Variance of additive functionals of stationary processes and stationary Markov Chains

George Deligiannidis
(joint with M. Peligrad and S. Utev)

Department of Statistics, Oxford

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Outline

Introduction

Let $\{X_n\}_{n\in\mathbb{Z}}$, be stationary and

$$\mathrm{E}\, X_i = 0, \qquad \mathrm{E}\, X_i^2 < \infty.$$
$$S_n := X_1 + \cdots + X_n$$
$$\mathrm{cov}(X_0, X_k) = \int_{-\pi}^{\pi} e^{ikt} F(dt).$$

Limit theorems for S_n usually require information about $\mathrm{var}(S_n)$.

For example one may assume that $\mathrm{var}(S_n)/n \to K > 0$.

What is known?

If covariances summable, then $\mathrm{var}(S_n) \sim n \sum R(k)$.
Related to Césaro summability of Fourier series through

$$\mathrm{var}(S_n) = \int_{-\pi}^{\pi} \frac{\sin^2(nt/2)}{\sin^2(t/2)} F(dt).$$

By Fejer's theorem, if spectral density exists, i.e. $F(dt) = f(t)dt$ and is continuous at 0, then

$$\mathrm{var}(S_n) \sim 2\pi f(0) n.$$

Only need f to have right and left limits (if infinite not of opposite sign).
Also many more sufficient conditions with mixing, proj. criteria and so on (see Bradley (2007)).

First necessary and sufficient condition

Theorem 1 (Hardy & Littlewood (1924))

$$\frac{\mathrm{var}(S_n)}{n} \to A, \quad \textit{if and only if} \quad \frac{1}{2t}\int_{-t}^{t} f(s)ds \to A.$$

Of course Hardy and Littlewood were not interested in stationary processes but in Césaro means of Fourier series.

First appearance in probability in Bryc & Dembo (1995).

Non-linear growth

Often var$(S_n) \sim n^{\alpha} l(n)$ for any $\alpha \in (0, 2)$, l slowly varying, e.g. random walk in random scenery, linear processes, long-range dependence.

If $\alpha > 1$ then slowly decaying correlations.

Partial results link the behaviour of the correlation, or of the spectral density to that of the variance.

eg if the correlations $\rho_n = n^{-d} l(n)$, then

$$\mathrm{var}(S_n) \sim C l(n) n^{2-d}.$$

Many such results appear in Samorodnitsky (2006).

NASC for Stationary processes

Here's our first main result.
Assume F defines a symmetric spectral measure.

Theorem 2 (D. & Utev (2013))

Let $l(x)$ *be positive and slowly varying at infinity, and* $\alpha \in (0, 2)$. *Then*

$$\mathrm{var}(S_n) \sim K_0 n^{\alpha} l(n) \quad \textit{if and only if}$$
$$\int_{-x}^{x} F(dx) \sim C(\alpha) K_0 x^{2-\alpha} l(1/x)$$

where $C(\alpha) = \Gamma(1+\alpha)\sin(\frac{\alpha\pi}{2})/[\pi(2-\alpha)]$.

In particular $\mathrm{var}(S_n) \sim K_0 n$ *if and only if* $\int_{-x}^{x} F(dx) \sim K_0 x/\pi$.

Proof

We write

$$\mathrm{var}(S_n) = \int_{-\pi}^{\pi} \frac{\sin^2(\frac{nt}{2})}{\sin^2(\frac{t}{2})} F(dt) =: \int_0^{\pi} I_n(t) G(dt)$$

where $I_n(t)/n$ is the Fejer kernel and $G(x) = \int_{-x}^{x} F(dx)$.

Positivity of the kernel $I_n(y)$ leads to the following very useful bounds:

Lemma 3

For any $A > 0$

$$\frac{4}{\pi^2} n^2 G(1/n) \leqslant \mathrm{var}(S_n) \leqslant G(\pi) + \frac{\pi^2}{4} n^2 G(A/n) + \pi^2 \int_{A/n}^{\pi} \frac{G(y)}{y^3} dy.$$

eg for lower bound, since $I_n(y) \geqslant 4n^2/\pi^2$ for $0 < y < 1/n$

$$\mathrm{var}(S_n) = \int_0^{\pi} I_n(y) G(dy) \geqslant \int_0^{1/n} \frac{4}{\pi^2} n^2 G(dy) \geqslant \frac{4}{\pi^2} n^2 G(1/n).$$

Equivalence of upper bounds

This offers a first glimpse of a necessary and sufficient condition.
In fact upper bounds for the spectral measure are equivalent to upper bounds for the variance. Using Lemma 3 one can show

Lemma 4 (Equivalence of upper bounds)

For $L \geqslant 0$ *slowly varying, TFAE:*

(a) $\mathrm{var}(S_n) = O(n^\gamma L(n))$, *as* $n \to \infty$,

(b) $G(x) = O(x^{2-\gamma} L(1/x))$ *as* $x \to 0$.

