The free path in a high velocity random flight process associated to a Lorentz gas in an external field

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November 20, 2015

Collaborators

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- The Lorentz gas model was introduced in 1905 by H.A. Lorentz as a model for the motion of electrons in metallic bodies.
- In this model, a point particle travels in an array of fixed convex scatterers. When the particle comes in contact with a scatterer it reflects specularly. This variant is referred to as the "hard core" version.
- Of the many variations on the hard core model, our study is motivated by the version in which an array of spherical scatterers is chosen randomly and the flight of the moving particle is determined by the action of an external field.

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Introduction

- The random scatterer Lorentz gas is difficult to study directly even in the absence of an external field.
- The Boltzmann-Grad limit is a low density limit in which the number of scatterers in a fixed box goes to infinity while, at the same time, the size of each scatterer goes to zero in such a way that the total volume of the scatterers in the box goes to zero.
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Markovian nature of the Boltzmann-Grad limit

- Re-collisions with scatterers become unlikely as the size of each scatterer goes to zero
- The Poisson nature of the scatterer locations which means that knowing the location of one scatterer does not give information about the locations of the other scatterers.

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- We are interested in the regime in which the particle's velocity is (typically) large.
- External field which accelerates the particle towards infinity. For example, the influence of a constant gravitational field has been studied for the random flight process in both a constant density of scatterers (Ravishankar and Triolo '99), a variable density of scatterers (Burdzy and Rizzolo '14), and for the periodic Lorentz gas in two dimensions (Chernov and Dolgopyat, JAMS '09).
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Boltzmann-Grad limit in $\mathbb{R}^3$

- Let $\mathcal{U} : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to (0, \infty)$ be differentiable. The function $\mathcal{U}$ will serve as the potential for a conservative force and g will be the density of scatterers.
- Fix an energy level E and consider a particle moving in the potential $\mathcal{U}$ with total energy E. By conservation of energy

$$\frac{m\|\mathbf{v}(t)\|^2}{2} + \mathcal{U}(\mathbf{y}(t)) = E,$$

where $\mathbf{y}$ is the particle's position and $\mathbf{v}$ is its velocity.

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- Assume spherical scatterers with radius $1/R$ are placed so their centers are the points of a Poisson process with intensity $R^2 g$.
- Further assume that for almost every initial condition the trajectory of a particle moving in the potential $\mathcal{U}$ with total energy E is open and has infinite length. The intuition for its necessity is that it is possible for a periodic orbit to avoid all of the scatterers in a random arrangement.

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Boltzmann-Grad limit in $\mathbb{R}^3$

If the initial position and velocity of the particle is absolutely continuous with respect to Lebesgue measure on the constant energy surface then the distribution of the position and velocity process of the particle converges as $R \to \infty$, in the sense of convergence of finite dimensional distributions, to a Markovian random flight process $(\mathbf{X}(t), \mathbf{V}(t))_{t \geq 0}$.

Boltzmann-Grad limit in $\mathbb{R}^3$

The process $(\mathbf{X}(t), \mathbf{V}(t))_{t\geq 0}$ can be constructed in the following way, which explains the name "random flight process". Let $(\mathbf{y}(\mathbf{x}_0, \mathbf{v}_0, t))_{t\geq 0}$ with $\mathbf{x}_0, \mathbf{v}_0 \in \mathbb{R}^3$ and $m\|\mathbf{v}_0\|^2/2 + \mathcal{U}(\mathbf{x}_0) = E$ denote the solution to the initial value problem

$$\begin{cases} m\mathbf{y}'' & = & -\nabla\mathcal{U}(\mathbf{y}) \\ \mathbf{y}(0) & = & \mathbf{x}_0 \\ \mathbf{y}'(0) & = & \mathbf{v}_0. \end{cases} \tag{1}$$

