The Growth Model: Busemann Functions, Shape, Geodesics, and Other Stories

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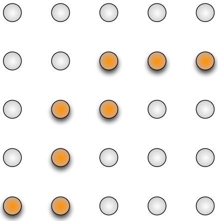
Lightning captured at 7,207 images per second (http://vimeo.com/28457062)

Random potential $\omega = (\omega_x)_{x \in \mathbb{Z}^2} \in \Omega$, $\mathbb{R}$-valued i.i.d., $2 + \varepsilon$ moments.

Up-right paths $x_{0,n} = (x_0, \ldots, x_n)$ take steps $e_1 = (1, 0)$ or $e_2 = (0, 1)$.



Passage time of path $x_{0,n}$ is $\sum_{i=0}^{n-1} \omega_{x_i}$.

Last Passage Percolation

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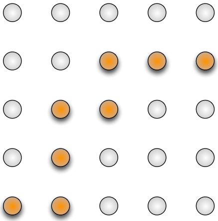

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Passage time of path $x_{0,n}$ is $\displaystyle\sum_{i=0}^{n-1} \omega_{x_i}$.

Point-to-point last passage time: $G_{x,y}(\omega) = \max\limits_{\substack{x_{0,n} \\ x_0=x, x_n=y}} \displaystyle\sum_{k=0}^{n-1} \omega_{x_k}$.

Connections to: Totally Asymmetric Simple Exclusion, Queuing Theory, Corner Growth Model, etc.

Shape Theorem

LLN says sum of i.i.d. grows linearly.

$G_{0,x}$ is not quite a sum of i.i.d.

It is however superadditive: $G_{x,y} + G_{y,z} \leq G_{x,z}$.

Then: outside one null set, for all $\xi \in \mathbb{R}_+^2$ and all $x_n/n \to \xi$ simultaneously

$g_{\mathrm{pp}}(\xi) = \lim_{n\to\infty} n^{-1} G_{0,x_n}$ exists, is deterministic, concave, homogenous ($g_{\mathrm{pp}}(c\xi) = cg_{\mathrm{pp}}(\xi)$) and continuous all the way to the boundary.

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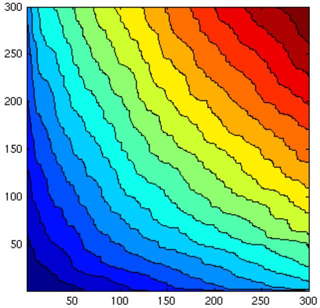

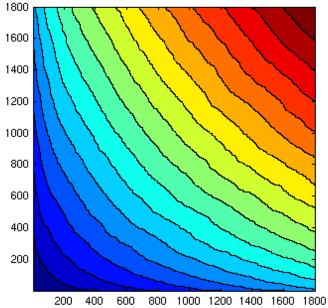

$\{x : G_{0,x} \leq t\}$

Geodesics

Path $x_{0,n}$ that maximizes $G_{x,y}$ is called a geodesic.

$x_{0,\infty}$ is a ξ-geodesic if $\forall n$ $x_{0,n}$ is a geodesic and $x_n/n \to \xi$.

Given $\xi \in \mathbb{R}^2_+$, is there an infinite ξ-geodesic?

Is it the limit of the geodesic from 0 to x_n as $n \to \infty$ and $x_n/n \to \xi$?

If ω_0 is continuous then finite geodesics are unique.

Is the infinite ξ-geodesic unique?

Do ξ-geodesics out of x and y coalesce (i.e. eventually merge)?

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Geodesics

Licea and Newman '96: answers are in the positive for standard first passage percolation (nearest-neighbor paths minimizing the passage time) if $g_{\text{pp}}(\xi)$ satisfies a global curvature assumption.

Problem: the curvature assumption has not been proved. Though conjectured to hold.

Damron and Hanson '14: Existence holds under just strict convexity or differentiability of g_{pp} (which presumably should be "easier" to prove).

Ferrari and Pimentel '05: answers are in the positive also for the last passage percolation model we are considering, but with ω_0 exponential.

The exponential model is one of the solvable models for which explicit computations are possible. In particular, an explicit formula is available for the shape $g_{\text{pp}}(\xi)$.

Would like to allow more general weight distributions.

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Understanding the shape

Consider a finite subset $V \subset \mathbb{Z}^2$ containing 0 (e.g. $\{u : |u| \le L\}$).

$\{G_{0,z_n-u} - G_{0,z_n} : u \in V\}$ describes the microscopic shape around z_n.

Expect this random vector to converge in distribution as $z_n/n \to \xi$.

Shifting by $-z_n$ and reflecting $\omega_x \mapsto \omega_{-x}$ turns the above into $\{G_{u,z_n} - G_{0,z_n} : u \in V\}$.