What about lower bounds?

For lower bounds the situation is slightly more complicated as we still require an upper bound.

Lemma 5 (Equivalence of lower bounds)

If

$$G(x) = O(x^{2-\gamma}L(1/x)), \qquad \textit{and} \qquad \mathrm{var}(S_n) > C_1 n^{\gamma} L(n),$$

then for some $C_2 > 0$, *we have* $G(x) > C_2 x^{2-\gamma} L(1/x)$.

Proof of Theorem 2

We are now pretty much ready to prove the theorem.
"$\Rightarrow$:" Assume $G(x) \sim x^{2-\gamma} L(1/x)$.

Fix $M \leqslant n$ and change variables

$$\begin{aligned} \operatorname{var}(S_n) &= \int_0^M \frac{\sin^2(y)}{n^2 \sin^2(y/n)} n^2 G(2dy/n) \\ &\quad + \int_M^{n\pi/2} \frac{\sin^2(y)}{n^2 \sin^2(y/n)} n^2 G(2dy/n) \\ &=: I_{n,M} + J_{n,M}. \end{aligned}$$

Using the classical Tauberian theorem we can show that

$$\frac{J_{n,M}}{n^{\gamma}L(n)} = O(\frac{1}{n^{\gamma}L(n)}) + O(M^{-\gamma}),$$

and is thus negligible.

For $I_{n,M}$ we first use weak convergence.
On the interval $[0, M]$ define the sequence of (almost probability) measures

$$\mu_n\{[0, y)\} := \frac{n^{2-\gamma}G(2y/n)}{L(n)(2M)^{2-\gamma}}.$$

Since $G(x) \sim x^{2-\gamma}L(1/x)$ as $n \to \infty$

$$\mu_n\{[0, y)\} \to \left(\frac{y}{M}\right)^{2-\gamma} =: \mu(y)$$

and thus $\mu_n \Rightarrow \mu$ weakly.

On the interval $[0, M]$ we have

$$\frac{\sin(y)^2}{n^2 \sin(y/n)^2} = \frac{\sin^2(y)}{y^2} + O\Big(\frac{M^2}{n^2}\Big).$$

Further since $\sin^2(y)/y^2$ is cts and bdd weak convergence implies that

$$\begin{aligned}
\frac{I_{n,M}}{g(n)} &= 2^{2-\gamma}(2-\gamma)\int_0^M \frac{\sin^2(y)}{y^{1+\gamma}}dy + \mathcal{E}_M(n) + O(M^{-\gamma}) \\
&= 2^{2-\gamma}(2-\gamma)\int_0^\infty \frac{\sin^2(y)}{y^{1+\gamma}}dy + \mathcal{E}_M(n) + O(M^{-\gamma}), \\
&= \frac{\sin(\gamma\pi/2)\Gamma(1+\gamma)}{\pi(2-\gamma)} + \mathcal{E}_M(n) + O(M^{-\gamma}),
\end{aligned}$$

where $\mathcal{E}_M(n) \to 0$ as $n \to \infty$ for all M.

Let first $n \to \infty$ and then $M \to \infty$ to complete the proof of "$\Rightarrow$".

Proof of "$\Leftarrow$"

Now suppose that $\mathrm{var}(S_n)/n^\gamma L(n) \to K$.

For any increasing integer sequence t_j we can write

$$\frac{\mathrm{var}(S_{t_j})}{t_j^\gamma L(t_j)} = \int_0^M \frac{\sin^2(y)}{y^2} \frac{t_j^{2-\gamma} G(2dy/t_j)}{L(t_j)} + O\Big(\frac{M^2}{t_j^\gamma}\Big) + O(M^{-\gamma}),$$

By Lemmas 4 and 5

$$C_1 x^{2-\gamma} L(1/x) \leqslant G(x) \leqslant C_2 x^{2-\gamma} L(1/x),$$

for some $0 < C_1 < C_2$.

