Palm Theory and Shift-Coupling

Hermann Thorisson
University of Iceland

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Mass-Stationarity of ξ

Let ξ be a random measure on $\mathbb{R}$ with $\xi(-\infty, 0] = \xi[0, \infty) = \infty$.

Write θ_t for the shift map: $\theta_t\xi = \xi(t + \cdot)$, $t \in \mathbb{R}$.

Recall that ξ is stationary if $\theta_t\xi \overset{D}{=} \xi$, $t \in \mathbb{R}$.

Definition for a simple point process ξ.

Put $T_0 = 0$ and for integers $n > 0$

$$T_n = \sup\{t > 0 : \xi[0, t) = n\}, \qquad T_{-n} = \sup\{t < 0 : \xi[t, 0) = n\}.$$

Call ξ mass-stationary if

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Definition for a diffuse random measure ξ : $\xi(\{t\}) = 0, \ t \in \mathbb{R}$.

Put $T_0 = 0$ and for real $r > 0$

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Further, let X be a random element in a space on which $\mathbb{R}$ acts.

For instance X could be a shift-measurable stochastic process $X = (X_s)_{s \in \mathbb{R}}$ and $\theta_t X = (X_{t+s})_{s \in \mathbb{R}}$. Write $\theta_t(X, \xi) = (\theta_t X, \theta_t \xi)$.

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Definition for a diffuse random measure ξ: $\xi(\{t\}) = 0$, $t \in \mathbb{R}$.

Call (X, ξ) mass-stationary if

$$\theta_{T_r}(X, \xi) \stackrel{D}{=} (X, \xi), \quad r \in \mathbb{R}.$$

Brownian motion is mass-stationary

Let $B = (B_s)_{s\in\mathbb{R}}$ be a two-sided standard Brownian motion. In particular, $B_0 = 0$ a.s.

The (diffuse) local time measure ℓ^x at $x \in \mathbb{R}$ can be defined by

$$\ell^x(A) := \lim_{h\to 0} \frac{1}{h} \int_A 1_{\{x \le B_s \le x+h\}} ds, \quad A \in \mathcal{B}.$$

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Theorem

The pair (B, ℓ^0) is mass-stationary: $\theta_{T_r} B \stackrel{D}{=} B$ for all $r \in \mathbb{R}$.

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When traveling in time according to the clock of local time at 0 we always see globally a two-sided Brownian motion.

Why does T_r work? — Some Palm theory

A pair (X, ξ) defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Palm version of a stationary $(\hat{X}, \hat{\xi})$ defined on some $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, if for each measurable $f \geq 0$ and each $A \in \mathcal{B}(\mathbb{R})$ with $0 < \lambda(A) < \infty$,

$$\mathbb{E}[f(X, \xi)] = \hat{\mathbb{E}}\Big[\int_A f(\theta_t(\hat{X}, \hat{\xi}))\hat{\xi}(dt)\Big]\Big/\lambda(A).$$

Here (X, ξ) and $(\hat{X}, \hat{\xi})$ are allowed to have distributions that are only σ-finite and not necessarily probability measures. The measure $\mathbb{P}$ is finite if and only if $\hat{\xi}$ has finite intensity.

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Theorem: Let ξ be diffuse. Then

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The key to the proof is that π_r is preserving in the sense that $\xi(\tau_{\pi_r} \in \cdot) = \xi$ where $\tau_{\pi_r}(s) = s + \pi_r(\theta_s\xi)$ for $s \in \mathbb{R}$.

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The pair (B, ℓ^0) is Palm version of the stationary $(\hat{B}, \hat{\ell}^0)$ where $\hat{B}$ has the distribution $\int_{\mathbb{R}} \mathbb{P}(x + B \in \cdot)dx$ (which is σ-finite).

Shift-coupling B and $x + B$?

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Question (*unbiased* two-sided Skorohod imbedding of x ?)

Is there a T such that $\theta_T B \overset{D}{=} x + B$ for $x \neq 0$?

That is: a T such that $\theta_T B - x$ is standard Brownian.

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Example (of a T that does NOT work)

Let $T = T_x$ be the hitting time of an $x \neq 0$

$$T_x := \inf\{t \geq 0 \colon B_t = x\}.$$

Then $(B_{T+s})_{s \geq 0} - x$ is one-sided standard Brownian
and $(B_{T+s})_{s \geq 0} - x$ is independent of $(B_{T-s})_{s \geq 0} - x$
but $(B_{T-s})_{s \geq 0} - x$ is NOT one-sided standard Brownian
(note that for all $s > 0$ small enough, $B_{T-s} - x \neq 0$).

Unbiased two-sided Skorohod imbedding of ν ?

Say that $x + B$ is two-sided Brownian with value x at 0.

More generally, say that a process $B' = (B'_s)_{s\in\mathbb{R}}$
is two-sided Brownian with distribution ν at 0
if B'_0 has distribution ν, and B'_0 is independent of $B' - B'_0$,
and $B' - B'_0$ is a two-sided standard Brownian motion.