We construct our process $((\mathbf{X}(t), \mathbf{V}(t)), t \geq 0)$ recursively as follows. Set $(\mathbf{X}(0), \mathbf{V}(0)) = (\mathbf{x}_0, \mathbf{v}_0)$ and let $T_0 = 0$. For $k \geq 1$, assuming we have defined $((\mathbf{X}(t), \mathbf{V}(t)))_{0\leq t\leq T_{k-1}}$, we let $\mathbf{U}_{k-1}$ be independent of this part of the path and uniformly distributed on $\mathbf{S}^2$ and let T_k

Boltzmann-Grad limit in $\mathbb{R}^3$

satisfy

$$\mathbb{P}\left(T_k - T_{k-1} > t \mid \mathbf{U}_{k-1}, ((\mathbf{X}_t, \mathbf{V}_t))_{0\le t\le T_{k-1}}\right)$$
$$= \exp\Bigg(- \int_0^t g\Big(\mathbf{y}(\mathbf{X}(T_{k-1}), \|V(T_{k-1})\|\mathbf{U}_{k-1}, s)\Big)$$
$$\Big\|\mathbf{y}'(\mathbf{X}(T_{k-1}), \|\mathbf{V}(T_{k-1})\|\mathbf{U}_{k-1}, s)\Big\|\, ds\Bigg). \quad (2)$$

For $t \in [T_{k-1}, T_k]$ we then define

$$\begin{aligned} \mathbf{X}(t) &:= \mathbf{y}(\mathbf{X}(T_{k-1}), \|\mathbf{V}(T_{k-1})\|\mathbf{U}_{k-1}, t - T_{k-1}) \\ \mathbf{V}(t) &:= \mathbf{y}'(\mathbf{X}(T_{k-1}), \|\mathbf{V}(T_{k-1})\|\mathbf{U}_{k-1}, t - T_{k-1}) \end{aligned} \quad (3)$$

Boltzmann-Grad limit in $\mathbb{R}^3$

- Intuitively, T_k defines the kth reflection of our particle by a scatterer. By studying the free path of the particle, we mean to study, in particular, the conditional law of $(\mathbf{X}(t))_{T_1 \leq t \leq T_2}$ given $\mathbf{X}(T_1) = \mathbf{x}$ under an appropriate scaling of the parameters in the model.
- Note that the path of the particle on the time interval $[T_1, T_2]$ is the path between two consecutive reflections. This is why we call it the *free path*.

Boltzmann-Grad limit in $\mathbb{R}^3$

If we define $\mathbf{X}_k = \mathbf{X}(T_k)$ and take $\mathbf{V}(0) = v(\mathbf{X}(0))\mathbf{U}$, with $\mathbf{U}$ uniformly distributed on $\mathbf{S}^2$, then $(\mathbf{X}_k)_{k\geq 0}$ is a Markov chain and its transition operator is

$$Pf(\mathbf{x}) = \mathbb{E}\left[f\left(\mathbf{y}\left(\mathbf{x}, v(\mathbf{x})\mathbf{U}, N(\mathbf{x}, \mathbf{U})\right)\right)\right], \tag{4}$$

where $N(\mathbf{x}, \mathbf{u})$ is a random variable with distribution

$$\mathbb{P}\left(N(\mathbf{x}, \mathbf{u}) > t\right) = \exp\left(-\int_0^t g[\mathbf{y}(\mathbf{x}, v(\mathbf{x})\mathbf{u}, s)]v[\mathbf{y}(\mathbf{x}, v(\mathbf{x})\mathbf{u}, s)]ds\right), \tag{5}$$

and conditional on $\mathbf{U} = \mathbf{u}$, $N(\mathbf{x}, \mathbf{U})$ is distributed like $N(\mathbf{x}, \mathbf{u})$.

Scaling

Suppose that $m = 2$, $\mathcal{U}(\mathbf{x}) = \|\mathbf{x}\|$, so that $\mathcal{U}$ produces a uniform acceleration towards the origin, and that $g(\mathbf{x}) \equiv 1$. Suppose we start the particle at the origin with speed $v_0 = 1$ moving in direction μ.