Thus for $y \leqslant M$

$$\frac{t_j^{2-\gamma} G(2M/t_j)}{L(t_j)} \leqslant CM^{2-\gamma} \frac{L(t_j/2M)}{L(t_j)} \leqslant CM^{2-\gamma}.$$

Therefore, by Helly's selection principle, we can find an increasing function h, defined on $[0, \infty)$ and a further subsequence $t_{j'}$ such that

$$F_{t_{j'}}(y) := \frac{t_{j'}^{2-\gamma} G(2y/t_{j'})}{L(t_{j'})} \to h(y), \tag{2.1}$$

at all continuity points of h.

From the bounds for G it must be that $h(y) \leqslant CM^{2-\gamma}$ for $y \leqslant M$ and since $\sin^2(y)/y^2$ cts and bdd on $[0, M]$ we have

$$\int_0^M \frac{\sin^2(y)}{y^2} F_{t_{j'}}(dy) \to \int_0^M \frac{\sin^2(y)}{y^2} h(dy).$$

By hypothesis $\mathrm{var}(S_n)/n^\gamma L(n) \to K$.
Therefore we have the identity for arbitrary M

$$K = \lim_{j'\to\infty} \frac{\mathrm{var}(S_{t_{j'}})}{t_{j'}^{\gamma} L(t_{j'})} = \int_0^M \frac{\sin^2(y)}{y^2} h(dy) + O(M^{-\gamma}),$$

and letting $M \to \infty$

$$K = \int_0^\infty \frac{\sin^2(y)}{y^2} h(dy).$$

At this stage h may depend on the subsequence t_j' chosen. To see why this is not actually the case we now exploit the regular variation of $\mathrm{var}(S_n)$.

Let $r > 0$ be arbitrary. Then by slow variation of L and a simple argument

$$F_{[rt_{j'}]}(y) := \frac{[rt_{j'}]^{2-\gamma} G(2y/[rt_{j'}])}{L([rt_{j'}])} \tag{2.2}$$

$$\sim r^{2-\gamma} \frac{t_{j'}^{2-\gamma} G\Big(2(y/r)/t_{j'}\Big)}{L(t_{j'})} \to r^{2-\gamma} h(y/r), \tag{2.3}$$

as $j' \to \infty$ at all good points y/r with h the same as before.

Since $\mathrm{var}(S_n)/g(n) \to K$, for any $r > 0$

$$K = \lim_{j' \to \infty} \frac{\mathrm{var}(S_{t_{j'}})}{g(t_{j'})} = \int_0^\infty \frac{\sin^2(y)}{y^2} r^{2-\gamma} h(dy/r).$$

A convolution equation

Combining all steps so far we arrive at the “convolution type equation"

$$\int_0^\infty \frac{\sin^2(ry)}{y^2} h(dy) = r^\gamma K, \tag{2.4}$$

where h may depend on the particular subsequence t_j'.

In the rest of the proof we show that the solution h must be unique, and thus cannot depend on the subsequence.

Solving the convolution equation

First define the auxiliary odd function, defined by

$$\psi(y) := \lim_{N\to\infty} \int_y^N x^{-2} h(dx), \qquad \psi(-y) := -\psi(y), \qquad y > 0.$$

Equation 2.4 allows us to compute the sine-transform of ψ as

$$\lim_{a\to\infty} \int_{-a}^{a} \sin(ry)\psi(y)dy = 2^{2-\gamma}\text{sgn}(r)|r|^{\gamma-1}K.$$

Unfortunately $\psi(y)$ behaves like $y^{-\gamma}$ so depending on γ this may not be L^1 or L^2 (eg for $\gamma = 1$). So parse ψ as a tempered distribution

$$\Psi[\phi] := \int_0^\infty \psi(y)(\phi(y) - \phi(-y))dy,$$

for Schwartz functions ϕ.

