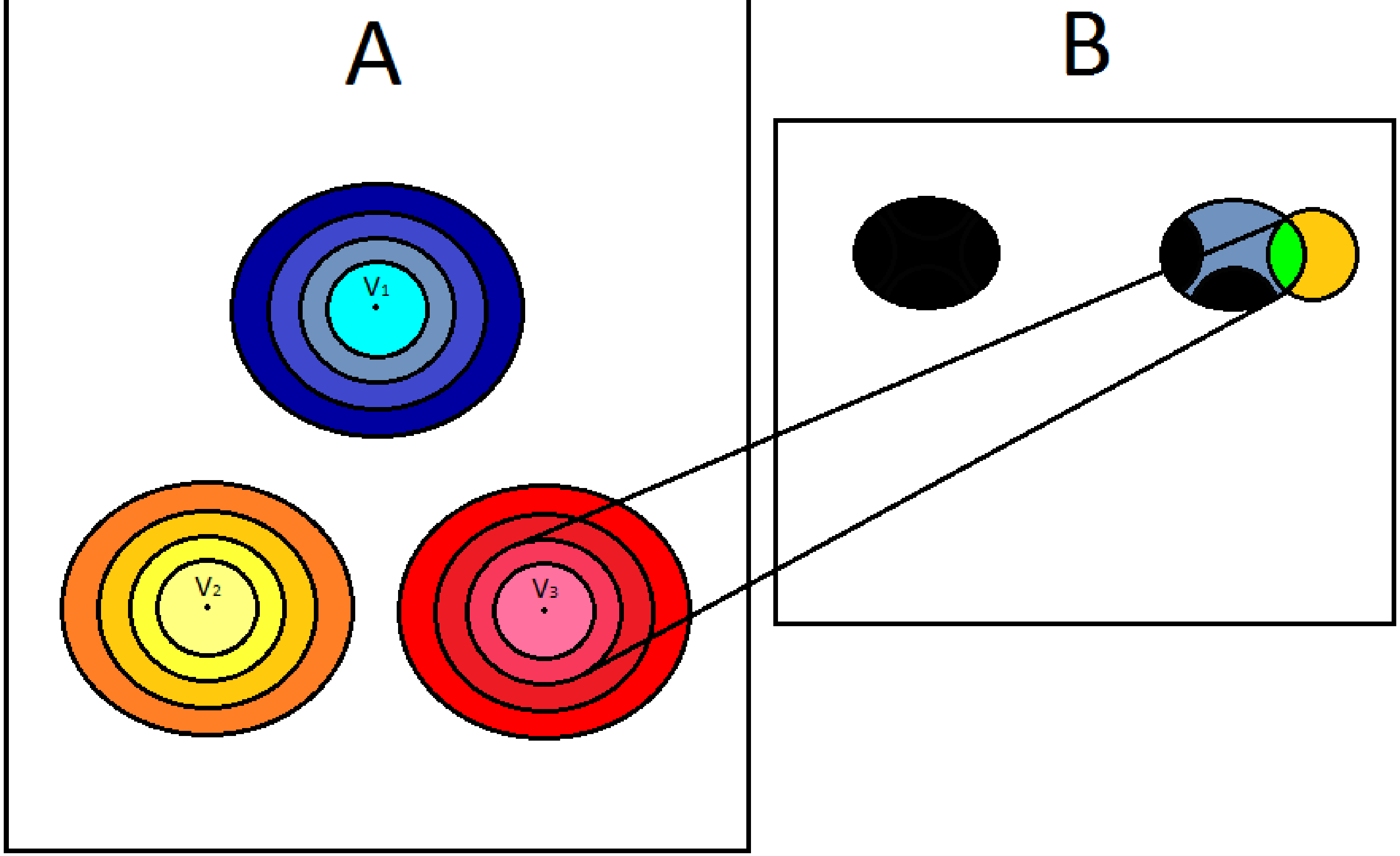

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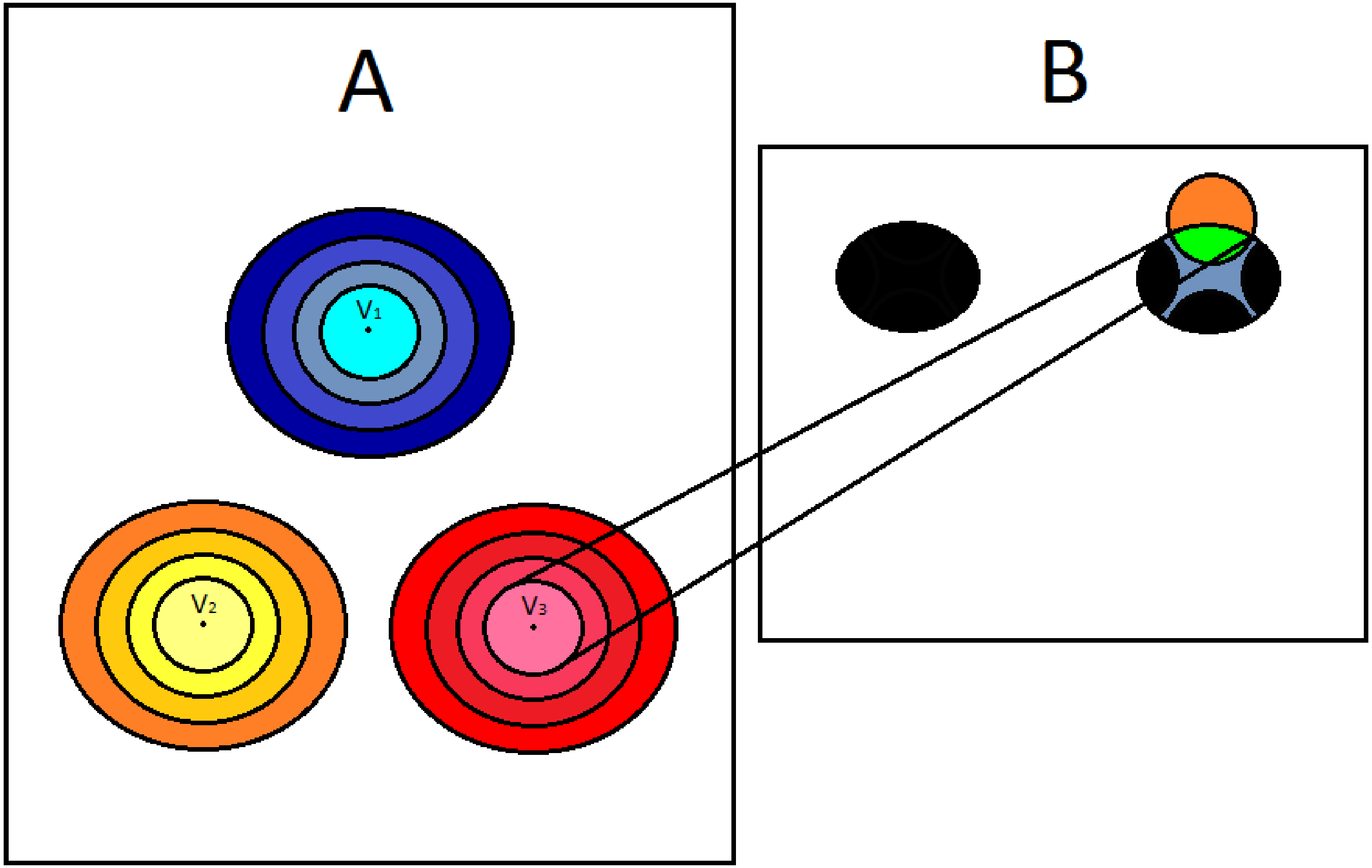

