Parameter estimation in a subcritical percolation model with colouring

Albert-Ludwigs-Universität Freiburg

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jointly with Felix Beck; arXiv:1604.08908 [math.ST]

Centre for Biological Systems Analysis (ZBSA)

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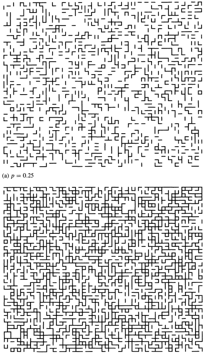
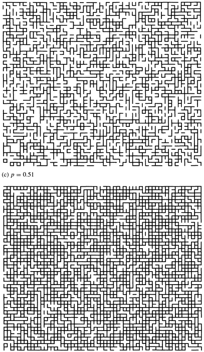
Bond percolation



(a) $p = 0.25$

(b) $p = 0.49$

Figure 1.2. Realizations of bond percolation on a 50×60 section of the square lattice for four different values of p. The pictures have been created using the same sequence of pseudorandom numbers, with the result that each graph is a subgraph of the next. Readers with good



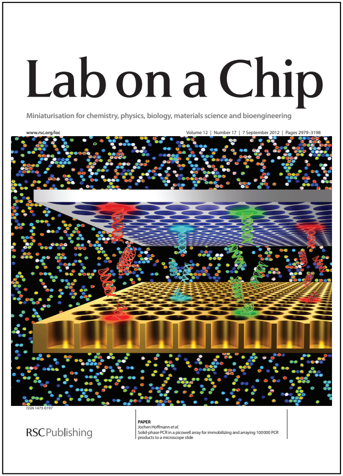
(c) $p = 0.51$

(d) $p = 0.75$

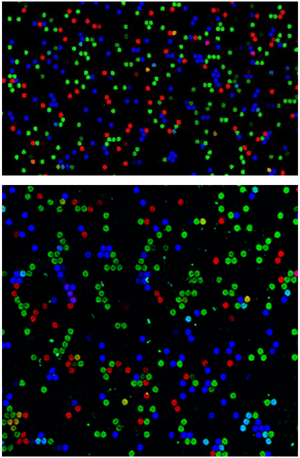
eyesight may care to check that there exist open paths joining the left to the right side when $p = 0.51$ but not when $p = 0.49$. The (random) value of p at which such paths appear for this realization is $0.5059\ldots$.

Geoffrey Grimmett; *Percolation*, Springer, 1999.

Cross-contamination rate estimation for digital PCR in lab-on-a-chip microfluidic devices


Lab on a Chip
Miniaturisation for chemistry, physics, biology, materials science and bioengineering
www.rsc.org/loc
Volume 12 | Number 17 | 7 September 2012 | Pages 2979–3198
ISSN 1473-0197
RSCPublishing
PAPER
Jochen Hoffmann *et al.*
Solid-phase PCR in a picowell array for immobilizing and arraying 100000 PCR products to a microscope slide

Hoffmann *et al.*; Lab on a Chip, **12**, 3049–3054, 2012.



Rath; MSc thesis, Univ. of Freiburg, 2014.

Modelling choices

Contamination:

(i) unidirectional (independent, directed edges $\xi_{i\to j}$ and $\xi_{j\to i}$ between any two adjacent vertices $i \sim j$), or

(ii) symmetric (undirected edges ξ_{ij}).

Open edges:

(1) independent Bernoulli variables, or

(2) locally correlated 0–1 random variables.

Contamination:

(A) confined to neighbours, or

(B) it might propagate via a series of open edges.

