Small-particle limits in a regularized Laplacian random growth model

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Overview

DLA aggregate formed on electrode in copper sulphate solution

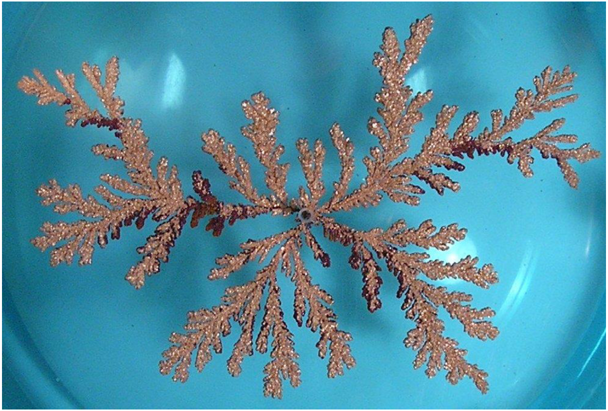


Photo by Kevin R Johnson

Dendrite growth formed by mineral deposit in sandstone

Photo by Alan Dickinson

Eden cluster formed by lichen growth

Photo by James Wearn

Lightning strike scar on a pavement

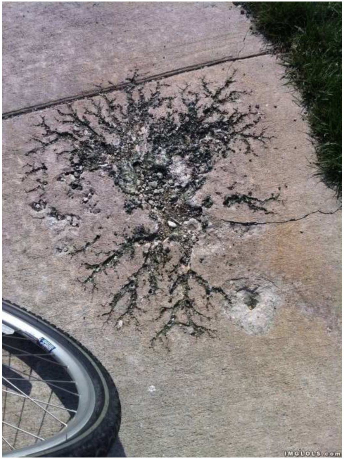


Photo by Adam Thompson

Electrical “tattoo” on survivor of lightning strike

From “Lichtenberg Figures Due to a Lightning Strike” by Yves Domart, MD, and Emmanuel Garet, MD

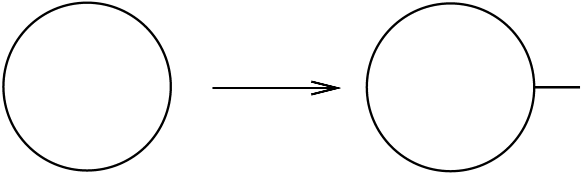
Hastings-Levitov planar random growth

- Family of models proposed by Hastings and Levitov (1998) for modelling planar random growth which occurs through the repeated aggregation of particles.
- Special cases include diffusion-limited aggregation (DLA), dielectric breakdown, and the Eden model for biological cell growth.
- Clusters are formed by iteratively composing conformal mappings, corresponding to the attachment of particles.
- Primary interest is asymptotic behaviour of large clusters. Natural to consider particle sizes that are very small compared to the overall size of the cluster and scaling limits where the particle diameters tend to zero while the number of particles grows at a rate that tends to infinity.

Conformal mapping representation of single particle

Let D_0 denote the exterior unit disk in the complex plane $\mathbb{C}$ and P denote a particle attached at the point 1.

We typically take P to be the "slit" $(1, d]$ and use the unique conformal mapping $f_P : D_0 \to D_0 \setminus (1, d]$ that fixes ∞ as a mathematical description of the particle.



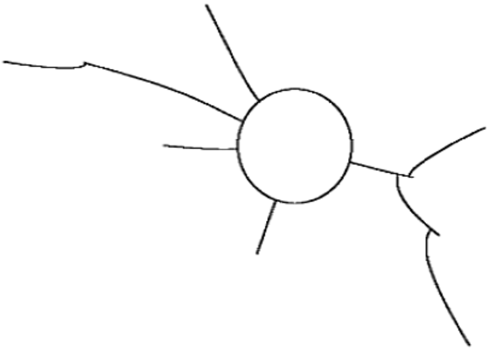
Conformal mapping representation of a cluster

Let $P_1, P_2, \dots$ be a sequence of particles with $\mathrm{diam}(P_j) = d_j$. Let $\theta_1, \theta_2, \dots$ be a sequence of angles. Define rotated copies $\{f_{P_j}^{\theta_j}\}$ of the maps $\{f_{P_j}\}$ so that $f_{P_j}^{\theta_j}(z) = e^{i\theta_j} f_{P_j}(e^{-i\theta_j} z)$. Take $\Phi_0(z) = z$, and recursively define

$$\Phi_n(z) = \Phi_{n-1} \circ f_{P_n}^{\theta_n}(z), \quad n = 1, 2, \dots.$$

This generates a sequence of conformal maps $\Phi_n : D_0 \to D_n = \mathbb{C} \setminus K_n$, where $K_{n-1} \subset K_n$ are growing compact sets, or clusters.