Again careful analysis to compute the Fourier transform of Ψ using the identity $\hat{\Psi}[\phi] = \Psi[\phi]$ and we find

$$\hat{\Psi}[\phi] = \int_{-\infty}^{\infty} \left(i2^{2-\gamma} K \operatorname{sgn}(t)|t|^{\gamma-1} \right) \phi(t) dt.$$

Fourier inversion allows us to identify ψ to be $KD(\gamma)y^{-\gamma}$, and therefore

$$h(x) = \frac{\gamma}{2-\gamma} KD(\gamma)x^{2-\gamma}, \quad D(\gamma) := \Gamma(\gamma)2^{2-\gamma}\frac{\sin(\gamma\pi/2)}{\pi}.$$

From this one can deduce that

$$\lim_{x\to 0} \frac{G(x)}{x^{2-\gamma}} L(1/x) = C(\gamma)K. \quad \square$$

Motivating question

This was motivated by a question by M. Peligrad:

Assume $\mathrm{var}(S_n)/n \to K$ along the subsequence $n_r = 2^r$. Does this imply convergence along the full sequence?

The answer is no!

Let $G(x) = 2^{-k}$, for $x \in (2^{-(k+1)}, 2^{-k}]$, for $k \geqslant 1$.

Then $\lim_{x\to 0} G(x)/x$ does not exist, as different subsequences give different limits, and therefore the limit of the full sequence $\mathrm{var}(S_n)/n$ cannot exist.
By direct calculation on the subsequence 2^r,

$$\frac{\mathrm{var}(S_{2^r})}{2^r} \to \sum_{k=0}^{\infty} \frac{\sin^2(2^k)}{2^k} + \sum_{k=1}^{\infty} 2^k \sin^2(2^{-k}) \in (0, \infty).$$

Functionals of Stationary Markov Chains

Suppose now that $X_n = g(\xi_n)$, where

- $(\xi_n)_{n\in\mathbb{Z}}$ stationary ergodic Markov chain with values in $(S, \mathcal{A})$,
- marginal π and transition kernel $Q(x, dy)$.
- $g \in \mathbf{L}^2(S, \pi)$ such that $\pi(g) = 0$.

Q also denotes the Markov transition operator

$$(Qg)(x) := \int_S g(s)Q(x, ds).$$

The chain is *reversible* iff Q is self-adjoint.

The chain will be called *normal* if Q is normal, ie $Q^*Q = QQ^*$.

Transition Spectral measure

In the context of Markov chains we distinguish between two different spectral measures.
The *transition operator* Q, acts on $\mathbf{L}^2(S, \pi)$, where S is the state space and π the stationary measure.

Assuming $Q^*Q = QQ^*$ by the spectral theorem there is a unique *transition spectral measure* ν, supported on the unit disc, such that

$$\mathrm{cov}(X_0, X_n) = \langle g, Q^n g\rangle = \int_D z^n \nu(dz). \tag{3.1}$$

where $D := \{z \in \mathbb{C} : |z| \leqslant 1\}$.

In the reversible case ν is concentrated on $[-1, 1]$.

Shift Spectral measure

The *shift operator* U acts on $\mathbf{L}^2(\Omega, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical probability space on which the MC is defined.

The spectral theorem applied to the unitary operator U implies the existence of a unique *shift spectral measure* F on $[-\pi, \pi]$ (or S^1) such that

$$\text{cov}(X_0, X_n) = \int_{-\pi}^{\pi} e^{int} F(d\, t).$$

Thus we can use Theorem 2 to analyse the variance in terms of the *shift spectral measure*.
Can we also analyse it in terms of the transition spectral measure?

What is known?

Kipnis & Varadhan (1986): $\sqrt{n}$-CLT for reversible MC if

$$\lim_{n\to\infty} \frac{\mathrm{var}(S_n)}{n} = \sigma_g^2, \quad \text{iff} \quad \sigma_g^2 := \int_{-1}^{1} \frac{1+t}{1-t}\nu(dt) < \infty. \tag{3.2}$$

Gordin & Lifšic (1981): $\sqrt{n}$-CLT for normal MC under

$$\int_D \frac{1}{|1-z|}\nu(dz) < \infty, \tag{3.3}$$

and if (3.3) then $\dfrac{\mathrm{var}(S_n)}{n} \to \sigma^2 := \displaystyle\int_D \frac{1-|z|^2}{|1-z|^2}\nu(dz).$

See also Tóth (1986); Derriennic & Lin (2001); Holzmann (2005).

Recently Zhao, Woodroofe & Volny (2010) and Longla, Peligrad & Peligrad (2012), studied reversible MC such that $\mathrm{var}(S_n) = nl(n)$.

