Change-Points in High-Dimensional Settings

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joint work with

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University of Bristol, 20.02.2015

Change-Point Setting

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where $\{(e_{1,t}, \ldots, e_{d,T})^T, t = 1, \ldots, T\}$ is i.i.d., $\operatorname{E} e_{i,j} = 0$, $0 < \operatorname{var} e_{i,t} < \infty$, $g(\cdot)$ Riemann-integrable.

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If we know, where to look, we can increase signal-to-noise ratio!

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Statistics based on projections

Let $\mathbf{p}_d$ be a (possibly random) projection vector!

Consider univariate time series:

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Standard univariate change-point statistics based on the centered partial sum process

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$$\left\{ \frac{U_{d,T}(x)}{\tau(\mathbf{p}_d)} : 0 \leqslant x \leqslant 1 \,|\, \mathbf{p}_d \right\} \overset{D[0,1]}{\longrightarrow} \{B(x) : 0 \leqslant x \leqslant 1\} \quad a.s.,$$

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Assumption on the error sequence

Let $\eta_{1,t}, \eta_{2,t}, \ldots$ independent (identically distributed across time t) with $\operatorname{E}\eta_{i,t} = 0$, $\operatorname{var}\eta_{i,t} = 1$ and $\operatorname{E}|\eta_{i,t}|^{\nu} \leqslant C < \infty$, $\nu > 2$.

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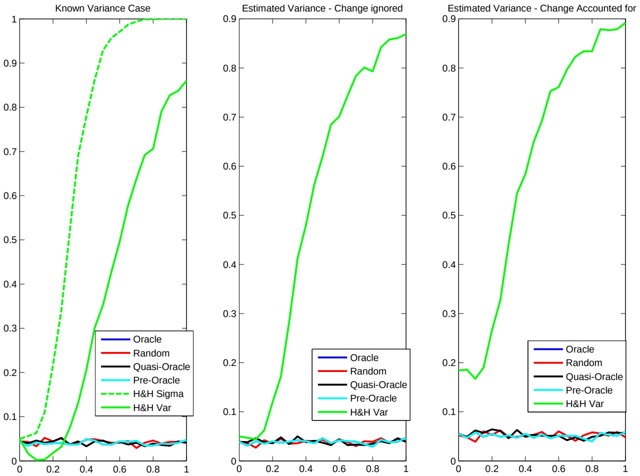
Empirical Size, $\mathcal{C}.3$, $s_j = 1$, $T = 100$, $d = 200$



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Asymptotics for contiguous alternatives

Theorem

Let

$$\mathcal{E}_1^2(\boldsymbol{\Delta}_d, \mathbf{p}_d) := \frac{\|\boldsymbol{\Delta}_d\|^2 \|\mathbf{p}_d\|^2 \cos^2(\alpha_{\boldsymbol{\Delta}_d, \mathbf{p}_d})}{\tau^2(\mathbf{p}_d)}.$$

a) If $\sqrt{T}\,\mathcal{E}_1(\boldsymbol{\Delta}_d, \mathbf{p}_d) \to \infty$ *a.s.*, then

$$\left\{ \frac{U_{d,T}(x)}{\tau(\mathbf{p}_d)\sqrt{T}\,\mathcal{E}_1(\boldsymbol{\Delta}, \mathbf{p}_d)} : 0 \leqslant x \leqslant 1 \,|\, \mathbf{p}_d \right\}$$

$$\xrightarrow{D[0,1]} \left\{ \int_0^x g(t)\,dt - x \int_0^1 g(t)\,dt : 0 \leqslant x \leqslant 1 \right\} \qquad a.s.$$

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$$\mathcal{E}_1^2(\boldsymbol{\Delta}_d, \mathbf{p}_d) := \frac{\|\boldsymbol{\Delta}_d\|^2 \|\mathbf{p}_d\|^2 \cos^2(\alpha_{\boldsymbol{\Delta}_d, \mathbf{p}_d})}{\tau^2(\mathbf{p}_d)}.$$

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$$\xrightarrow{D[0,1]} \left\{ \int_0^x g(t)\,dt - x \int_0^1 g(t)\,dt : 0 \leqslant x \leqslant 1 \right\} \qquad a.s.$$