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Example (of another T that does NOT work)

Consider $T \equiv t$ where $t \neq 0$. Let ν be the distribution of B_t.
Put $B' = \theta_t B$. Then $B'_0 = B_t$ has distribution ν and
$B' - B'_0 = \theta_t B - B_t$ is a two-sided standard Brownian motion.
But B'_0 is NOT independent of $B' - B'_0$ since $B'_{-t} - B'_0 = -B'_0$.

A time T^ν that works for unbiased imbedding

For $x \in \mathbb{R}$ define $T^x = \inf\{t > 0 \colon \ell^0([0,t]) = \ell^x([0,t])\}$.

Theorem

If $x \neq 0$ then $\theta_{T^x} B$ is two-sided Brownian with value x at 0:

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Times T^x and T^ν that work for unbiased imbedding

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Theorem

If $x \neq 0$ then $\theta_{T^x} B$ is two-sided Brownian with value x at 0:

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For a probability measure ν on $\mathbb{R}$ define the local time at ν by

$$\ell^\nu = \int \ell^x \, \nu(dx)$$

and set

$$T^\nu := \inf\{t > 0 \colon \ell^0([0,t]) = \ell^\nu([0,t])\}.$$

Theorem (*unbiased* two-sided Skorohod imbedding)

If $\nu(\{0\}) = 0$

then $\theta_{T^\nu} B$ is two-sided Brownian with distribution ν at 0.

Why does T^ν work? — Shift-coupling Palm versions

Let $\hat{X}$ be stationary ergodic and the measures $\hat{\xi}$ and $\hat{\eta}$ invariant
i.e. $\exists$ measurable maps $f, g : \theta_t\hat{\xi} = f(\theta_t\hat{X}), \theta_t\hat{\eta} = g(\theta_t\hat{X}), t \in \mathbb{R}$.
Let $\hat{\xi}$ and $\hat{\eta}$ have the same finite intensity.

The above conditions hold for $(\hat{X}, \hat{\xi}, \hat{\eta}) = (\hat{B}, \hat{\ell}^0, \hat{\ell}^\nu)$.

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Theorem ($\hat{X}$ stationary ergodic, $\hat{\xi}$ and $\hat{\eta}$ invariant, intensity$<\infty$)

If (X, ξ) is Palm version of $(\hat{X}, \hat{\xi})$ and (X', η') of $(\hat{X}, \hat{\eta})$ then

$$\theta_{\pi(X)}X \overset{D}{=} X' \iff \pi \text{ balances } \xi \text{ and } \eta$$

Further, if ξ is diffuse and ξ and η are mutually singular then

$$\pi(X) := \inf\{t > 0 : \xi([0, t]) = \eta([0, t])\} \quad \text{balances } \xi \text{ and } \eta$$

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Recall that $T^\nu = \inf\{t > 0 : \ell^0([0, t]) = \ell^\nu([0, t])\}$. Note also that if $\nu(\{0\}) = 0$ then the diffuse $\hat{\ell}^0$ and ℓ^ν are mutually singular.

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In general, if B' is two-sided Brownian with distribution ν at 0 then $(B', \hat{\ell}'^\nu)$ is Palm version of the stationary $(\hat{B}, \hat{\ell}^\nu)$.

Take $(\hat{X}, \hat{\xi}, \hat{\eta}) = (\hat{B}, \hat{\ell}^0, \hat{\ell}^\nu)$ and $\nu(\{0\}) = 0$ to obtain $\theta_{T^\nu} B \overset{D}{=} B'$.

The Brownian Bridge

The Slepian process $(B_{s+1} - B_s)_{s\in\mathbb{R}}$ is stationary ergodic.

This process has a local-time-at-zero measure, denote it η.
Set $X_s = (B_{s+u} - B_s)_{0\le u\le 1}$ and $X = (X_s)_{s\in\mathbb{R}}$.
With (X', η') Palm version of (X, η), X'_0 is a Brownian bridge.

Let ξ be Lebesgue measure, $\xi = \lambda$.
Since X is stationary, (X, ξ) is Palm version of itself.

The measures ξ and η are diffuse and mutually singular.

So the conditions of the shift-coupling are satisfied. Set

$$T = \inf\{t > 0 \colon \eta([0, t]) = t\}$$

to obtain $\theta_T X \overset{D}{=} X'$.

Thus $X_T \overset{D}{=} X'_0$, that is, $(B_{T+u} - B_T)_{0\le u\le 1}$ is a Brownian bridge.

Mass-Stationarity — General random measures on $\mathbb{R}$

Setting

Let ξ be a random measure on $\mathbb{R}$.

Let X be a random element in a space on which $\mathbb{R}$ acts.

Write θ_t for the shift map placing a new origin at $t \in \mathbb{R}$.