We will address the following two issues:

(i) How are the two spectral measures related?

(ii) Can we get necessary and sufficient conditions for $\text{var}(S_n) \sim Cn^{\alpha} l(n)$ in terms of the *transition spectral measure* ν?

(iii) For reversible: is it true $\text{var}(S_n) \sim Cn$ if and only if $\nu\{(1-x, 1)\} \sim C'x$?

Question 3.1 (Open?)

Is the Kipnis-Varadhan condition necessary for a $\sqrt{n}$-CLT in the reversible case?

Transition vs shift spectral measure

Let ∂D and D_0 be the boundary and the interior of the unit disc.

Theorem 6

The shift spectral measure has the representation

$$F(dt) = \nu|_{\partial D}(dt) + f(t)dt, \quad \textit{where}$$
$$f(t) = \frac{1}{2\pi}\int_{D_0} \frac{1-|z|^2}{|1-ze^{it}|^2}\nu_0(dz). \tag{3.4}$$

In other words if ν is supported on D_0, the spectral density exists.

See also Häggström & Rosenthal (2007); Jewel & Bloomfield (1983); Derriennic & Lin (2001).

In reversible case f is simpler

$$f(t) = \frac{1}{2\pi}\int_{-1}^{1} \frac{1-\lambda^2}{1+\lambda^2-2\lambda\cos t}d\nu(\lambda)$$

Proof of Theorem 6.

For $t \in [-\pi, \pi]$ let

$$f(t) := \frac{1}{2\pi} \int_{D_0} \left[1 + \sum_{k=1}^{\infty} (z^k e^{itk} + \bar{z}^k e^{-itk}\right] \nu(dz)$$
$$= \frac{1}{2\pi} \int_{D_0} \frac{1 - |z|^2}{|1 - ze^{it}|^2} \nu(dz).$$

An easy calculation shows that

$$\int_0^{2\pi} e^{ikt} f(t) dt = \int_{D_0} z^k \nu(dz),$$

and the result follows easily. □

Link through Brownian motion I

The presence of the Poisson kernel hints at a link with harmonic measure.

Let $(B_t^z)_{t \geqslant 0}$ be standard planar Brownian motion in $\mathbb{C}$, started at the point z.

Let $Z \sim \nu$ be a random point in $D := \{z : |z| \leqslant 1\}$ distributed according to ν(normalized);
let $\tau_D^Z := \inf\{t > 0 : B_t^Z \notin D\}$. Let $\alpha \in (0, 2)$.

Theorem 7

The shift spectral measure can be expressed as the harmonic measure of Brownian motion in the disc started from ν. That is for any Borel $A \subset [-\pi, \pi]$

$$F(A) = P\{\arg(B^Z_{\tau_D^Z}) \in A\} = \int_D P\{\arg(B^z_{\tau_D}) \in A\}\nu(dz).$$

Can we find NASC?

Having obtained the link between the two spectral measures we look for necessary and sufficient conditions for $\mathrm{var}(S_n)$ to be regularly varying in terms of the *transition* spectral measure.

We begin with reversible Markov chains.

The asymptotically linear case is known since Kipnis & Varadhan (1986):

$$\lim \frac{\mathrm{var}(S_n)}{n} = \int_{-1}^{1} \frac{1+t}{1-t} \nu(dt).$$

NASC for reversible MCs

Proposition 8

Assume that Q *is self-adjoint and that* ν *has no atoms at* ± 1. *Then the shift spectral measure* F *is absolutely continuous and the following relations are equivalent. Let* $\alpha \in \geqslant 1$

1. $\text{var}(S_n) \sim n^{\alpha} l(n)$ *as* $n \to \infty$;
2. $\int_{-1}^{1-x} \frac{1}{1-t} \nu(dt) \sim \frac{\alpha(\alpha-1)}{2\Gamma(3-\alpha)} x^{1-\alpha} l(\frac{1}{x})$ *and* $\alpha \geqslant 1$.
3. $\nu(1-x, 1] \sim \dfrac{\alpha(\alpha-1)}{2\Gamma(3-\alpha)} x^{2-\alpha} l(1/x)$ *as* $x \to 0_+$, *and* $\alpha > 1$.

Remark 9

So if $\alpha = 1$ ν does not have to be regularly varying at 1.