Implications for change-point tests

Corollary (Asymptotic power one)

If $\sqrt{T}\,\mathcal{E}_1(\mathbf{\Delta}_d, \mathbf{p}_d) \to \infty$ *a.s., then for* $g(\cdot) \neq c$ *it holds*

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Corollary

For the AMOC-situation $g(x) = 1_{\{x>\vartheta\}}$, *the estimator*

$$\widehat{\vartheta}_T = \left\lfloor \frac{\arg\ max_k |U_{d,T}(k/T)|}{T} \right\rfloor$$

is consistent, i.e. $P\left(\left|\widehat{\vartheta}_T - \vartheta\right| \geqslant \epsilon \,|\, \mathbf{p}_d\right) \to 0 \qquad a.s.$

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$$\left\{ \frac{U_{d,T}(x)}{\tau(\mathbf{p}_d)} - s_d C_1 \left(\int_0^x g(t)\,dt - x \int_0^1 g(t)\,dt \right) : 0 \leqslant x \leqslant 1 \,|\, \mathbf{p}_d \right\} \xrightarrow{D[0,1]} \{B(x)\} \qquad a.s.,$$

where $s_d = \operatorname{sgn}(\mathbf{\Delta}_d^T \mathbf{p}_d)$.

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Proposition

If Σ is invertible, then

$$\mathcal{E}_1(\boldsymbol{\Delta}_d, \mathbf{p}_d) = \|\Sigma^{-1/2}\boldsymbol{\Delta}_d\| \cos(\alpha_{\Sigma^{-1/2}\boldsymbol{\Delta}_d, \Sigma^{1/2}\mathbf{p}_d}).$$

High-dimensional efficiency only depends on magnitude of change $\|\Sigma^{-1/2}\boldsymbol{\Delta}_d\|$ and angle between projection and change.

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Consider a random uniform projection $\mathbf{r}_d$ on the d-dimensional unit sphere and $\mathbf{r}_{\Sigma,d} = \Sigma^{-1/2}\mathbf{r}_d$.

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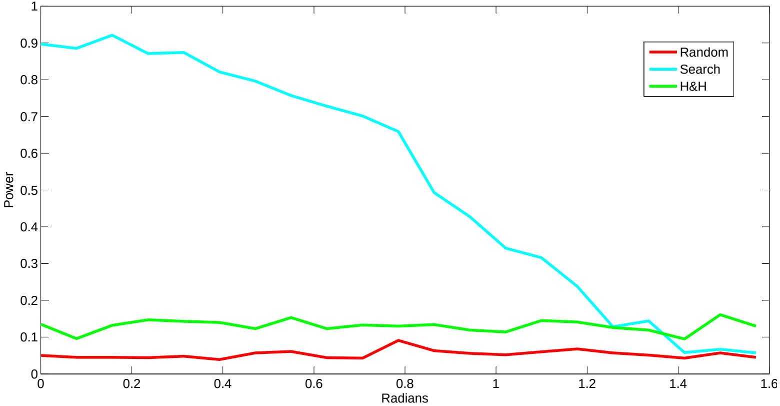
Empirical Power: Different Angles, $T = 100$, $d = 200$



$\mathcal{C}.1$: $s_j = 1$, i.e. $\Sigma = I_d$

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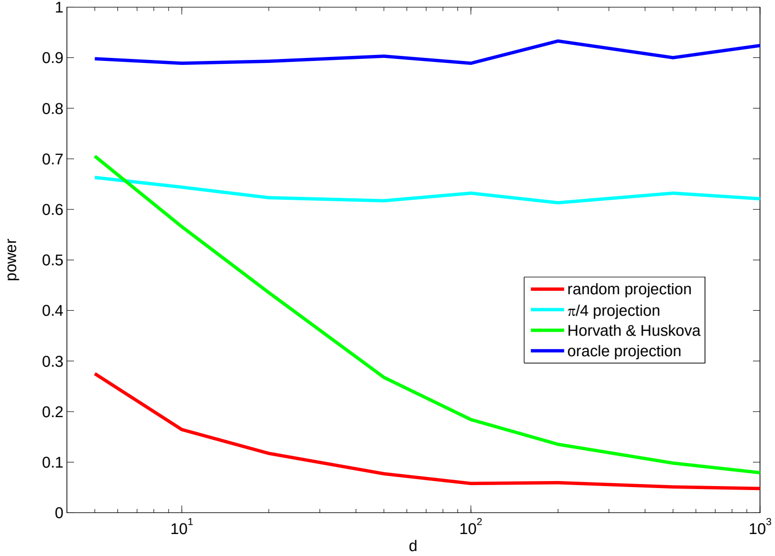