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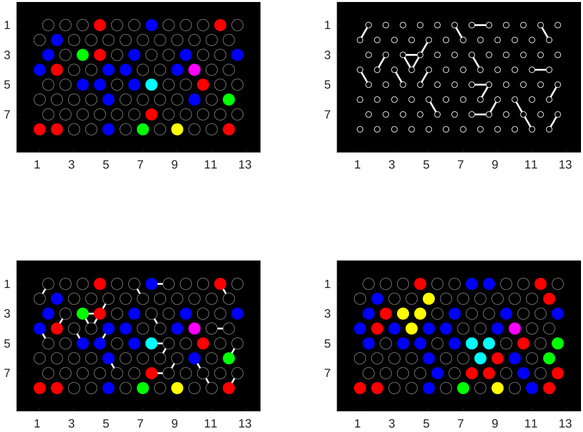
The process



The strategy

- Process: index set I ($n_I := |I|$), colours $\ell \in \{1, 2, \ldots, n_c\}$

$$Y_i^\ell := X_i^\ell \vee \bigvee_{j:\ j \leftrightarrow i} X_j^\ell, \qquad Y_i^\ell \in \{0, 1\}$$

- Goal: estimate $\theta = (\lambda^1, \ldots, \lambda^{n_c}, \mu)$ from the data $(Y_i^\ell)_{i \in I, \ell \in \{1,2,\ldots,n_c\}}$

1. Objective: the method of simulated moments (MSM) is strongly consistent.
2. We prove a strong law of large numbers (SLLN) with weakly dependent variables.
3. To upper bound dependence (i.e. correlations bw. vertices), we use the FKG and BK inequalities of percolation theory.

The method of simulated moments (MSM)

- Data $\mathscr{Y} = (\mathscr{Y}_i)_{i \in I}$ originates from a distribution which is parameterised by the unknown $\theta_0 \in \Theta$ (the *true parameter value*).
- Moments: K is some n_m-dimensional function of the individual observations Y_i. $k(\theta) := \mathrm{E}_\theta[K(Y_i)]$
- Identifiability: $\mathrm{E}_{\theta_0}[K(\mathscr{Y}_i)] = k(\theta) \iff \theta = \theta_0$
- MSM: $k(\theta)$ is not available in analytical form but there exists an unbiased estimator $\widetilde{k}(U_i^s, \theta)$. $(U_i^s)_{i \in I, s \in \{1,\dots,n_s\}}$ is some source of randomness, typically vectors of independent $\mathrm{U}[0,1]$.
- $\Omega \in \mathbb{R}^{n_m \times n_m}$ is a symmetric, positive definite matrix; $\alpha(\eta) = \eta^{\mathrm{T}} \Omega \eta$ a quadratic form. The MSM estimator is

$$\hat{\theta}_{n_s,n_l} := \arg\min_{\theta \in \Theta} \alpha \left(\frac{1}{n_l} \sum_{i=1}^{n_l} \left(K(\mathscr{Y}_i) - \frac{1}{n_s} \sum_{s=1}^{n_s} \widetilde{k}(U_i^s, \theta) \right) \right)$$

The method of simulated moments (MSM)

Proposition

$\Omega \in \mathbb{R}^{n_m \times n_m}$ is symmetric, positive definite; $\alpha(\eta) = \eta^{\mathrm{T}}\Omega\eta$.
The MSM estimator is

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If n_s is fixed and n_l tends to infinity, and the almost sure convergence guaranteed by the SLLN

$$\frac{1}{n_l}\sum_{i=1}^{n_l}\widetilde{k}(U_i^s,\theta) \underset{n_l\to\infty}{\longrightarrow} k(\theta)$$

is uniform in $\theta \in \Theta$ for every s, then $\hat{\theta}_{n_s,n_l}$ is strongly consistent (i.e. $\hat{\theta}_{n_s,n_l}$ converges to θ_0 almost surely).

Application to our percolation model

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- Variables (Y_i^ℓ) are neither identically distributed (*boundary!*) nor independent. — SLLN is not a given.
- Moments we use: $Y_i^\ell, \quad Y_i^\ell Y_j^\ell$ for $i \sim j$
- Identifiability:
  If $(\lambda^1, \dots, \lambda^{n_c}) = h \in \{0,1\}^{n_c}$, then for any choice of μ, $(Y_i)_{i \in I}$ is identically h.
  Similarly, if ($\mu = 1$ and ($\lambda^\ell > 0 \iff h_\ell = 1$)), then $(Y_i)_{i \in I}$ is identically h (with high probability as $n_I \to \infty$).