Cluster formed by iteratively composing slit mappings



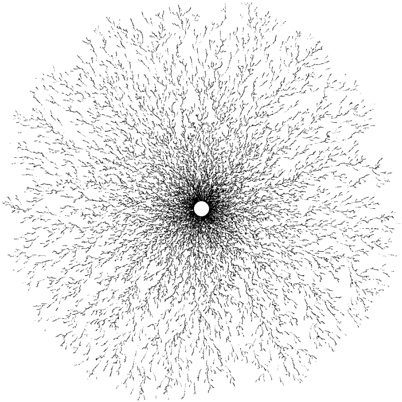
Hastings-Levitov family of clusters

By choosing the sequences $\{\theta_j\}$ and $\{d_j\}$ in different ways, it is possible to describe a wide class of growth models.

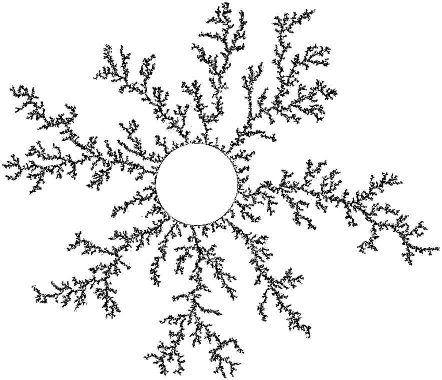
In the Hastings-Levitov family of models $\mathrm{HL}(\alpha)$, $\alpha \in [0, 2]$, the θ_j are chosen to be independent uniform random variables on the unit circle which corresponds to the attachment point at the nth step being distributed according to harmonic measure at infinity for K_{n-1}; the diameters are taken as $d_j = d/|\Phi'_{j-1}(e^{i\theta_j})|^{\alpha/2}$. Heuristically, the case $\alpha = 1$ corresponds to the Eden model and the case $\alpha = 2$ is a candidate for off-lattice DLA.

Although $\alpha = 0$ is not physical, it is mathematically the most tractable and a detailed study of scaling limits was carried out by Norris and T. (2012).

HL(0) cluster with 25,000 particles for $d = 0.02$



HL(2) cluster with 25,000 particles for $d = 0.02$



Scaling limits for HL(0)

Let $c = \log f_P'(\infty)$ be the logarithmic capacity of a single particle. For slit maps, this relates to the diameter d by

$$e^c = 1 + \frac{d^2}{4(1+d)}$$

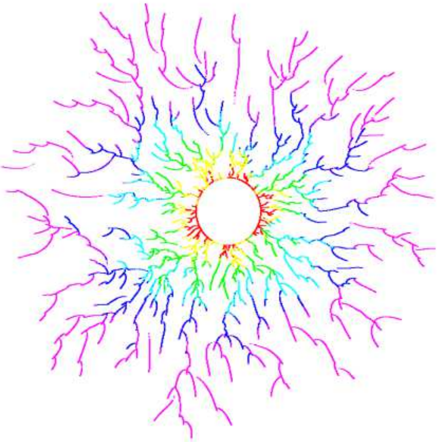
so $c \asymp d^2/4$.

In the limit as $d \to 0$,

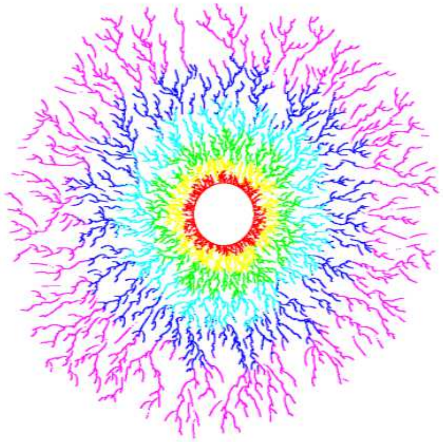
$$\sup_{n \leq d^{-6}} |\Phi_n(z) - e^{cn} z| \to 0.$$

Geometrically, the cluster after n arrivals approximates a ball of radius e^{cn} and the n^{th} particle is located close to $e^{cn+i\theta_n}$.