Misscaled Projections

In high dimensional settings: Covariance structure not known and not estimable!

Theorem

a) For a misscaled random projection $\mathbf{r}_{\mathbf{M},d} = \mathbf{M}^{-1/2}\mathbf{r}_d$:

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- The projection ${}_q\mathbf{o} = (\Delta_1/\sigma_1^2, \ldots, \Delta_d/\sigma_d^2)^T$ is called **quasi-oracle**, if $\sigma_j^2 > 0$, $j = 1, \ldots, d$.
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Remark:

It can happen, that all the efficiency of all three oracle projections is of the same order as for the random projection!

Example:

$\mathcal{C}.3$: Mixed dependence with common factor:

If $\boldsymbol{\Delta}_d \sim \boldsymbol{\Phi}$ projection maximizes not only the signal but also noise!

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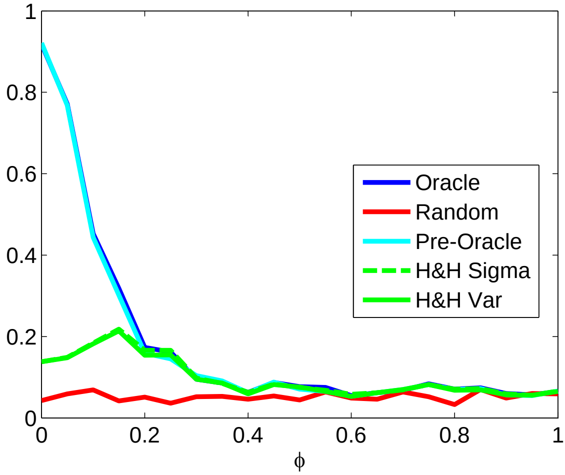

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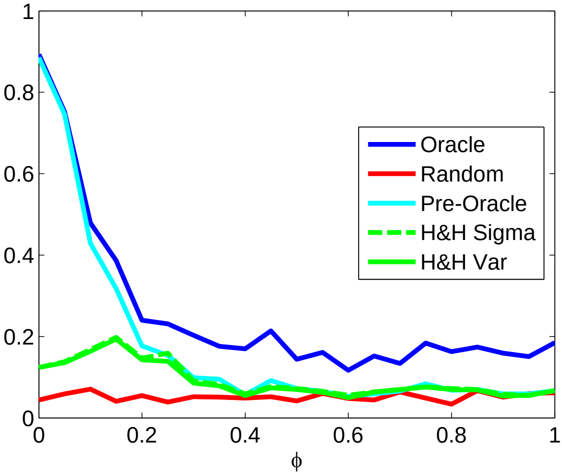

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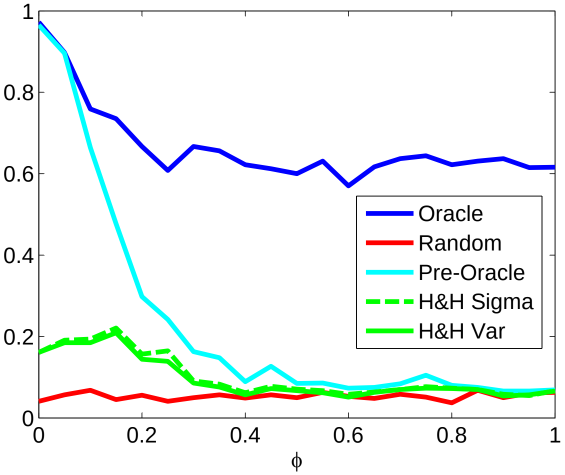