MSM for the percolation model

Theorem (Main result)

$I_2 := \{(i,j) \in I \times I \mid i \sim j, i < j\}$;
$\Omega \in \mathbb{R}^{2n_c \times 2n_c}$ is symmetric, positive definite; $\alpha(\eta) = \eta^{\mathrm{T}}\Omega\eta$ a quadratic form.

For triangular lattice: Θ a compact subset of $([0,1]^{n_c} \setminus \{0,1\}^{n_c}) \times [0,1/5[$.
(For the square lattice case, replace $1/5$ with $1/3$.)
When n_s is fixed and $n_l \to \infty$, then

$$\hat{\theta}_{n_s,n_l} := \arg\min_{\theta\in\Theta} \alpha \begin{pmatrix} \left(\frac{1}{n_l}\sum_{i\in I}\left(\mathscr{Y}_i^{\ell} - \frac{1}{n_s}\sum_{s=1}^{n_s} Y_i^{\ell,s}\right)\right)_{\ell\in\{1,\dots,n_c\}} \\ \left(\frac{1}{|I_2|}\sum_{(i,j)\in I_2}\left(\mathscr{Y}_i^{\ell}\mathscr{Y}_j^{\ell} - \frac{1}{n_s}\sum_{s=1}^{n_s} Y_i^{\ell,s}Y_j^{\ell,s}\right)\right)_{\ell\in\{1,\dots,n_c\}} \end{pmatrix}$$

$$= \arg\min_{\theta\in\Theta} \alpha \begin{pmatrix} \left(\bar{\mathscr{Y}}^{\ell} - \frac{1}{n_s}\sum_{s=1}^{n_s} \bar{Y}^{\ell,s}\right)_{\ell\in\{1,\dots,n_c\}} \\ \left(\bar{\mathscr{Z}}^{\ell} - \frac{1}{n_s}\sum_{s=1}^{n_s} \bar{Z}^{\ell,s}\right)_{\ell\in\{1,\dots,n_c\}} \end{pmatrix}$$

is strongly consistent.

Goal: strong law of large numbers (SLLN)

Goal: as $n_I \to \infty$, for $i \sim j$, almost surely, uniformly in $\theta \in \Theta$,

$$\frac{1}{n_I} \sum_{i \in I} Y_i^\ell \longrightarrow \mathrm{E}_\theta Y_i^\ell$$

and

$$\frac{1}{|I_2|} \sum_{(i,j) \in I_2} Y_i^\ell Y_j^\ell \longrightarrow \mathrm{E}_\theta \left[Y_i^\ell Y_j^\ell \right].$$

This would ensure that almost surely, uniformly in $\theta \in \Theta$,

$$\alpha \begin{pmatrix} \left(\bar{\mathscr{Y}}^\ell - \frac{1}{n_s} \sum_{s=1}^{n_s} \bar{Y}^{\ell,s} \right)_{\ell \in \{1,\dots,n_c\}} \\ \left(\bar{\mathscr{Z}}^\ell - \frac{1}{n_s} \sum_{s=1}^{n_s} \bar{Z}^{\ell,s} \right)_{\ell \in \{1,\dots,n_c\}} \end{pmatrix} \xrightarrow[n_I \to \infty]{} \alpha \begin{pmatrix} \left(\mathrm{E}_{\theta_0} Y_i^\ell - \mathrm{E}_\theta Y_i^\ell \right)_{\ell \in \{1,\dots,n_c\}} \\ \left(\mathrm{E}_{\theta_0} [Y_i^\ell Y_j^\ell] - \mathrm{E}_\theta [Y_i^\ell Y_j^\ell] \right)_{\ell \in \{1,\dots,n_c\}} \end{pmatrix}$$

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$$\frac{1}{n_I}\sum_{i\in I} Y_i^\ell - \frac{1}{n_I}\sum_{i\in I} \mathrm{E}_\theta Y_i^\ell \longrightarrow 0$$

$$\text{and} \qquad \frac{1}{|I_2|}\sum_{(i,j)\in I_2} Y_i^\ell Y_j^\ell - \frac{1}{|I_2|}\sum_{(i,j)\in I_2} \mathrm{E}_\theta\left[Y_i^\ell Y_j^\ell\right] \longrightarrow 0.$$