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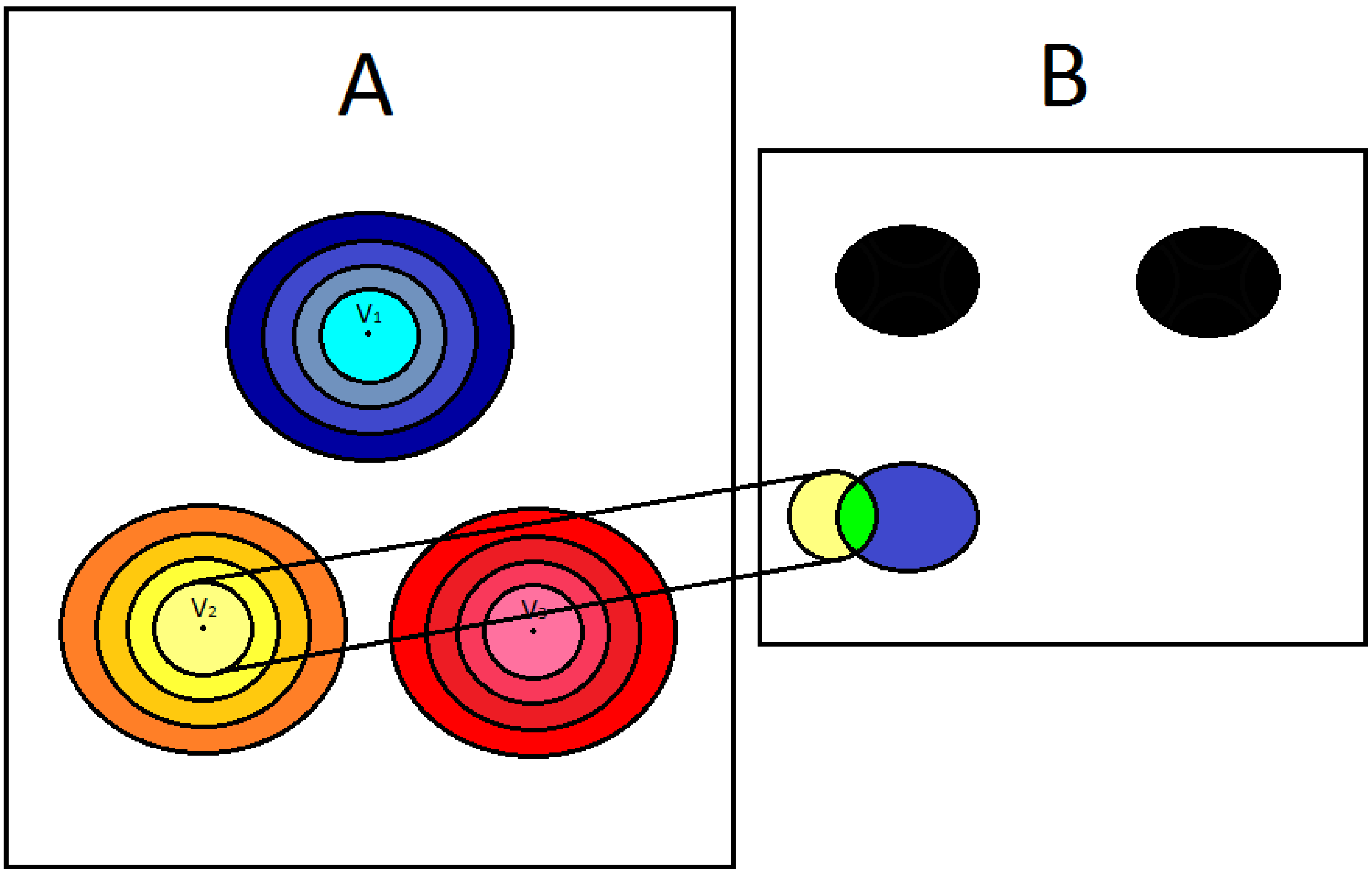

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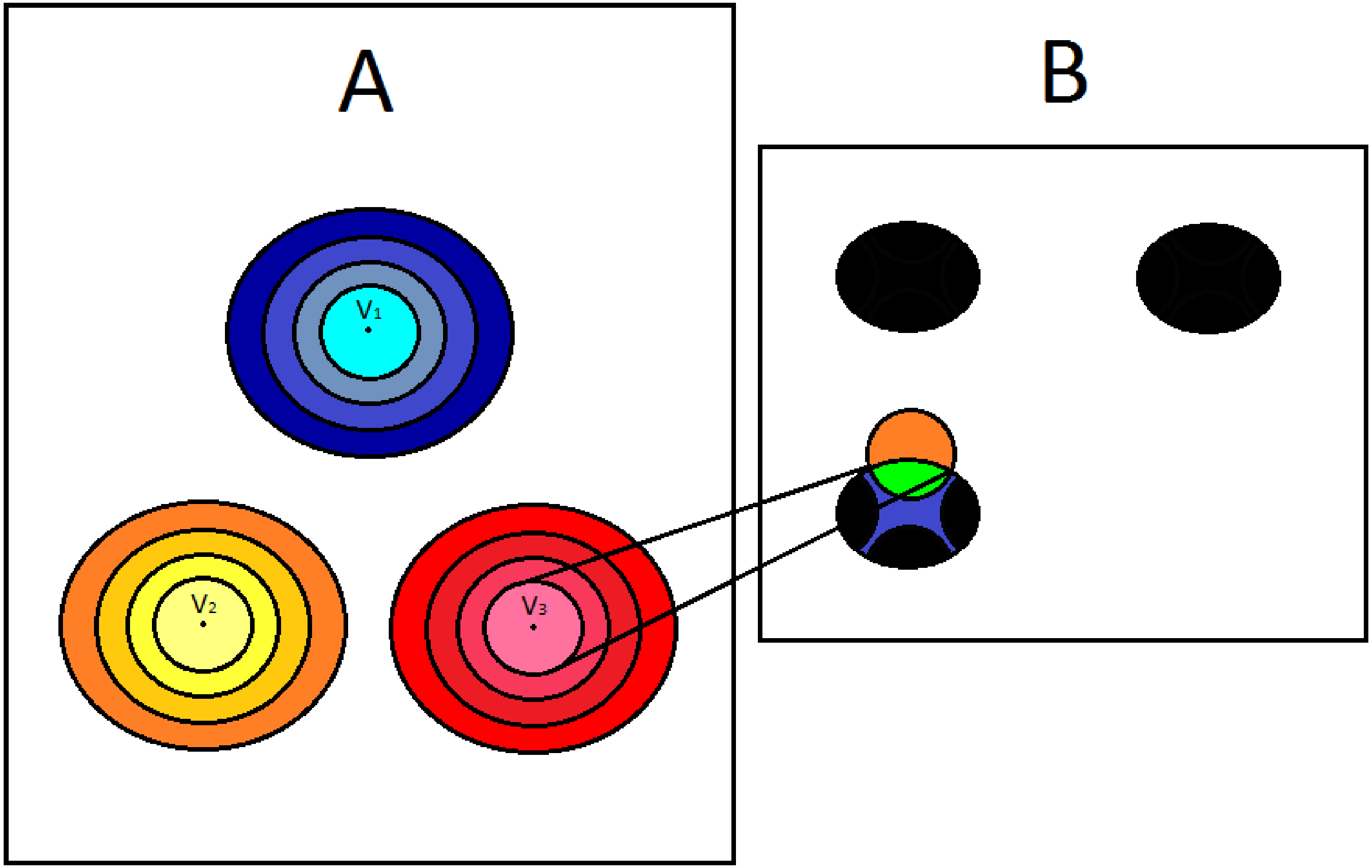