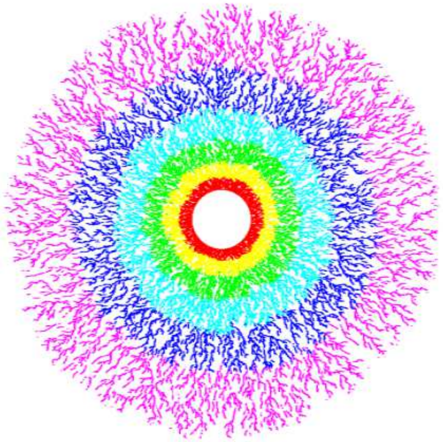
HL(0) cluster after 800 arrivals with $d = 0.1$



HL(0) cluster after 5,000 arrivals with $d = 0.04$



HL(0) cluster after 20,000 arrivals with $d = 0.02$



The harmonic measure flow

For the mapping associated to a particle P, write g_P for the inverse mapping from $D_1 \to D_0$. Set $\Gamma_n = g_{P_n} \circ \cdots \circ g_{P_1}$, where $g_{P_n} = (f_{P_n}^{\theta_n})^{-1}$, so that $\Gamma_n : D_n \to D_0$.

The map Γ_n extends continuously to the boundary ∂D_n and gives a natural parametrization of the boundary by the unit circle. It has the property that, for $\xi, \eta \in \partial D_n$, the normalized harmonic measure ω (from ∞) of the positively oriented boundary segment from ξ to η is given by $\Gamma_n(\eta)/\Gamma_n(\xi) = e^{2\pi i \omega}$.

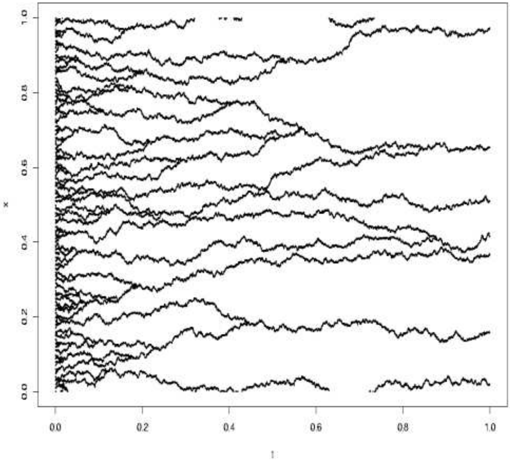
Loosely speaking, the lifting of this extension to the real line is the ‘harmonic measure flow’ on the unit circle and this flow describes the evolution of the harmonic measure on the cluster boundary, as particles are added to the cluster.

Convergence to the coalescing Brownian flow

The 'harmonic measure flow' arising from the $\mathrm{HL}(0)$ model cluster with particles of diameters d arriving at rate $\rho(P) = \left(\int_0^1 (\gamma_P(x) - x)^2\right)^{-1} = \Theta(d^{-3})$ can be shown to converge to the coalescing Brownian flow (also called the Brownian web) on the circle.

In particular, if $x_1, \dots, x_n$ is a positively oriented set of points on the circle with $x_0 = x_n$, then in the limit as $d \to 0$, the harmonic measure of the boundary segment of all fingers attached between x_{k-1} and x_k evolves like $B_t^k - B_t^k$, where $(B_t^1, \dots, B_t^n)_{t \geq 0}$ is a family of coalescing Brownian motions on the circle starting from $(x_1, \dots, x_n)$.