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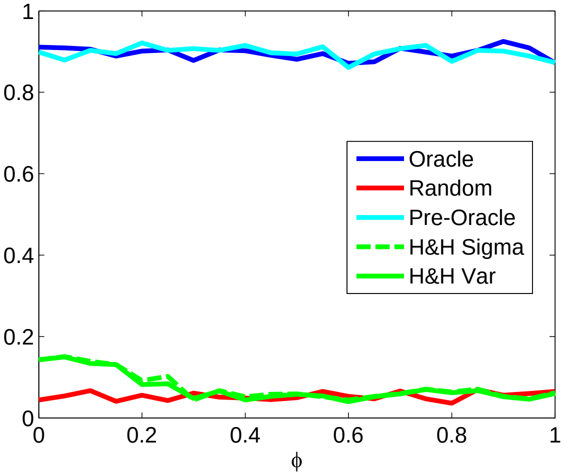

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Comparison with Panel-Data-Statistics

Null asymptotics for independent panels

Theorem (Horváth, Hušková (2012))

If the panels are independent, $\sigma_i^2 = \operatorname{var} e_{i,t} \geqslant c > 0$ for all i and $\mathrm{E}\,|e_{i,t}|^\nu \leqslant C < \infty$ for some $\nu > 4$ and $\frac{d}{T^2} \to 0$,
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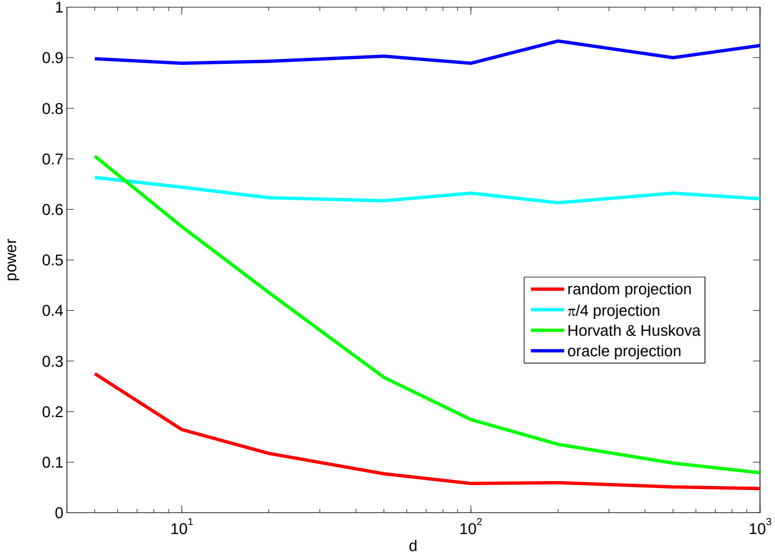

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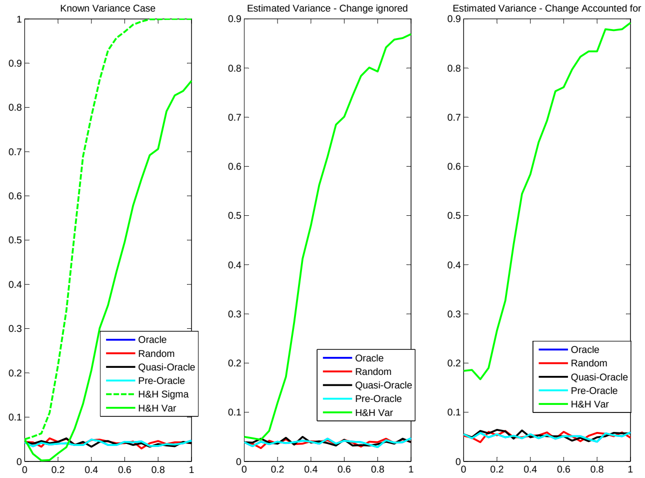