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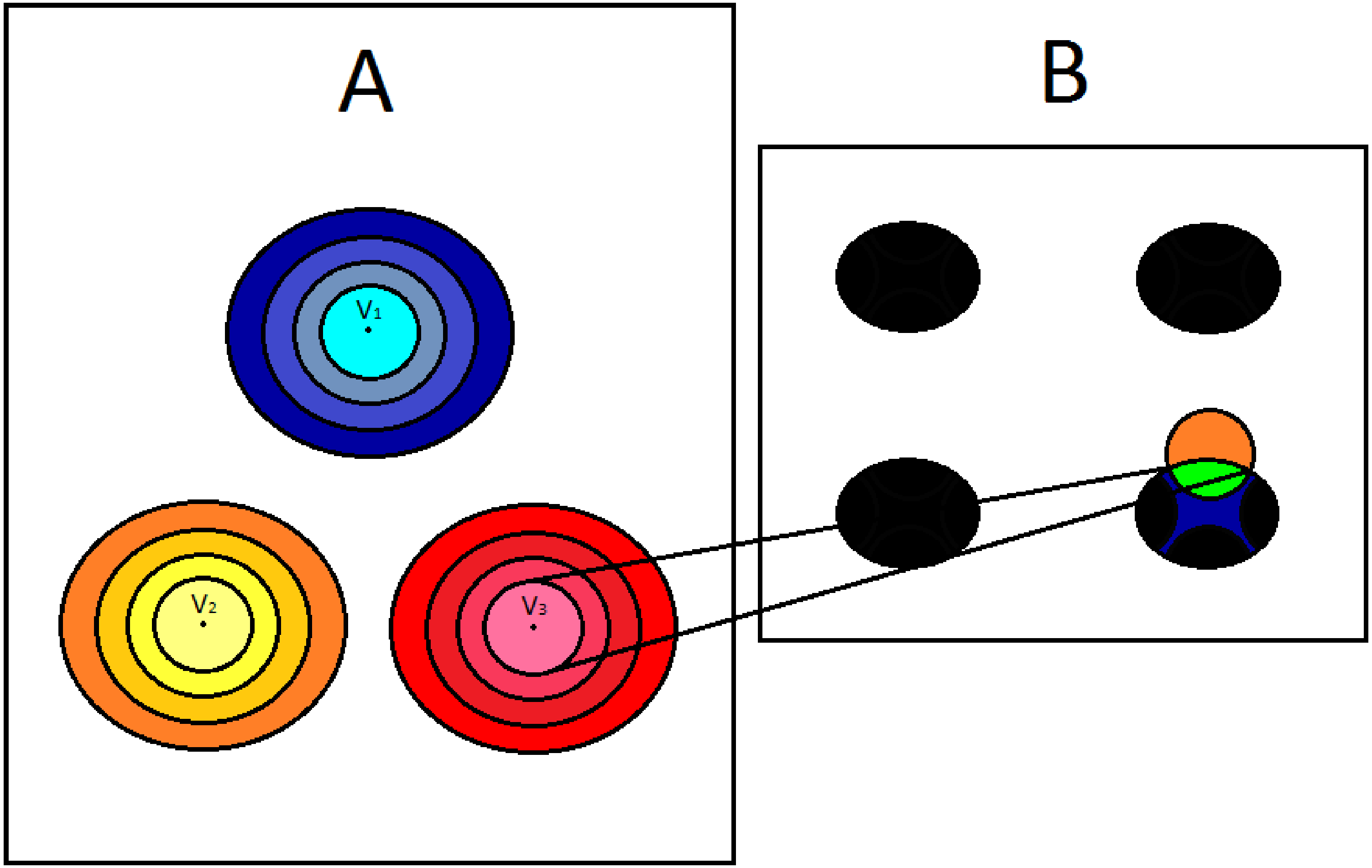

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