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Strong law of large numbers (SLLN)

Proposition (SLLN for our percolation model)

Let Θ be a compact subset of $[0,1]^{n_c} \times [0,1/5[$ (triangular lattice: $\mu < 1/5$; square lattice: $\mu < 1/3$). If Y is generated with parameter value $\theta \in \Theta$, then

$$\frac{1}{n_I}\left(\sum_{i \in I} Y_i^\ell - \sum_{i \in I} \mathrm{E}_\theta Y_i^\ell\right) \underset{n_I \to \infty}{\longrightarrow} 0$$

almost surely, uniformly in $\theta \in \Theta$.

Proof

Let $Y_i := Y_i^\ell$ for fixed $\ell \in \{1, \ldots, n_c\}$.
Let $a > 1$. Define the lacunary sequence $k_n := [a^n]$.
Let $S_k := \sum_{i=1}^{k} Y_i$.

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Strong law of large numbers (SLLN)

By Chebyshov's inequality for the $\theta(k_n)$ for every $k_n = [a^n]$ where the supremum on the compact set Θ is achieved, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathrm{P}\left(\sup_{\theta\in\Theta} \left| \frac{S_{k_n} - \mathrm{E} S_{k_n}}{k_n} \right| > \varepsilon \right) = \sum_{n=1}^{\infty} \mathrm{P}\left(\left| \frac{S_{k_n}(\theta(k_n)) - \mathrm{E}_{\theta(k_n)} S_{k_n}}{k_n} \right| > \varepsilon \right)$$
$$\leq \sum_{n=1}^{\infty} \frac{\sup_{\theta\in\Theta} \mathrm{Var}\, S_{k_n}}{\varepsilon^2 k_n^2}$$
$$\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{k_n^2} \sup_{\theta\in\Theta} \sum_{i=1}^{k_n} \mathrm{Var}\, Y_i$$
$$+ \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{2}{k_n^2} \sup_{\theta\in\Theta} \sum_{1\leq i<j\leq k_n} (\mathrm{E}[Y_i Y_j] - \mathrm{E} Y_i \mathrm{E} Y_j).$$

If this is finite, then by the Borel–Cantelli lemma, as $n \to \infty$,

$$\sup_{\theta\in\Theta} \left| \frac{S_{k_n} - \mathrm{E} S_{k_n}}{k_n} \right| \to 0 \quad \text{a.s.}$$

Strong law of large numbers (SLLN)

$$\frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{k_n^2}\sup_{\theta\in\Theta}\sum_{i=1}^{k_n}\mathrm{Var}\,Y_i + \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{2}{k_n^2}\sup_{\theta\in\Theta}\sum_{1\le i<j\le k_n}(\mathrm{E}[Y_iY_j]-\mathrm{E}Y_i\,\mathrm{E}Y_j)$$

Lemma

If $1 < a$, $k_n = [a^n]$, then

$$\sum_{n=1}^{\infty}\frac{1}{k_n} < \infty.$$

Lemma

As $n \to \infty$, it holds

$$\sup_{\theta\in\Theta}\left|\sum_{1\le i<j\le n}(\mathrm{E}[Y_iY_j]-\mathrm{E}Y_i\,\mathrm{E}Y_j)\right| = \mathcal{O}(n).$$

Strong law of large numbers (SLLN)

$$\frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{1}{k_n^2}\underbrace{\sup_{\theta\in\Theta}\sum_{i=1}^{k_n}\underbrace{\operatorname{Var} Y_i}_{\leq 1}}_{\leq k_n}+\frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{2}{k_n^2}\sup_{\theta\in\Theta}\sum_{1\leq i<j\leq k_n}(\mathrm{E}[Y_iY_j]-\mathrm{E}Y_i\mathrm{E}Y_j)$$

Lemma

If $1 < a$, $k_n = [a^n]$, *then*

$$\sum_{n=1}^{\infty}\frac{1}{k_n}<\infty.$$

Lemma

As $n\to\infty$, *it holds*

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Strong law of large numbers (SLLN)

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Quantifying (in)dependence of faraway vertices

Geoffrey Grimmett; *Percolation*, Springer, 1999.