By conservation of energy the total energy of the particle will be

$$E(v_0) = mv_0^2/2 = 1$$

and as a result the maximum distance the particle can move away from the origin is 1.

Scaling

Consequently, there is no way to rescale the trajectory of the particle to obtain a diffusive limit: Any scaling of space will cause the trajectory to degenerate, and scaling time alone will produce jumps.

However, we can obtain a diffusive limit if we let v_0 (and, consequently, E) go to infinity. Treating v_0 as a parameter, the energy becomes a function of v_0, specifically $E(v_0) = v_0^2$ is the maximum distance the particle can travel from the origin.

Scaling

Let $(\mathbf{X}^{v_0}(t))_{t\geq 0}$ be the trajectory of the random flight process with initial conditions $\mathbf{X}^{v_0}(0) = \mathbf{0}$ and initial velocity $\dot{\mathbf{X}}^{v_0}(0) = v_0\mathbf{U}$, where $\mathbf{U}$ is uniformly distributed on $\mathbf{S}^2$. In order to obtain a diffusive limit for $(\mathbf{X}^{v_0}(t))_{t\geq 0}$, we rescale space so that its maximum distance from the origin remains constant. In particular, we look at $(v_0^{-2}\mathbf{X}^{v_0}(t))_{t\geq 0}$, which travels at most distance 1 from the origin.

Scaling

By Brownian scaling, in order to obtain a diffusive limit, we expect that we should scale time so that there are approximately v_0^4 reflections per unit time. Typically, we expect the particle to be distance of order v_0^2 from the origin, and if $\mathbf{X}^{v_0}(t) = v_0^2\mathbf{x}$ and t is a reflection time, the time until the next reflection is approximately

$$\frac{1}{\left\|\dot{\mathbf{X}}^{v_0}(t)\right\|} = \frac{1}{\sqrt{v_0^2 - v_0^2\|\mathbf{x}\|}} = \frac{1}{v_0\sqrt{1 - \|\mathbf{x}\|}}.$$

Thus the amount of time for v_0^4 reflections to occur is of the order v_0^3. This suggests looking for a diffusive limit of $(v_0^{-2}\mathbf{X}^{v_0}(v_0^3 t))_{t\geq 0}$ as v_0 tends to infinity, which is what we undertake.

Scaling

The path of the particle $(\mathbf{X}^n(t))_{t\geq 0}$ evolves as the position of a random flight process where the density of scatterers is

$$g_n := \sqrt{n} g,$$

the potential energy of the field

$$\mathcal{U}_n := \frac{1}{\sqrt{n}} \mathcal{U}$$

and the total energy of the particle

$$E_n := \frac{1}{\sqrt{n}} E,$$

where g, $\mathcal{U}$, and E are fixed and independent of n and $g(\mathbf{x}) = g(\|\mathbf{x}\|)$ and $\mathcal{U}(\mathbf{x}) = \mathcal{U}(\|\mathbf{x}\|)$. That is, the density of scatterers and potential are spherically symmetric

Scaling

We can write the speed as a function of E and $\mathcal{U}$; consequently the speed v is also rescaled as

$$v_n(r) := \frac{1}{n^{1/4}} v(r).$$

The trajectory of the particle with these rescaled parameters we denote by

$$(\mathbf{y}_n(\mathbf{x}, v_n(\mathbf{x})\mathbf{u}, t))_{t \geq 0}$$

In general, subscripts or superscripts of n refer to distributions relative to these scaled parameters.

Scaling

To make the connection with our heuristic arguments, if we take $v_0 = n^{1/4}$, $g = 1$ and $\mathcal{U}(\mathbf{x}) = \|\mathbf{x}\|$ then

$$(\mathbf{X}^n(t))_{t\geq 0} =_d (v_0^{-2}\mathbf{X}^{v_0}(t))_{t\geq 0}.$$

Thus the scaling we do here accounts for increasing the initial speed and scaling space to keep the particle's trajectory contained. This is the scaling under which we analyze the free path of the particle.