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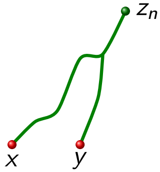
Busemann functions: $B^\xi(x, y; \omega) = \lim_{z_n/n \to \xi} (G_{x,z_n} - G_{y,z_n})$.

Limit exists if e.g. geodesics coalesce.



Understanding geodesics

Note that $G_{x,z_n} = \omega_x + \max(G_{x+e_1,z_n}, G_{x+e_2,z_n})$.

So $(G_{x,z_n} - G_{x+e_1,z_n}) \wedge (G_{x,z_n} - G_{x+e_2,z_n}) = \omega_x$ almost surely.

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$n \to \infty$ gives $B^\xi(x, x+e_1) \wedge B^\xi(x, x+e_2) = \omega_x$ almost surely.

The above suggests that ξ-geodesic out of x should follow the smallest $B^\xi(x, x+z)$, $z \in \{e_1, e_2\}$.

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Busemann functions

Consider g_{pp} as a concave function on $\mathcal{U} = \{(t, 1-t) : t \in (0,1)\}$.

Given $\xi \in \mathcal{U}$ let $[\underline{\xi}, \overline{\xi}] \subset \mathcal{U}$ be the maximal (possibly degenerate) interval containing ξ on which g_{pp} is linear.

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Standing assumptions: $\mathbb{P}\{\omega_0 \geq c\} = 1$, ω_x i.i.d. with $2 + \varepsilon$ moments, $\underline{\xi}$ and $\overline{\xi}$ are points of differentiability.

Theorem. $B^{\xi}(x, y; \omega) = \lim_{z_n/n \to \xi} (G_{x,z_n} - G_{y,z_n})$ exists **a.s.**

Furthermore: Same limit for all ξ in the same linear segment.

Corollary. If g_{pp} is differentiable, limits exist $\forall \xi$. (No convexity needed.)

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Remark. $\omega_0 \geq c$ only because we use results from queuing where service times were assumed nonnegative. All the queuing results seem to go through without this assumption.

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Properties

L^1: $\mathbb{E}[|B^\xi(x,y)|] < \infty$. (Comes from construction.)

Stationary: $B^\xi(x,y;T_z\omega) = B^\xi(x+z,y+z;\omega)$ $((T_z\omega)_x = \omega_{x+z})$

Cocycle: $B^\xi(x,y) + B^\xi(y,z) = B^\xi(x,z)$.

The space of L^1 stationary cocycles: $\mathscr{C}$.

Potential recovery: $B^\xi(0,e_1) \wedge B^\xi(0,e_2) = \omega_0$ almost surely.

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Geodesics

If $B \in \mathscr{C}$ then a *B-geodesic* is a path that follows the minimal $B(x, x+z)$, $z \in \{e_1, e_2\}$. (In case of ties, OK to go either way.)

Theorem. If B recovers potential ω ($B(0, e_1) \wedge B(0, e_2) = \omega_0$ a.s.) then a B-geodesic is a geodesic: every finite piece of it is a geodesic.

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Given $\xi \in \mathcal{U}$, recall the maximal linear segment $[\underline{\xi}, \overline{\xi}]$.

A geodesic $x_{0,\infty}$ is directed in $[\underline{\xi}, \overline{\xi}]$ if all limit points of x_n/n belong to this interval.

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A geodesic $x_{0,\infty}$ is directed in $[\underline{\xi}, \overline{\xi}]$ if all limit points of x_n/n belong to this interval.

Theorem.

a) Any B^{ξ}-geodesic is directed in $[\underline{\xi}, \overline{\xi}]$.

b) Any geodesic directed in $[\underline{\xi}, \overline{\xi}]$ is a B^{ξ}-geodesic.

c) The B^{ξ}-geodesic with e_2-tie breaks is the topmost of all geodesics directed in $[\underline{\xi}, \overline{\xi}]$. Similarly for rightmost.

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Geodesics

Corollary. If g_{pp} is differentiable everywhere, then every geodesic is directed in $[\underline{\xi}, \overline{\xi}]$ for some ξ.

Remark. Can also handle corners, but will omit.

Thus can show: If g_{pp} is strictly concave, then every geodesic has a direction ξ, i.e. $\lim x_n/n$ exists almost surely.

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Thus can show: If g_{pp} is strictly concave, then every geodesic has a direction ξ, i.e. $\lim x_n/n$ exists almost surely.

Theorem. Assume also $\mathbb{P}\{\omega_0 \le r\}$ is continuous in r. Then $\mathbb{P}\{B^\xi(0, e_1) = B^\xi(0, e_2)\} = 0$.

Corollary. If ω_0 is continuous, then there exists a unique geodesic directed in $[\underline{\xi}, \overline{\xi}]$ out of every point $x \in \mathbb{Z}^2$.