Probability space $(\{0,1\}^S, \mathscr{F}, \mathrm{P})$ ($|S| \leq \aleph_0$);
events $\mathscr{F}$: σ-algebra generated by the finite-dimensional cylinder sets;
the measure is a product measure $\mathrm{P} = \prod_{s \in S} \nu_s$,

ν_s is given by $\big(p(s)\big)_{s\in S} \in [0,1]^S$ via

$$\nu_s\big(\omega(s)=1\big) = p(s), \quad \nu_s\big(\omega(s)=0\big) = 1 - p(s)$$

for sample vectors $\big(\omega(s)\big)_{s\in S} \in \{0,1\}^S$.

A colour $\ell \in \{1,\ldots,n_c\}$ is already fixed.
Insert a loop edge for every vertex, $p(s) := \lambda^\ell$.
For edges of the lattice, $p(s) := \mu$.

An event $A \in \mathscr{F}$ is increasing $:\iff \Big((\omega \leq \omega', \omega \in A) \Rightarrow \omega' \in A\Big)$.

FKG inequality (Fortuin, Kasteleyn, Ginibre; 1971)

If $A, B \in \mathscr{F}$ are increasing events, then $\mathrm{P}(A \cap B) \geq \mathrm{P}(A)\mathrm{P}(B)$.

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Some percolation theory

An event $A \in \mathscr{F}$ is increasing $:\iff \big((\omega \le \omega', \omega \in A) \Rightarrow \omega' \in A\big)$.

Let $e_1, e_2, \ldots, e_N$ be N distinct edges, $A, B \in \mathscr{F}$ two increasing events which depend on the states of these N edges $\omega = (\omega(e_1), \ldots, \omega(e_N))$ only.

$$J(\omega) := \left\{ e_i \,\middle|\, i \in \{1, \ldots, N\},\ \omega(e_i) = 1 \right\}$$

For A, B increasing, A and B occur disjointly:
$A \circ B := \{\omega \in \{0,1\}^S \mid$ there exists an $H \subseteq J(\omega)$ such that ω' determined by $J(\omega') = H$ belongs to A, and ω'' determined by $J(\omega'') = J(\omega) \setminus H$ belongs to $B\}$.

$A \circ B$ is also increasing and $A \circ B \subseteq A \cap B$.

BK inequality (van den Berg, Kesten; 1985)

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An event $A \in \mathscr{F}$ is increasing $:\iff \Big((\omega \le \omega', \omega \in A) \Rightarrow \omega' \in A\Big)$.

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$$J(\omega) := \left\{ e_i \,\middle|\, i \in \{1, \ldots, N\},\ \omega(e_i) = 1 \right\}$$

For A, B increasing, A and B occur disjointly:
$A \circ B := \{\omega \in \{0,1\}^S \mid$ there exists an $H \subseteq J(\omega)$ such that ω' determined by $J(\omega') = H$ belongs to A, and ω'' determined by $J(\omega'') = J(\omega) \setminus H$ belongs to $B\}$.

$A \circ B$ is also increasing and $A \circ B \subseteq A \cap B$.

BK inequality (van den Berg, Kesten; 1985)

If $A, B \in \mathscr{F}$ are increasing events, then $\mathrm{P}(A \circ B) \le \mathrm{P}(A)\mathrm{P}(B)$.

Some percolation theory

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Some percolation theory

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FKG inequality (Fortuin, Kasteleyn, Ginibre; 1971)

If $A, B \in \mathscr{F}$ are increasing events, then $\mathrm{P}(A \cap B) \ge \mathrm{P}(A)\mathrm{P}(B)$.

$$J(\omega) := \{e_i \,|\, i \in \{1, \dots, N\}, \ \omega(e_i) = 1\}$$

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If $A, B \in \mathscr{F}$ are increasing events, then $\mathrm{P}(A \circ B) \le \mathrm{P}(A)\mathrm{P}(B)$.

