LONG RANGE DEPENDENCE IN ANALYSIS AND NUMBER THEORY

Continued fractions $x \in (0,1)$ irrational

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \qquad \text{expansion}$$

$$a_1(x) = [1/x], \quad a_{n+1}(x) = a_1(T^n x), \quad Tx = \{1/x\}$$

Gauss (1821) Letter to Laplace

$$\mu(x \in (0,1) : T^n x \le t) \longrightarrow \frac{1}{\log 2} \int_0^t \frac{1}{1+u} du$$

and asked for the speed of convergence

Kusmin (1928) $r_n = O(e^{-c\sqrt{n}})$

In today's language: The sequence (a_n) is stationary and ψ-mixing with respect to the Gaussian measure with nearly exponential speed

vous en occuper quelques momens: dans ce cas je suis sur que vous trouverés une solution plus complete. Le voici. Soit M une quantité inconnue entre les limites 0 et 1, pour laquelle tou[te]s les valeurs sont ou egalement probables ou plus ou moins selon une loi donnée: qu'on la suppose convertie en une fraction continue

$$M = \cfrac{1}{a' + \cfrac{1}{a'' + \text{etc.}}}$$

Quelle est la probabilité, qu'en s'arretant dans le developpement à un terme fini, $a^{(n)}$, la fraction suivante

$$\cfrac{1}{a^{(n+1)} + \cfrac{1}{a^{(n+2)} + \text{etc.}}}$$

soit entre les limites 0 et x? Je la designe par $P(n, x)$ et j'ai en supposant pour M toutes les valeurs egalement probables

$$P(0, x) = x;$$

$P(1, x)$ est une fonction transcendente dependante de la fonction

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x}$$

que Euler nome inexplicable et sur la quelle je viens de donner plusieurs recherches dans un memoire presenté à notre societé des sc[iences] qui sera bientot imprimé[*)]. Mais pour les cas ou n est plus grand, la valeur exacte de $P(n, x)$ semble intraitable. Cependant j'ai trouvé par des raisonnemens tres simples que pour n infini on a

$$P(n, x) = \frac{\log(1 + x)}{\log 2}.$$

Mais les efforts que j'ai fait lors de mes recherches pour assigner

$$P(n, x) - \frac{\log(1 + x)}{\log 2}$$

pour une valeur tres grande de n, mais pas infinie, ont été infructueux.

. . . .

[*) *Disquisitiones circa seriem etc.*, der Gesellschaft der Wissenschaften vorgelegt am 30. Januar 1812, Werke III, S. 123; siehe insbesondere S. 154 ff.]

Khinchin (1923): Small denominator problem For almost all α

$$\left|\alpha - \frac{p}{q}\right| < \frac{f(q)}{q^2} \qquad \text{i.o.} \quad \text{iff} \quad \sum_{k=1}^{\infty} \frac{f(k)}{k} = \infty$$

Erdős-Kac (1939) $\omega(n) =$ # of prime divisors of n

$$\frac{1}{N}\#\left\{n \le N : \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \le x\right\} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$$

Salem-Zygmund (1947) $n_{k+1}/n_k \ge q > 1$

$$N^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x \xrightarrow{d} N(0, 1/2)$$

Rosenblatt (1956) Start of "purely probabilistic" theory

Mandelbrot, Van Ness (1968) Study of the R/S statistic of water level of river Nile ($H \approx 0.77$ instead of $H = 0.5$)

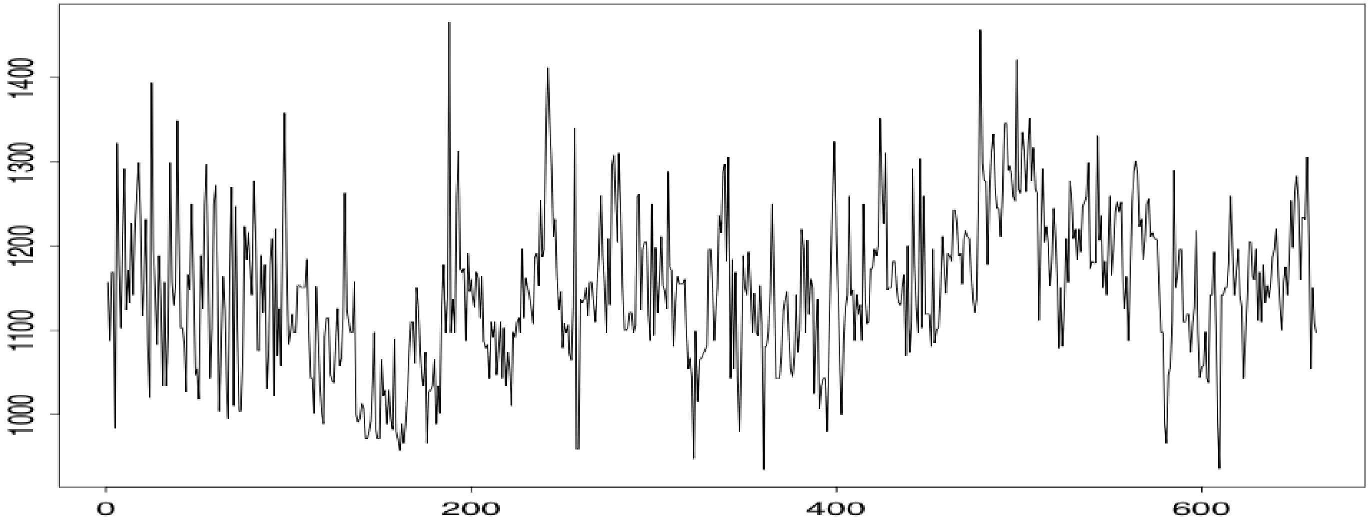


Figure 1.1. Annual minima of the water level in the Nile river for the years 622 to 1281, measured at the Roda gauge near Cairo.