Definition (extended to cover all random measures on $\mathbb{R}$)

Call (X, ξ) mass-stationary if for all bounded λ-continuity sets $C \subseteq \mathbb{R}$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C) \stackrel{D}{=} (X, \xi, U_C)$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

For ξ diffuse, this is equivalent to $\theta_{T_r}(X, \xi) \stackrel{D}{=} (X, \xi), \quad r \in \mathbb{R}$.

And for ξ simple point process, to $\theta_{T_n}(X, \xi) \stackrel{D}{=} (X, \xi), \quad n \in \mathbb{Z}$.

Mass-Stationarity — General random measures on $\mathbb{R}$

Setting

Let ξ be a random measure on $\mathbb{R}$.

Let X be a random element in a space on which $\mathbb{R}$ acts.

Write θ_t for the shift map placing a new origin at $t \in \mathbb{R}$.

Definition (extended to cover all random measures on $\mathbb{R}$)

Call (X, ξ) mass-stationary if for all bounded λ-continuity sets $C \subseteq \mathbb{R}$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C) \overset{D}{=} (X, \xi, U_C)$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Theorem: Let ξ be a general random measure on $\mathbb{R}$. Then

(X, ξ) mass-stationary $\Longleftrightarrow$ (X, ξ) Palm version of stationary pair

Mass-Stationarity — General random measures on $\mathbb{R}$

Setting

Let ξ be a random measure on $\mathbb{R}$.

Let X be a random element in a space on which $\mathbb{R}$ acts.

Write θ_t for the shift map placing a new origin at $t \in \mathbb{R}$.

Definition (equivalent to (X, ξ) being Palm of a stationary pair)

Call (X, ξ) mass-stationary if for all bounded λ-continuity sets $C \subseteq \mathbb{R}$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C) \overset{D}{=} (X, \xi, U_C)$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Mass-Stationarity — General random measures on $\mathbb{R}$

Setting

Let ξ be a random measure on $\mathbb{R}$.

Let X be a random element in a space on which $\mathbb{R}$ acts.

Write θ_t for the shift map placing a new origin at $t \in \mathbb{R}$.

Definition (equivalent to (X, ξ) being Palm of a stationary pair)

Call (X, ξ) mass-stationary if for all bounded λ-continuity sets $C \subseteq \mathbb{R}$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C) \overset{D}{=} (X, \xi, U_C)$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Mass-Stationarity — General random measures on G

Setting

Let G be a locally compact second countable topological group with left-invariant Haar measure λ. For instance, $G = \mathbb{R}^d$.

Let ξ be a random measure on G.

Let X be a random element in a space on which G acts.

Write θ_t for the shift map placing a new origin at $t \in G$.

Definition (equivalent to (X, ξ) being Palm of a stationary pair)

Call (X, ξ) mass-stationary if for all bounded λ-continuity sets $C \subseteq G$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C^{-1}) \stackrel{D}{=} (X, \xi, U_C^{-1})$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Mass-Stationarity when G is compact

Definition (from previous slide) for general G

Call (X, ξ) mass-stationary if for all bounded λ-continuity sets $C \subseteq G$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C^{-1}) \stackrel{D}{=} (X, \xi, U_C^{-1})$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Note that when G is compact then $\mathbb{P}(V_G \in \cdot \mid X, \xi) = \xi(\cdot \mid G)$.

Mass-Stationarity when G is compact

Definition (from previous slide) for general G :

Call (X, ξ) mass-stationary if for all bounded λ-continuity sets $C \subseteq G$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C^{-1}) \stackrel{D}{=} (X, \xi, U_C^{-1})$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Note that when G is compact then $\mathbb{P}(V_G \in \cdot \mid X, \xi) = \xi(\cdot \mid G)$.

Theorem for compact G :

Let G be compact and V be a random element in G such that

$$\mathbb{P}(V \in \cdot \mid X, \xi) = \xi(\cdot \mid G).$$

Then

$$(X, \xi) \text{ mass-stationary} \iff \theta_V(X, \xi) \stackrel{D}{=} (X, \xi)$$

Shift-coupling Palm versions when G is general

Definition (from previous slides) for general G :

Call (X, ξ) mass-stationary if for all bounded λ-continuity sets $C \subseteq G$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C^{-1}) \stackrel{D}{=} (X, \xi, U_C^{-1})$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Theorem ($\hat{X}$ stationary ergodic, $\hat{\xi}$ and $\hat{\eta}$ invariant, intensity$<\infty$)

If (X, ξ) and (X', η') are Palm versions of $(\hat{X}, \hat{\xi})$ and $(\hat{X}, \hat{\eta})$ then

$$\theta_{\pi(X)} X \stackrel{D}{=} X' \iff \pi \text{ balances } \xi \text{ and } \eta$$

Recall: if $G = \mathbb{R}$ and ξ, η diffuse and mutually singular then

$$\pi(X) := \inf\{t > 0 \colon \xi([0, t]) = \eta([0, t])\} \quad \text{balances } \xi \text{ and } \eta$$

This type of result is now being extended to $G = \mathbb{R}^d$.

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