Proof

Assume $\alpha \in [1,2)$ and in particular that $\mathrm{var}(S_n)/n \to \infty$.
Let $C_1(n) := \sum_{i=0}^{n-1} \int_0^1 x^i \nu(dx)$. Then it is easy to see that

$$\mathrm{var}(S_n) \sim 2 \sum_{k=1}^{n} C_1(k).$$

Since $C_1(k)$ is increasing by the Tauberian theorem

$$\mathrm{var}(S_n) \sim n^{\gamma} L(n), \qquad \text{iff} \qquad C_1(n) \sim \frac{\gamma}{2} n^{\gamma-1} L(n).$$

Since

$$C_1(n) = \int_0^1 \frac{1-x^n}{1-x} \nu(dx),$$

the result follows after integration by parts and change of variables.

Example for non $\sqrt{n}$-CLT's I

We next give an example of a Metropolis-Hastings type chain that satisfies a CLT with super-diffusive normaliser.

Example 10

Let $E = \{x : |x| \leqslant 1\}$ and ν a symmetric probability measure on E such that

$$\theta := \int_{-1}^{1} \frac{\nu(dx)}{1 - |x|} < \infty.$$

Define the transition kernel

$$Q(x, A) = |x|\delta_x(A) + (1 - |x|)\nu(A).$$

So if the current state is x you propose from $\nu(\cdot)$ and you accept with probability $1 - |x|$.

Example for non $\sqrt{n}$-CLT's II

This kernel defines a Markov chain $\{\xi_k\}$, which is reversible wrt

$$\mu(dx) = \frac{\nu(dx)}{\theta(1-|x|)}.$$

For any odd g

$$Q^k g(x) = |x|^k g(x),$$

and thus letting $g(x) = \mathrm{sgn}(x)$ we have

$$(g, Q^k g) = \int_{-1}^{1} |x|^k \mu(dx) = 2\int_0^1 x^k \mu(dx),$$

and thus the transition spectral measure is given by 2μ and is supported on $[0, 1]$. Define

Example for non $\sqrt{n}$-CLT's III

$$V(x) := \int_0^{1-x} \frac{\nu(dy)}{1-y} \sim \frac{1}{2} h(\frac{1}{x}).$$

We have the following result

Theorem 11

If $V(x)$ is slowly varying at 0, then

$$\frac{1}{nh(n)} \sum_{i=1}^{n} \mathrm{sgn}(\xi_i) \Rightarrow N(0,1).$$

Normal MC's

Having covered reversible, let us now have a look at the normal case. Using representation in terms of harmonic measure and Theorem 2 we get the following result for free.

Theorem 12

The following statements are equivalent:

(i) $\operatorname{var}(S_n) \sim n^{\alpha} l(n)$ *as* $n \to \infty$;

(ii) $P\left\{ B^{Z}_{\tau^{Z}_{D}} \in (-x, x) \right\} \sim C(\alpha) x^{2-\alpha} l(1/x)/\nu(D)$ *as* $x \to 0$.

Normal MC's

Define

$$\sigma^2 = \int_{\mathbb{D}} \frac{1-|z|^2}{|1-z|^2} \nu(dz). \tag{3.5}$$

The following result clarifies the linear case.

Theorem 13

Assume $\liminf_{n\to\infty} \text{var}(S_n)/n > 0$. *Then the following are equivalent*

(a) $\text{var}(S_n)/n \to K < \infty$;

(b) $\sigma^2 < \infty$, *and* $\nu(U_x)/x \to (K - \sigma^2)/\pi$,

where

$$U_x = \{z = (1-r)e^{iu} \in \mathbb{D} \;:\; 0 \leqslant r \leqslant |u| \leqslant x\}.$$

U_x appears in Cuny & Lin (2009).

Sketch of Proof I

Proof begins with martingale decomposition

$$S_n = \mathbb{E}_0(S_n) + \sum_{i=1}^{n} \mathbb{E}_i(S_n - S_{i-1}) - \mathbb{E}_{i-1}(S_n - S_{i-1}).$$

and its spectral representation

$$\mathrm{var}(S_n) = \int_D \frac{|1 - z^n|^2}{|1 - z|^2} \nu(dz) + \sum_{j=1}^{n} \int_{D_0} \frac{|1 - z^j|^2 (1 - |z|^2)}{|1 - z|^2} \nu(dz) + O(1).$$

The second term easily results in the $\sigma^2 < \infty$ condition.