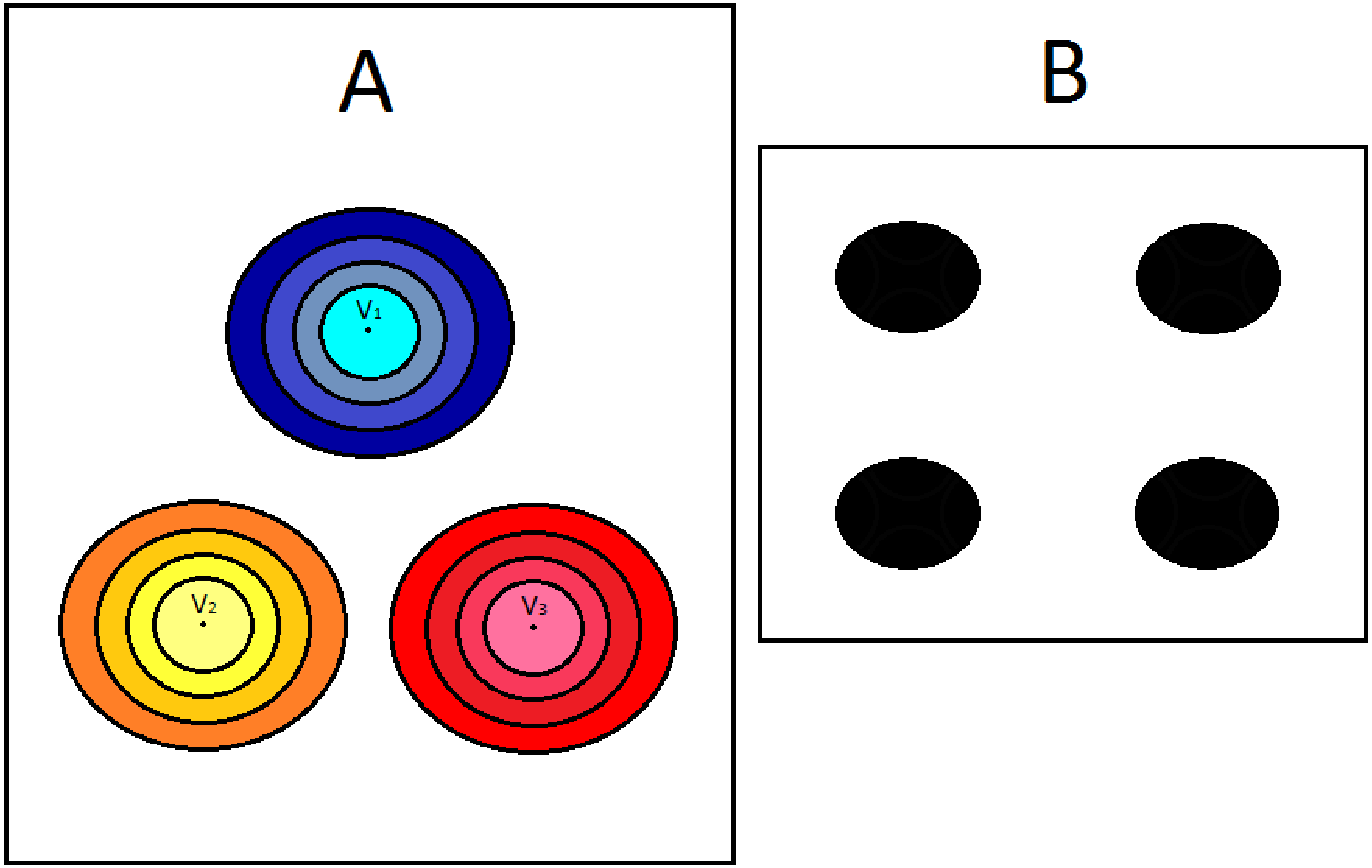

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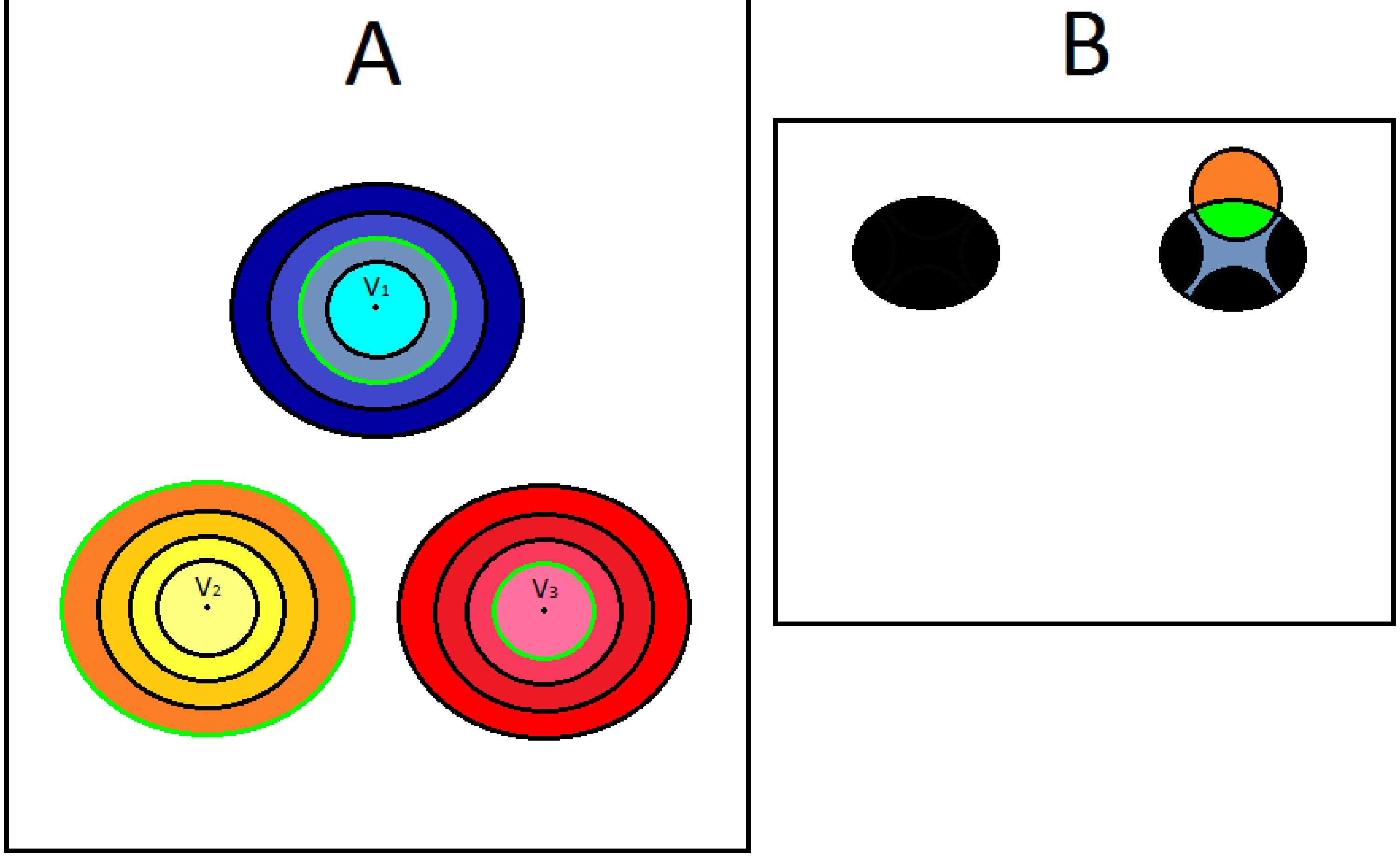

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