Quantifying (in)dependence of faraway vertices

Lemma

As $n \to \infty$, it holds

$$\sup_{\theta \in \Theta} \left| \sum_{1 \le i < j \le n} (\mathrm{E}[Y_i Y_j] - \mathrm{E}Y_i \mathrm{E}Y_j) \right| = \mathcal{O}(n).$$

In the lattice graph extended with loop edges, the event $\{Y_i = 1\}$ is *increasing*.

By the

FKG inequality

If $A, B \in \mathscr{F}$ are increasing events, then $\mathrm{P}(A \cap B) \geq \mathrm{P}(A)\mathrm{P}(B)$.

,

$$\mathrm{E}[Y_i Y_j] - \mathrm{E}Y_i \mathrm{E}Y_j = \mathrm{P}(Y_i Y_j = 1) - \mathrm{P}(Y_i = 1)\mathrm{P}(Y_j = 1) \geq 0.$$

Quantifying (in)dependence of faraway vertices

We have

$$\mathrm{E}[Y_iY_j] - \mathrm{E}Y_i\mathrm{E}Y_j = \mathrm{P}(Y_iY_j = 1) - \mathrm{P}(Y_i = 1)\mathrm{P}(Y_j = 1) \overset{\text{FKG}}{\geq} 0.$$

By the

BK inequality

If $A, B \in \mathscr{F}$ are increasing events, then $\mathrm{P}(A \circ B) \leq \mathrm{P}(A)\mathrm{P}(B)$.

,

$$\begin{aligned}\mathrm{P}(Y_iY_j = 1) - \mathrm{P}(Y_i = 1)\mathrm{P}(Y_j = 1) &= \mathrm{P}\big(\{Y_i = 1\} \circ \{Y_j = 1\}\big) - \mathrm{P}(Y_i = 1)\mathrm{P}(Y_j = 1) \\ &\quad + \mathrm{P}\Big(\{Y_iY_j = 1\} \setminus \{Y_i = 1\} \circ \{Y_j = 1\}\Big) \\ &\overset{\text{BK}}{\leq} \mathrm{P}\Big(\{Y_iY_j = 1\} \setminus \{Y_i = 1\} \circ \{Y_j = 1\}\Big).\end{aligned}$$

Cooccurrence of $\{Y_i = 1\}$ and $\{Y_j = 1\}$ which is not disjoint is one where i and j are in the same component:

$$\{Y_iY_j = 1\} \setminus \{Y_i = 1\} \circ \{Y_j = 1\} \subseteq \{i \leftrightarrow j\}.$$

Quantifying (in)dependence of faraway vertices

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Goal: on the triangular lattice, for every $\varepsilon > 0$, *uniformly* for $\mu \in [0, 1/5 - \varepsilon]$,

$$\sum_{1 \le i < j \le n} \mathrm{P}(i \leftrightarrow j) = \mathcal{O}(n)$$

$\mathscr{W}_k :=$ {paths (i.e. self-avoiding walks) on the triangular lattice with length k and beginning in a fixed vertex i}

$$|\mathscr{W}_k| \le 6 \times 5^{k-1}$$

$$\mathrm{E}[\#\text{paths from } i] = \sum_{k=1}^{\infty} \sum_{\gamma \in \mathscr{W}_k} \mu^k = \sum_{k=1}^{\infty} |\mathscr{W}_k| \mu^k \le \sum_{k=1}^{\infty} 6 \times 5^{k-1} \mu^k = 6\mu \frac{1}{1 - 5\mu} < \infty$$

By allowing any paths on the infinite lattice $\supseteq I$,

$$\sum_{1 \le i < j \le n} \mathrm{P}(i \leftrightarrow j) \le \sum_{1 \le i \le n} \sum_{k=1}^{\infty} \sum_{\substack{j \text{ is endpoint} \\ \text{of } \gamma \in \mathscr{W}_k}} \mu^k$$

$$\le \sum_{1 \le i \le n} \frac{6\mu}{1 - 5\mu} = \frac{6\mu}{1 - 5\mu} n.$$

In summary

As $n \to \infty$, by the FKG and BK inequalities,

$$\sup_{\theta\in\Theta} \left| \sum_{1\le i<j\le n} (\mathrm{E}[Y_i Y_j] - \mathrm{E}Y_i \mathrm{E}Y_j) \right| = \mathscr{O}(n).$$