Scaling

We assume that for all $n \in \mathbb{N}$ the process $(\|\mathbf{X}^n(t)\|, t \geq 0)$ evolves in a domain $\mathcal{D} \subset \mathbb{R}_+$ where $\mathcal{D} = \mathbb{R}_+$ or $\mathcal{D} = [h_-, h_+]$, where $0 \leq h_- < h_+ < \infty$. The domain $\mathcal{D}$ is chosen so that $E - U(r) > 0$ for all $r \in \mathcal{D}^\circ$. This is equivalent to $v(r) > 0$ for all $r \in \mathcal{D}^\circ$.

Main Result

Our first result considers the asymptotic behavior of the steps of the free path Markov chain $((\mathbf{X}_k^n, \mathcal{T}_k^n))_{k\geq 0}$, where $\mathbf{X}_k^n$ is the location of the particle at the time of the k'th reflection and, anticipating our diffusion approximations, $n^{3/4}\mathcal{T}_k^n$ is the time of the k'th collision. This Markov chain has transition operator

$$Q_n f(\mathbf{x}, t) = \mathbb{E}\left[f\left(\mathbf{y}\left[\mathbf{x}, v(\mathbf{x})\mathbf{U}, N^n(\mathbf{x}, \mathbf{U})\right], t + n^{-3/4} N^n(\mathbf{x}, \mathbf{U})\right)\right].$$

The following result characterizes the asymptotic mean and covariance structure of the free path chain.

Main Result

Let

$$\mu_n(\mathbf{x}, t) := n\mathbb{E}\left[(\mathbf{X}_1^n, \mathcal{T}_1^n) - (\mathbf{x}, t) \mid (\mathbf{X}_0^n, \mathcal{T}_0^n) = (\mathbf{x}, t)\right]$$

be the scaled drift of $(\mathbf{X}_k^n, \mathcal{T}_k^n)_{k\geq 0}$, and let S be a compact subset of $\mathcal{D}^\circ$.

Furthermore, let

$$\sigma_{n,ij}^2(\mathbf{x}, t) := n\mathbb{E}\left[\left(X_{1,i}^n - x_i\right)\left(X_{1,j}^n - x_j\right) \mid (\mathbf{X}_0^n, \mathcal{T}_0^n) = (\mathbf{x}, t)\right]$$

$$\sigma_{n,it}^2(\mathbf{x}, t) := n\mathbb{E}\left[\left(X_{1,i}^n - x_i\right)(\mathcal{T}_1^n - t) \mid (\mathbf{X}_0^n, \mathcal{T}_0^n) = (\mathbf{x}, t)\right]$$

$$\sigma_{n,t}^2(\mathbf{x}, t) := n\mathbb{E}\left[(\mathcal{T}_1^n - t)^2 \mid (\mathbf{X}_0^n, \mathcal{T}_0^n) = (\mathbf{x}, t)\right].$$

Theorem 1

Let S be a compact subset of $\mathcal{D}^\circ$. *Then*

$$\lim_{n\to\infty}\left|\mu_n(\mathbf{x},t)-\left(\frac{-1}{3g(\mathbf{x})^2}\left[\frac{2\nabla\mathcal{U}(\mathbf{x})}{mv(\mathbf{x})^2}+\frac{\nabla g(\mathbf{x})}{g(\mathbf{x})}\right],\ \frac{1}{g(\mathbf{x})v(\mathbf{x})}\right)\right|=0 \tag{6}$$

uniformly on $\{\mathbf{x}:\|\mathbf{x}\|\in S\}\times[0,\infty)$.