Theorem. Topmost $[\underline{\xi}, \overline{\xi}]$-directed geodesics coalesce, rightmost $[\underline{\xi}, \overline{\xi}]$-geodesics coalesce, and when ω_0 is continuous, $[\underline{\xi}, \overline{\xi}]$-geodesics coalesce.

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Variational formula

Until recently, the only description of $g_{pp}(\xi)$ was from superadditivity: $g_{pp}(\xi) = \sup_n n^{-1}\mathbb{E}[G_{0,[n\xi]}]$ (e.g. if $\xi \in \mathbb{Z}_+^2$).

Going through random polymer models:

Theorem. $g_{pp}(\xi) = \inf_{B\in\mathscr{C}} \operatorname{ess\,sup}\{\omega_0 - B(0,e_1;\omega) \wedge B(0,e_2;\omega) + \overline{B}\cdot\xi\}$.

($\overline{B} = (\mathbb{E}[B(0,e_1)], \mathbb{E}[B(0,e_2)])$ and $\mathscr{C}$ is class of L^1 stationary cocycles.)

Such formulas are important in statistical mechanics: their solutions are expected to describe the infinite-volume system (i.e. geodesics and shape as $n \to \infty$).

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Shape

Theorem. Under the standing assumptions, B^ξ solves the variational formula for $g_{pp}(\xi)$. In fact, the essential supremum is not needed and we have almost surely

$$g_{pp}(\xi) = \omega_0 - B^\xi(0, e_1, \omega) \wedge B^\xi(0, e_2, \omega) + \overline{B^\xi} \cdot \xi.$$

Corollary. Due to potential recovery, we have $g_{pp}(\xi) = \overline{B^\xi} \cdot \xi$.

Using some calculus one then gets that $\overline{B^\xi} = \nabla g_{pp}(\xi)$.

Nice interpretation: average microscopic gradient is macroscopic gradient.

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Solvable models

When ω_0 are exponential or geometric we in fact can calculate explicitly the distributions of $B^\xi(0, e_1)$ and $B^\xi(0, e_2)$ for all $\xi \in \mathcal{U}$.

For example, if ω_0 is exponential with rate 1, then $B^\xi(0, e_1)$ is exponential with rate α and $B^\xi(0, e_2)$ is exponential with rate $1 - \alpha$, where

$$\alpha = \frac{\sqrt{\xi_1}}{\sqrt{\xi \cdot e_1} + \sqrt{\xi \cdot e_2}}.$$

Then $\overline{B^\xi} = (\mathbb{E}[B^\xi(0, e_1)], \mathbb{E}[B^\xi(0, e_2)]) = (\frac{1}{\alpha}, \frac{1}{1-\alpha})$

and $g_{\mathrm{pp}}(\xi) = \overline{B^\xi} \cdot \xi = (\sqrt{\xi \cdot e_1} + \sqrt{\xi \cdot e_2})^2$.

This is the known formula derived by Rost '81.

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Fluctuations

CLT says that if $X_{0,n}$ has increments e_1 or e_2 equally likely, then it fluctuates from its average (straight line from 0 to $(n/2, n/2)$) by $n^{1/2}$.

Limit distribution of $(X_n - (n/2, n/2))/n^{1/2}$ is Gaussian.

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Say ω_0 is continuous.

What are the fluctuations of the geodesic from 0 to $[n\xi]$?

Conjecture: with enough moments on ω_0 geodesic has fluctuations of order $n^{2/3}$.

Superdiffusivity is because the path goes "out of its way" looking for high values of the potential.

On the other hand, $G_{0,[n\xi]}$ should have $n^{1/3}$ fluctuations.

Limit distributions related to Tracy-Widom from random matrices.

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Fluctuations

Models with these fluctuation exponents are said to belong to the Kardar-Parisi-Zhang (KPZ) universality class.

Johansson '00 proved LPP with exponential weights is in the KPZ class.

Again: solvability of the model was key.

Fluctuations

When ω_0 is exponential or geometric, $B^\xi(ne_1, (n+1)e_1)$ are i.i.d. and so are $B^\xi(ne_2, (n+1)e_2)$.

Balázs, Cator, and Seppäläinen '06 used this to prove the $n^{2/3}$ fluctuations of the geodesic and $n^{1/3}$ fluctuations of the last passage time, in the exponential weights case, with less technology than Johansson's proof of the Tracy-Widom limit.

More generally, CLT exponents for fluctuations of $B^\xi(0, ne_1)$ and $B^\xi(0, ne_2)$ imply information about fluctuation exponents of last passage quantities. (The above BCS result is one way to achieve this.)

Now we have a promising route to proving universality of KPZ fluctuations for general weight distributions.

Thank You

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