Harmonic measure flow for $\mathrm{HL}(0)$ with $d = 0.02$



Fine scale structure of the cluster

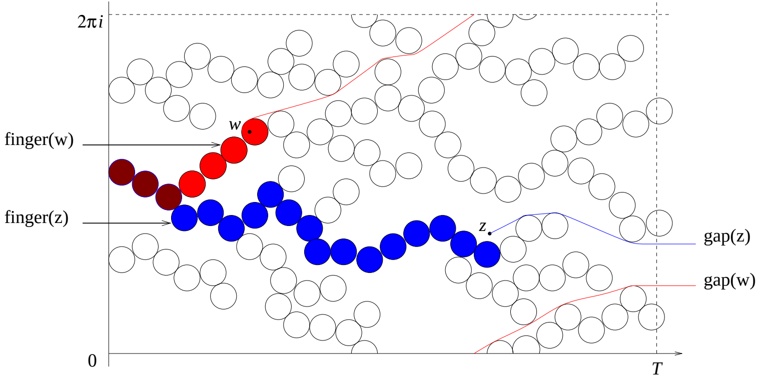
A finer scale analysis of the fingers and gaps in the cluster can be carried out by working in logarithmic space:

$$\widetilde{K}_n = \{z \in \mathbb{C} : e^z \in K_n\} \subseteq \mathbb{R}_+ \times \mathbb{R} \quad \text{(time-space)}.$$

For $\mathrm{Re}(z) \geq 0$, let finger(z) be the nearest particle to z in $\widetilde{K}_n$, together with all its "parent" particles.

Let gap(z) denote the unique minimal length path from the nearest point to z in $\overline{\widetilde{K}_n^c}$ to ∞ that does not leave $\overline{\widetilde{K}_n^c}$.

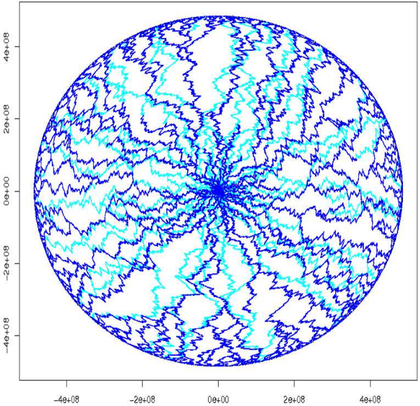
Diagram illustrating fingers and gap paths



Global limit result

For any fixed $T > 0$ and finite $E \subset [0, T] \times \mathbb{R}$, let $N = \lfloor (cd)^{-1} T \rfloor$ so that K_N is approximately a disc of radius $e^{T/d}$. Under a rescaling of "time" by d, the gap paths in $\widetilde{K}_N$ starting from points in E converge to coalescing periodic Brownian motions starting from E and the fingers converge to coalescing periodic backwards Brownian motions starting from E.

Global limit approximation of fingers and gap paths for $T = 1$ and $d = 0.05$



Infinite branches

As all (backward) Brownian motions on the circle starting at a fixed time coalesce into a single Brownian motion, the HL(0) cluster has a single infinite finger, or equivalently one common ancestor.

An open question is

"What is the smallest value of α for which $\mathrm{HL}(\alpha)$ has more than one infinite branch?"

We shall give a partial answer to this question.

Regularized Hastings-Levitov growth model

For $\alpha > 0$, we construct a regularized version of $\mathrm{HL}(\alpha)$, which we call $\mathrm{HL}(\alpha, \sigma)$, by chosing the θ_j to be independent uniform random variables on the unit circle as above, but

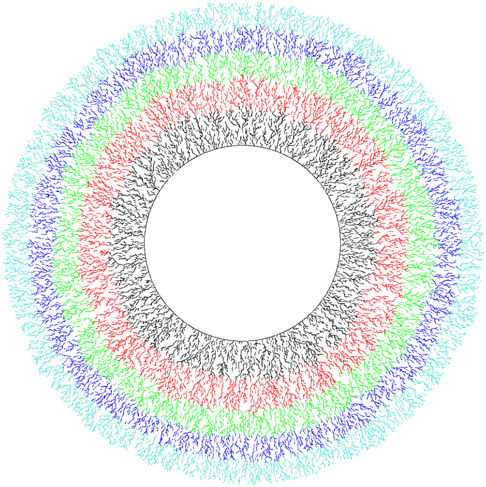
$$c_j = c/|\Phi'_{j-1}(e^{\sigma + i\theta_j})|^{\alpha}.$$

Setting $\sigma = 0$ recovers the original Hastings-Levitov models.