Theorem 2

Let S be a compact subset of $\mathcal{D}^\circ$. Then

1. $\lim_{n\to\infty} \sup_{(\mathbf{x},t):\|\mathbf{x}\|\in S, t\geq 0} \left| \sigma^2_{n,ij}(\mathbf{x},t) - \frac{2}{3g(\mathbf{x})^2}\delta_{ij} \right| = 0$, *where* $\delta_{ij} = \mathbb{1}\{i=j\}$.
2. $\lim_{n\to\infty} \sup_{(\mathbf{x},t):\|\mathbf{x}\|\in S, t\geq 0} \sigma^2_{n,it}(\mathbf{x},t) = 0.$
3. $\lim_{n\to\infty} \sup_{(\mathbf{x},t):\|\mathbf{x}\|\in S, t\geq 0} \sigma^2_{n,t}(\mathbf{x},t) = 0.$

Remarks on the main result

Heuristically, this says that both $\mathcal{U}_n$ and g_n impart a drift towards areas where the corresponding function has a smaller value. That is, the particle prefers to move towards areas where it travels quickly and where there are few scatterers, which may be competing influences.

Remarks on the main result

- It is interesting to note that the potential $\mathcal{U}$ does not appear in the covariance terms. It can, and will, affect the diffusion coefficient of the limiting diffusion only through an overall time change.
- We can obtain the following diffusion approximation, which separates out the effects of the particle's position at times of reflection from the effects of the speed at which the particle is moving. Let $\mathcal{D}_3 = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| \in \mathcal{D}\}$.

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Diffusion approximation

Let $(\mathcal{X}_t)_{t\geq 0}$ be a diffusion on $\mathcal{D}_3$ whose generator G acts on functions $f \in C^2(\mathcal{D}_3)$ with compact support in $\mathcal{D}_3^\circ$ by

$$Gf(\mathbf{x}) = \frac{1}{3g^2(\mathbf{x})}\Delta f(\mathbf{x}) - \frac{1}{3g^2(\mathbf{x})}\left(\frac{\nabla g(\mathbf{x})}{g(\mathbf{x})} + \frac{2\nabla \mathcal{U}(\mathbf{x})}{mv(\mathbf{x})^2}\right)\cdot \nabla f(\mathbf{x}) \quad (7)$$

and killed if/when $\mathcal{R} := \|\mathcal{X}\|$ hits the boundary of $\mathcal{D}$. Consider any $l, u \in \mathcal{D}^\circ$ with $l < u$ and start the process $((\mathbf{X}_k^n, \mathcal{T}_k^n), k \in \mathbb{N}_0)$ at $(\mathbf{x}, 0)$, where $l < \|\mathbf{x}\| < u$. Define the stopping times

$$\tau_{l,u}^n := \inf\{k \in \mathbb{N}_0 : \|\mathbf{X}_k^n\| \notin [l, u]\}$$

and

$$\tau_{l,u} := \inf\{t \geq 0 : \|\mathcal{X}_t\| \notin [l, u]\}.$$

Diffusion approximation

Theorem 3

Then, as $n \to \infty$*, the family of continuous time processes* $((\mathbf{X}^n_{\lfloor nt \rfloor \wedge \tau^n_{l,u}}, \mathcal{T}^n_{\lfloor nt \rfloor \wedge \tau^n_{l,u}}), t \geq 0)$ *converges in distribution on the Skorokhod space* $D(\mathbb{R}_+, \mathbb{R}^3 \times \mathbb{R}_+)$ *to the diffusion*

$$\left(\left(\mathcal{X}_{t \wedge \tau_{l,u}}, \int_0^{t \wedge \tau_{l,u}} \frac{ds}{g(\mathcal{X}_s) v(\mathcal{X}_s)} \right), t \geq 0 \right)$$

with initial position $(\mathbf{x}, 0)$*. This convergence happens jointly with the convergence of the hitting times,* $n^{-1} \tau^n_{l,u} \Rightarrow \tau_{l,u}$*.*

Diffusion approximation

The convergence above also holds without stopping near a boundary point of $\mathcal{D}$ that is inaccessible for the diffusion $\mathcal{R} = \|\mathcal{X}\|$. In particular, the boundaries are inaccessible for the constant acceleration towards the origin and for Newtonian gravity centered at the origin.