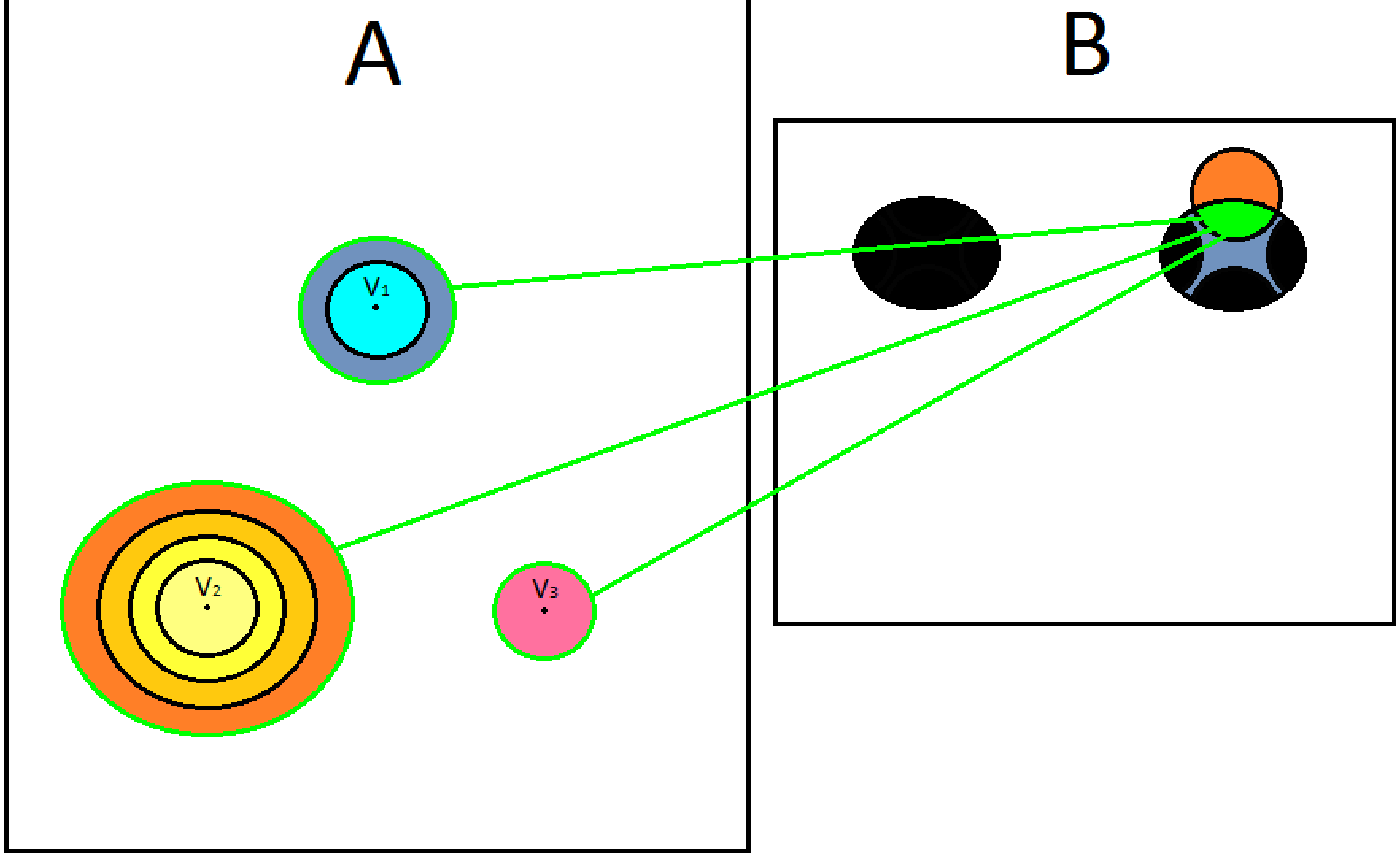

Upper Bound/Lower Bound

- Theorem $\forall \varepsilon > 0, k > 2$:

$$\mathbb{P}\left(-(k-1+\varepsilon) < \frac{nW_k - (k-1)\log n}{\log\log n} < -(k-2-\varepsilon)\right) \to 1 \;\; as\;\, n \to \infty$$

- Conjecture

$$\frac{nW_k - (k-1)\log n}{\log\log n} \xrightarrow{p} -(k-1) \;\; as\;\, n \to \infty$$

Conclusions

- Obtained limiting results for weight of extremal Steiner trees in random edge-weighted graphs
- Made progress towards characterising fluctuations of typical Steiner tree
- Fluctuations progress implies some non-trivial distribution of limiting tree shapes