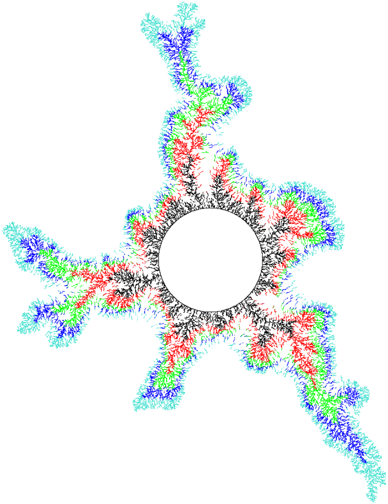
Setting $\sigma = \infty$ results in a deterministic sequence of diameters, whose capacities are given to leading order by $c_j = c/(1 + \alpha c(j-1))$.

Models retain the complicated long range dependencies of the non-regularized models, however the regularization makes it easier to control the effects of these dependencies.

HL(2,1) cluster with 25,000 particles for $d = 0.02$



HL(2,0.02) cluster with 25,000 particles for $c = 10^{-4}$



Convergence of capacities

Let

$$c_n^* = \frac{c}{1 + \alpha c(n-1)}$$

be the capacity corresponding to $\sigma = \infty$.

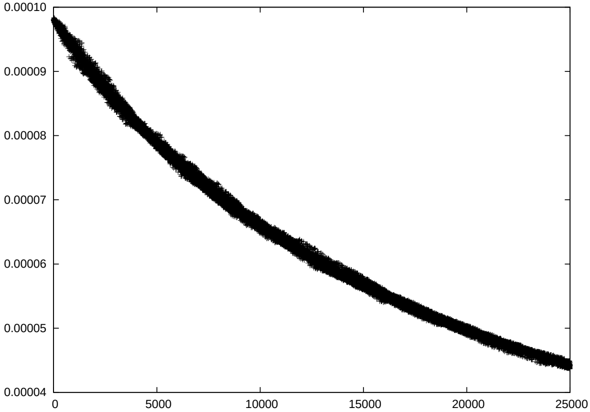
Suppose that $\sigma \gg (\log 1/c)^{-1/2}$. Then, for all $T > 0$,

$$\sup_{n \leq T/c} |\log(c_n/c_n^*)| \to 0$$

in probability as $c \to 0$.

This enables an analysis of $\mathrm{HL}(\alpha, \sigma)$ through coupling with "semi-deterministic" clusters corresponding to particles with capacities c_n^* which can be studied using the same techniques as HL(0).

Capacity sequence for HL(0.5,0.2) cluster with 25,000 particles for $c = 10^{-4}$



Convergence of clusters

For all $T > 0$, as $c \to 0$,

$$\sup_{n<T/c} \left|\Phi_n(z) - (1+\alpha cn)^{1/\alpha} z\right| \to 0.$$

Geometrically the $\mathrm{HL}(\alpha, \sigma)$ cluster after n arrivals approximates a ball of radius $(1+\alpha cn)^{1/\alpha}$.

Note that, as $\alpha \to 0$, the result for HL(0) is recovered.

Convergence of the harmonic measure flow

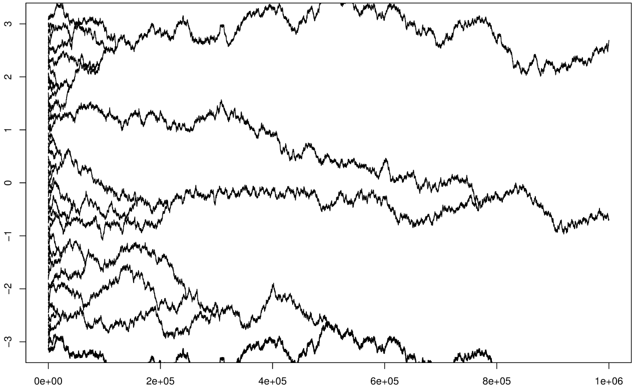
Rescale the harmonic measure flow to correspond to particles arriving at rate $c^{-3/2}$.