Diffusion approximation

- Although our models can naturally be viewed as transport processes, our results on the free path of the particle suggest an approach to diffusive limits that has more in common with that used to study Continuous Time Random Walks.
- The advantage of this approach is that, on the diffusive scale, we may easily distinguish between the effects of the particle's displacement between collisions and the effects of the speed at which the particle is traveling.

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Process on its natural time scale

The full trajectory $(\mathbf{X}^n(t))_{t\geq 0}$ of the particle with energy E_n moving in potential $\mathcal{U}_n$ and scattering density g_n is given by

$$\mathbf{X}^n(t) := \mathbf{y}_n\left(\mathbf{X}_k^n, v(\mathbf{X}_k^n)\mathbf{U}_k^n, t - n^{3/4}\mathcal{T}_k^n\right) \tag{8}$$

for $t \in [n^{3/4}\mathcal{T}_k^n, n^{3/4}\mathcal{T}_{k+1}^n)$. Fix $l, u \in \mathcal{D}^\circ$ with $l < u$. Suppose that $\|\mathbf{X}^n(0)\| \in (l, u)$ and define

$$\iota_{l,u}^n := \inf\{t : \|\mathbf{X}^n(t)\| \notin [l, u]\}.$$

Process on its natural time scale

Suppose $\mathcal{X}$ is the process with generator

$$Gf(\mathbf{x}) = \frac{1}{3g^2(\mathbf{x})}\Delta f(\mathbf{x}) - \frac{1}{3g^2(\mathbf{x})}\left(\frac{\nabla g(\mathbf{x})}{g(\mathbf{x})} + \frac{2\nabla \mathcal{U}(\mathbf{x})}{mv(\mathbf{x})^2}\right)\cdot \nabla f(\mathbf{x}) \quad (9)$$

and Ω is the time change given by

$$\Omega(t) := \mathcal{I}\left(\int_0^{\cdot} \frac{ds}{g(\mathcal{X}(s\wedge\tau_{l,u}))v(\mathcal{X}(s\wedge\tau_{l,u}))}\right)(t)$$

where $\mathcal{I}$ is the inverse operator defined by
$\mathcal{I}(f)(t) = \inf\{s : f(s) > t\}$

Process on its natural time scale

Theorem 4

We have the following convergence in distribution on the Skorokhod space $D(\mathbb{R}_+, \mathbb{R}^3)$*:*

$$\left(\mathbf{X}^n((n^{3/4}t) \wedge \iota^n_{l,u})\right)_{t\geq 0} \to (\mathcal{X}(\Omega(t) \wedge \tau_{l,u}))_{t\geq 0}\,.$$

Furthermore, the time changed process $(\mathcal{X}(\Omega(t)), t \geq 0))$ *is a diffusion process whose generator* $\mathscr{G}$ *acts on functions* $f \in C^2(\mathcal{D}_3)$ *with compact support in* $\mathcal{D}_3^\circ$ *by*

$$\begin{aligned}\mathscr{G}f(\mathbf{x}) &= g(\mathbf{x})v(\mathbf{x})Gf(\mathbf{x}) \\ &= \frac{v(\mathbf{x})}{3g(\mathbf{x})}\Delta f(\mathbf{x}) - \frac{v(\mathbf{x})}{3g(\mathbf{x})}\left(\frac{\nabla g(\mathbf{x})}{g(\mathbf{x})} + \frac{2\nabla\mathcal{U}(\mathbf{x})}{mv(\mathbf{x})^2}\right)\cdot\nabla f(\mathbf{x})\end{aligned}$$

Process on its natural time scale

The cutoffs are necessary because the generality of the potentials and scattering densities we allow permit very different behaviors at the boundary. In some cases we expect the boundaries to be inaccessible, while in other cases we expect the boundaries to be reflecting and in yet others, like $\mathcal{U}(r) = -r^{-2}$, the origin should trap the particle. We leave a detailed investigation of the boundary as an open problem.