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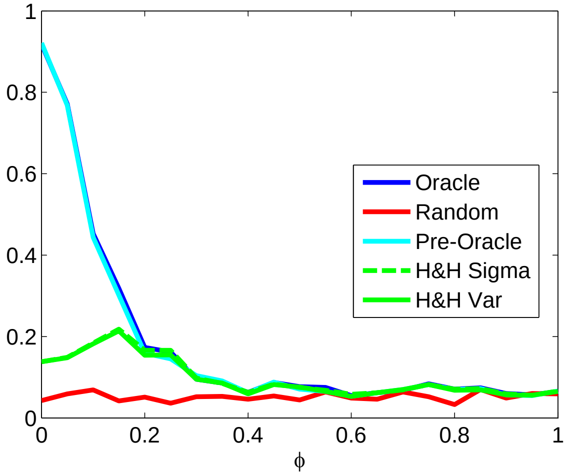

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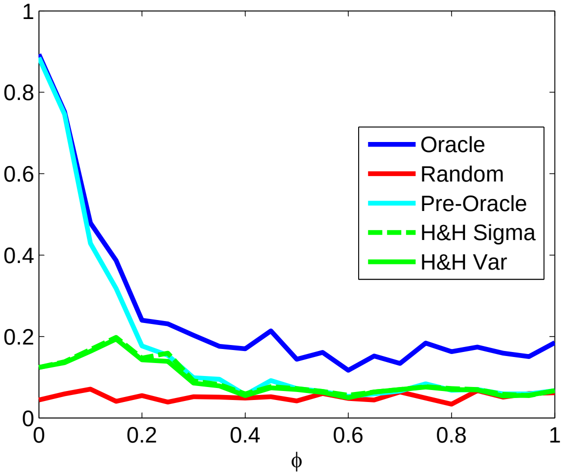

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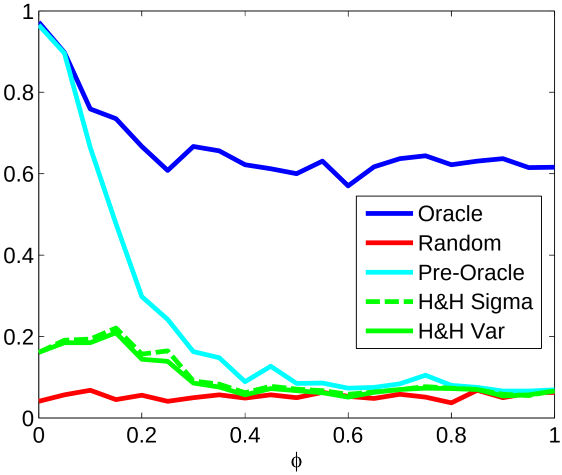
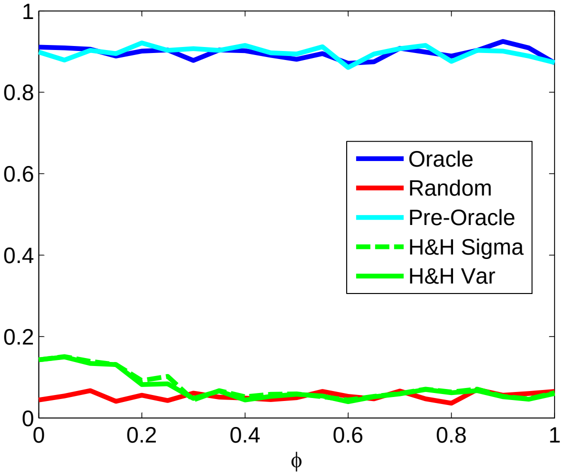


Angle between $\mathbf{\Delta}_d$ and $\mathbf{\Phi}_d$ is $\pi/4$ radians.

Empirical Power: $s_j = 1,\ T = 100,\ d = 200$



Angle between $\boldsymbol{\Delta}_d$ and $\boldsymbol{\Phi}_d$ is $\pi/2$ radians.

Outlook: Different projection methods

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- Multivariate change-point tests in lower dimensional space:
  Only information about subspace not actual directions used.

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For further reading:

Aston, Kirch
Change-points in high dimensional settings.
Preprint, 2014.

Horváth , Hušková
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Thank you very much for your attention!

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Thank you very much for your attention!