- If $\alpha c^{-1/2} \to 0$ then the limit is the coalescing Brownian flow on the circle with diffusivity $16/(3\pi)$.
- If $\alpha c^{-1/2} \to a \in (0, \infty)$ then the limit is a time change of the coalescing Brownian flow on the circle stopped at $32/(3\pi a)$ with time change given by

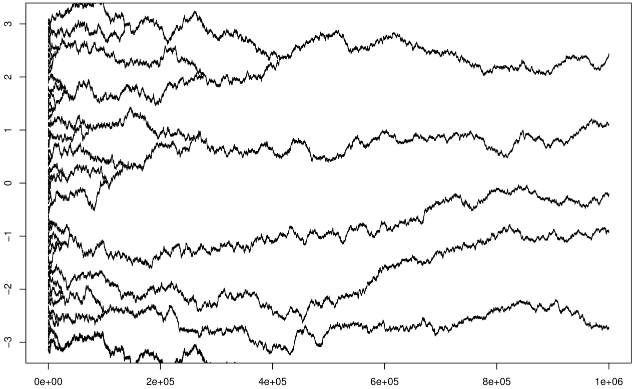
$$t \mapsto \frac{32}{3\pi a}\left(1 - \frac{1}{\sqrt{1+at}}\right).$$

- If $\alpha c^{-1/2} \to \infty$ (sufficiently slowly) then the limit is the identity.

Harmonic measure flow for $\mathrm{HL}(10^{-4}, \infty)$ with $c = 10^{-4}$



Harmonic measure flow for $\mathrm{HL}(10^{-2}, \infty)$ with $c = 10^{-4}$



Coalescing Brownian motions and Kingman's coalscent

The following result is due to Bertoin and Le Gall (2005).

Suppose that $X_0^1, \ldots, X_0^p$ are uniformly distributed on the unit circle (identified with $[0, 1)$) and suppose that $X_t^1, \ldots, X_t^p$ are coalescing Brownian motions with diffusion coefficient $\sqrt{1/12}$ starting from $X_0^1, \ldots, X_0^p$. Define a partition Π_t^p on $\{1, 2, \ldots, p\}$ by $i \sim j$ if and only if $X_t^i = X_t^j$. Then the process Π_t^p is Kingman's coalescent.

Numbers of common ancestors

Suppose that $\sigma \gg (\log c^{-1})^{-1/2}$ and $\alpha c^{-1/2} \to a$ for some $a \in [0, \infty)$. Let B be the number of common ancestors in the limit $\mathrm{HL}(\alpha, \sigma)$ cluster as $c \to 0$.

If $a = 0$, then $B = 1$ a.s.

Suppose $a > 0$ and let τ_j be the time of coalescence into j partitions in Kingman's coalescent. Then,

$$\mathbb{P}(B \leq j) = \mathbb{P}(\tau_j \leq 8/(9\pi a)),$$

and, in particular, the distribution of B is stochastically increasing in a.

Numbers of common ancestors

Furthermore, the distribution of τ_j can be explicitly calculated by

$$\tau_j \sim \sum_{k=j+1}^{\infty} E_k$$

where E_k are independent exponential random variables with rates $k(k-1)/2$. Conditional on $B = j$, the positions of the j common ancestors are that of j independent uniform points on $[0, 2\pi)$.

Summary of scaling limits of $\mathrm{HL}(\alpha, \sigma)$

Suppose that $\sigma \gg (\log c^{-1})^{-1/2}$. Then in the limit as $c \to 0$, the $\mathrm{HL}(\alpha, \sigma)$ cluster is a disk with internal structure consisting of

- one infinite branch if $\alpha \ll c^{1/2}$;
- a random number of infinite branches, whose distribution is stochastically increasing in a, if $\alpha c^{-1/2} \to a \in (0, \infty)$;
- deterministic radial growth if $\alpha \gg c^{1/2}$.

References

[1] J. Bertoin, J.-F. Le Gall, *Stochastic flows associated to coalescent processes II: Stochastic differential equations*, Ann. Inst. H. Poincaré Probabilites et Statistiques, 41, 307-333 (2005).

[2] M.B. Hastings and L.S. Levitov, *Laplacian growth as one-dimensional turbulence,* Physica D 116, 244-252 (1998).

[3] F. Johansson Viklund, A. Sola, A. Turner, *Small particle limits in a regularized Laplacian random growth model*, to appear in Communications in Mathematical Physics. Available at arxiv:1309.2194.

[4] J. Norris, A. Turner, *Hastings-Levitov aggregation in the small-particle limit,* Communications in Mathematical Physics, 316, 809-841 (2012). Available at arxiv:1106.3546.