Process on its natural time scale

If we let $\mathcal{U} \equiv 0, g \equiv 2/3, 2E/m = 1$ we get that the limit of the trajectory of the particle is a diffusion with generator

$$\begin{aligned}\mathscr{G} f(\mathbf{x}) &= \frac{v(\mathbf{x})}{3g(\mathbf{x})}\Delta f(\mathbf{x}) - \frac{v(\mathbf{x})}{3g(\mathbf{x})}\left(\frac{\nabla g(\mathbf{x})}{g(\mathbf{x})} + \frac{2\nabla \mathcal{U}(\mathbf{x})}{mv(\mathbf{x})^2}\right)\cdot \nabla f(\mathbf{x}) \\ &= \frac{1}{2}\Delta f(\mathbf{x}).\end{aligned}$$

As a result, the limiting diffusion is a Brownian motion in $\mathbb{R}^3$.

Main difficulties /technicalities of proofs

Our proofs are complicated by the fact that for a general potential $\mathcal{U}$ it is not possible to explicitly find the trajectory of the particle in the absence of collisions and this makes it difficult to determine how long this path spends in domains where it is traveling slowly.

Main difficulties /technicalities of proofs

The next lemma shows that the scaled trajectories converge to the starting point, uniformly over a fixed time interval.

Lemma 5

Fix $T > 0$. Then

$$\lim_{n\to\infty} \sup_{(\mathbf{x},\mathbf{u},t)\in\{\mathbf{x}:\|\mathbf{x}\|\in S\}\times\mathbf{S}^2\times[0,T]} \|\mathbf{y}_n(\mathbf{x},\mathbf{u},t)-\mathbf{x}\| = 0$$

$$\lim_{n\to\infty} \sup_{(\mathbf{x},\mathbf{u},t)\in\{\mathbf{x}:\|\mathbf{x}\|\in S\}\times\mathbf{S}^2\times[0,T]} \left\| \sqrt{n}\ddot{\mathbf{y}}_n(\mathbf{x},\mathbf{u},t) - \frac{\nabla\mathcal{U}(\mathbf{x})}{m} \right\| = 0$$

Main difficulties /technicalities of proofs

- Since we assume $\mathcal{U}$ and g are spherically symmetric we can work in polar coordinates (r, α).
- Let θ be the kth reflection angle and let $N^{(n)}(r_0, \theta)$ be the random variable that is distributed like $T^n_{k+1} - T^n_k$.
- For many of our proofs we require estimates that show the time between reflections approaches 0 with high probability as the scaling factor n goes to infinity.

Main difficulties /technicalities of proofs

The following is a key Lemma

Lemma 6

The family

$$\{n^{1/4} N^{(n)}(r_0, \theta) : (r_0, \theta, n) \in S \times \ [-\pi, \pi] \times \mathbb{N}\}$$

is bounded in L^p *for* $1 \leq p < \infty$.

Main difficulties /technicalities of proofs

Lemma 7

Let $\mathbf{U}$ *be independent and uniformly distributed on* $\mathbf{S}^2$. *Then*

$$\lim_{n\to\infty} \sup_{\mathbf{x}:\|\mathbf{x}\|\in S} \left\| n^{3/4}\mathbb{E}\left[N^{(n)}(\mathbf{x},\mathbf{U})\mathbf{U}\right] - \frac{1}{3g(\mathbf{x})^2 v(\mathbf{x})}\left(\frac{\nabla\mathcal{U}(\mathbf{x})}{mv(\mathbf{x})^2} - \frac{\nabla g(\mathbf{x})}{g(\mathbf{x})}\right)\right\| = 0. \quad (10)$$

Thank you for your patience!