Sketch of Proof II

First term is essentially $\mathsf{E}\left[\mathsf{E}_0(S_n)^2\right]$.
The proof consists in several approximation steps that essentially show that

$$\frac{1}{n}\mathsf{E}\left[\mathsf{E}_0(S_n)^2\right] = \int_0^{\pi} \frac{\sin^2(nt/2)}{\sin^2(t/2)} G(dt) + o(1),$$

where the distribution function G is given by $G(x) := \nu(U_x)$, with

$$U_x = \{z = (1-r)e^{iu} \in D \ : \ 0 \leqslant r \leqslant |u| \leqslant x\}.$$

Essentially we sweep the measure ν out towards the boundary of D.
Then we can apply Theorem 2 to the measure $\tilde{F}(x) = \nu(U_x)$ to complete the result.

In fact by looking at the proof it can be easily seen that in the super linear case we can say more.

Corollary 14

Assume $\sigma^2 < \infty$ *and* $\alpha \geqslant 1$. *Then with* $C(\alpha)$ *as defined in Theorem 2.*

$$\mathrm{var}(S_n) \sim n^{\alpha} l(n), \text{ as } n \to \infty, \quad \textit{iff} \quad \nu(U_x) \sim C(\alpha) x^{2-\alpha} l(1/x) \text{ as } x \to 0^+.$$

Continuous time

Stationary Markov process $\{\xi_t\}_{t\geqslant 0}$, with values in $(S, \mathcal{A})$;
for $g \in L^2_0(\pi)$ let $T_t g(x) := \mathbb{E}[g(\xi_t)|\xi_0 = x]$.
$T_t = e^{Lt}$, where L is assumed normal, so that spectrum is supported on $\{z \in \mathbb{C} : \Re(z) \leqslant 0\}$, such that

$$\mathrm{cov}(f(\xi_t), f(\xi_0)) = \int_{\Re(z)\leqslant 0} e^{zt}\nu(dz).$$

Finally define

$$S_T(g) := \int_{s=0}^{T} g(\xi_s)ds.$$

Again there is also a *shift spectral measure* F on $(-\infty, \infty)$ such that

$$\mathrm{cov}(f(\xi_0), f(\xi_t)) = \int_{-\infty}^{\infty} e^{iut}F(du).$$

Link between spectral measures

Write $(B_t^z)_{t\geqslant 0}$ for a standard planar Brownian motion in $\mathbb{C}$, started at the point $z \in \mathbb{H}^- := \{z \in \mathbb{C} : \mathfrak{R}(z) \leqslant 0\}$.
Let $Z \sim \nu$ be a random point in $\mathbb{H}^-$ and

$$\tau_{\mathbb{H}^-}^Z := \inf\{t \geqslant 0 : B_t^Z \notin \mathbb{H}^-\}.$$

Then for $A \in \mathfrak{B}(\mathbb{R})$ the shift spectral measure is can be expressed as

$$F(A) = P(B^Z(\tau_{\mathbb{H}}^Z) \in A).$$

Theorem 15

For $\alpha \in (0,2)$ *and* L *slowly varying the following are equivalent:*

(a) $\mathrm{var}(S_T) \sim T^\alpha L(T)$,

(b) $P\{B_{\tau_{\mathbb{H}^-}^Z}^Z \in (-ix, ix)\} \sim C(\alpha) x^{2-\alpha} L(1/x)/\nu(\mathbb{H}^-)$.

Necessary and sufficient conditions

Theorem 16

Let

$$\sigma^2 := -2 \int_{\mathbb{H}^-} \Re(1/z)\nu(dz) < \infty.$$

The following are equivalent:

(i) $\mathrm{var}(S_T(g))/T \to L = \sigma^2 + K$, *where* $K > 0$*;*

(ii) $\sigma^2 < \infty$ *and* $\nu(U_x)/x \to K/\pi$ *as* $x \to 0^+$*.*

In addition, if $\sigma^2 < \infty$, $\liminf_{T\to\infty} \mathrm{var}(S_T)/T = \infty$, *and* $\alpha \geqslant 1$ *then*

$$\mathrm{var}(S_T) \sim T^\alpha h(T) \quad \textit{iff} \quad \nu(U_x) \sim C(\alpha)x^{2-\alpha}h(1/x).$$

Thank you for your attention!

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