Hence for $k_n = [a^n]$,

$$\sum_{n=1}^{\infty} \mathrm{P}\left(\sup_{\theta\in\Theta} \left| \frac{S_{k_n} - \mathrm{E}S_{k_n}}{k_n} \right| > \varepsilon \right) < \infty.$$

Further, consider $k_n \le n_l < k_{n+1}$. Then

$$\frac{1}{n_l} \left(\sum_{i\in I} Y_i^\ell - \sum_{i\in I} \mathrm{E}_\theta Y_i^\ell \right) \xrightarrow[n_l \to \infty]{} 0$$

almost surely, uniformly in $\theta \in \Theta$, where Θ is a compact subset of $[0,1]^{n_c} \times [0,1/5[$ ($\mu < 1/3$ for the square lattice).

In summary

As $n \to \infty$, by the FKG and BK inequalities,

$$\sup_{\theta\in\Theta}\left|\sum_{1\le i<j\le n}(\mathrm{E}[Y_iY_j]-\mathrm{E}Y_i\mathrm{E}Y_j)\right| = \mathscr{O}(n).$$

Hence for $k_n = [a^n]$,

$$\sum_{n=1}^{\infty}\mathrm{P}\left(\sup_{\theta\in\Theta}\left|\frac{S_{k_n}-\mathrm{E}S_{k_n}}{k_n}\right|>\varepsilon\right)<\infty.$$

Further, consider $k_n \le n_l < k_{n+1}$. Then

$$\frac{1}{n_l}\left(\sum_{i\in I}Y_i^\ell-\sum_{i\in I}\mathrm{E}_\theta Y_i^\ell\right)\underset{n_l\to\infty}{\longrightarrow}0$$

almost surely, uniformly in $\theta\in\Theta$, where Θ is a compact subset of $[0,1]^{n_c}\times[0,1/5[$ ($\mu<1/3$ for the square lattice). For the strong consistence of $\hat{\theta}_{n_s,n_l}$, repeat for

$$\frac{1}{|I_2|}\left(\sum_{(i,j)\in I_2}Y_i^\ell Y_j^\ell-\sum_{(i,j)\in I_2}\mathrm{E}_\theta[Y_i^\ell Y_j^\ell]\right)\underset{n_l\to\infty}{\longrightarrow}0.$$

Implementation

$n_c = 3$ colours. Set $\alpha(\eta) = \eta^{\mathrm{T}}\Omega\eta$ by

$$\Omega = \mathrm{diag}\Big((\bar{\mathscr{Y}}^1)^{-2},\ldots,(\bar{\mathscr{Y}}^{n_c})^{-2},(\bar{\mathscr{Z}}^1)^{-2},\ldots,(\bar{\mathscr{Z}}^{n_c})^{-2}\Big).$$

$$\alpha\left(\begin{array}{c}\left(\bar{\mathscr{Y}}^\ell - \frac{1}{n_s}\sum_{s=1}^{n_s}\bar{Y}^{\ell,s}\right)_{\ell\in\{1,\ldots,n_c\}}\\ \left(\bar{\mathscr{Z}}^\ell - \frac{1}{n_s}\sum_{s=1}^{n_s}\bar{Z}^{\ell,s}\right)_{\ell\in\{1,\ldots,n_c\}}\end{array}\right) \longrightarrow \min$$

Common random numbers for different $\theta = (\lambda^1,\ldots,\lambda^{n_c},\mu)\in\Theta$.

Method 1: $(U_i^{\ell,s}),(V_{ij}^s) \sim U[0,1]$ $\quad(\ell\in\{1,\ldots,n_c\}, s\in\{1,\ldots,n_s\}, i\in I, (i,j)\in I_2)$

$$X_i^{\ell,s} := \begin{cases}1 & \text{if } U_i^{\ell,s} < \lambda^\ell,\\ 0 & \text{otherwise,}\end{cases} \qquad \xi_{ij}^s := \begin{cases}1 & \text{if } V_{ij}^s < \mu,\\ 0 & \text{otherwise.}\end{cases}$$

Method 2: $(\sigma^{\ell,s}) \sim U[S_{n_I}]$, $(\tau^s) \sim U[S_{|I_2|}]$ $\quad(\ell\in\{1,\ldots,n_c\}, s\in\{1,\ldots,n_s\})$

$$X_i^{\ell,s} := \begin{cases}1 & \text{if } \sigma^{\ell,s}(i) \le \lfloor\lambda^\ell n_I\rceil,\\ 0 & \text{otherwise,}\end{cases} \qquad \xi_{ij}^s := \begin{cases}1 & \text{if } \tau^s\big((i,j)\big) \le \lfloor\mu|I_2|\rceil,\\ 0 & \text{otherwise.}\end{cases}$$

Results

1. $n_l = 25 \times 25 = 625$ vertices ($|I_2| = 1776$)

n_s	n_{opt}	$\mu_{\max}$	θ_0	$\hat{\theta}^{(M1)}_{n_s,n_l}$	$d^{(M1)}$	$\hat{\theta}^{(M2)}_{n_s,n_l}$	$d^{(M2)}$	$\alpha_{\hat{\theta}^{(M2)}_{n_s,n_l}}$
10	10	0.1	0.1	0.1287	28.7%	0.1223	22.3%	0.0126
			0.05	0.0597	19.4%	0.0605	21%	
			0.07	0.0614	12.29%	0.0587	16.14%	
			0.06	0.0428	28.67%	0.0436	27.33%	

2. $n_l = 500 \times 500 = 250{,}000$ vertices ($|I_2| = 748{,}001$)

n_s	n_{opt}	$\mu_{\max}$	θ_0	$\hat{\theta}^{(M1)}_{n_s,n_l}$	$d^{(M1)}$	$\hat{\theta}^{(M2)}_{n_s,n_l}$	$d^{(M2)}$	$\alpha_{\hat{\theta}^{(M2)}_{n_s,n_l}}$
2	3	0.04	0.03	0.0295	1.67%	0.0293	2.33%	0.0011
			0.04	0.0402	0.5%	0.0401	0.25%	
			0.05	0.0522	4.4%	0.0520	4%	
			0.02	0.0192	4%	0.0195	2.5%	

d is the relative bias: $\left|1 - \hat{\theta}_{n_s,n_l} \big/ \theta_0\right| \times 100\%$

Summary

- Model chosen: symmetric (undirected edges ξ_{ij}); edges are independent Bernoulli variables; contamination propagates via a series of open edges.
- Method of simulated moments is strongly consistent as $n_l \to \infty$ but n_s bounded.
- Unusual: sample is large but neither independent nor identically distributed.
- Proof method:
  1. The method of simulated moments (MSM) is strongly consistent.
  2. Proved a strong law of large numbers (SLLN) with weakly dependent variables.
  3. FKG and BK inequalities of percolation theory used to upper bound dependence (i.e. correlations bw. vertices).

Open problems

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1. Confidence intervals? Under regularity conditions (the estimator is continuously differentiable with respect to θ), $\sqrt{n_l}(\hat{\theta}_{n_s,n_l} - \theta_0)$ is asymptotically normal with known limiting variance.
   - It is possible to choose Ω optimally, i.e. to minimise this asymptotic variance.
2. Beyond estimating μ, estimate the proportion of vertices which are in a non-trivial component.
3. Largest μ for which SLLN holds? (cf. $1/5$ for triangular, $1/3$ for square lattice) Will this MSM work in the entire subcritical regime?
4. Maximum likelihood estimation; computing the probability of a configuration (and esp. of an *animal*).
5. Model fit? Locally positively correlated open edges might be needed; e.g. Ising model for the edges (increases degrees of freedom by 1).

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