Data show **long range dependence/long memory**

(X_n) stationary Gaussian, covariance $r_n \sim n^{-\alpha}$, $Ef(X_1)^2 < \infty$

Limit distribution of $\frac{1}{A_N} \sum_{k=1}^N f(X_k)$

$\alpha > 1$ Gaussian

$\alpha < 1$ nongaussian, depending on the first nonzero coefficient in the Hermite-expansion of f

(a) Sensitive dependence on f
(b) Combinatorial formalism for moments and cumulants (**diagram formula**)
(c) Unusual behavior of empirical process (semi-deterministic instead of Gaussian)

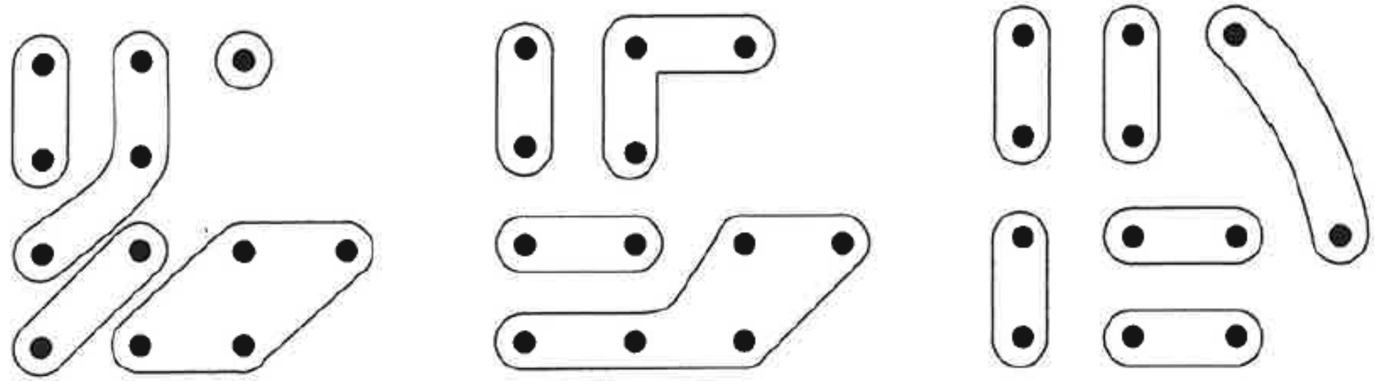


Figure 2. (a) $\gamma \in \Gamma_W^c$; (b) $\gamma \notin \Gamma_W^c$, (c) $\gamma \in \Gamma_W^{\mathcal{N}}$.

With (c.3) in mind, we call a diagram $\gamma = (V_1, \ldots, V_r)$ *Gaussian* if $|V_1| = \cdots = |V_r| = 2$. Obviously, this implies $2r = |W|$, i.e., the total number of elements of W must be even. Write Γ_W^c, $\Gamma_W^{\mathcal{N}}$ for the set of all connected diagrams and all Gaussian diagrams, respectively.

Proposition 2.1 (The Diagram Formula for usual products).

$$EX^W = E\prod_{j=1}^{k} X^{W_i} = \sum_{\gamma=(V)_r \in \Gamma_W} \chi(X^{,V_1}) \cdots \chi(X^{,V_r}), \tag{2.3}$$

$$\chi(X^{W_1}, \ldots, X^{W_k}) = \sum_{\gamma=(V)_r \in \Gamma_W^c} \chi(X^{,V_1}) \cdots \chi(X^{,V_r}). \tag{2.4}$$

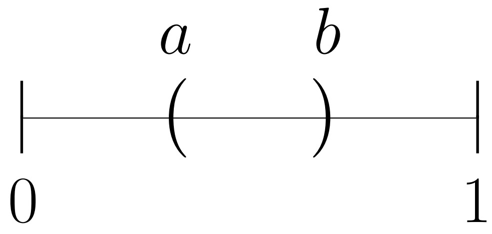
Probability theory of $\{n\alpha\}$

Kronecker (1876) $\{n\alpha\}$ is dense if α is irrational

Sierpinski, Bohl, Weyl (1910) Uniformly distributed mod 1

$(x_n) \subset (0,1)$ is UD if

$$\frac{1}{N}\#\big\{k \le N : \ x_k \in (a,b)\big\} \longrightarrow b-a$$



$$D_N(x_1,\ldots,x_N) = \sup_{0\le a<b\le 1}\left|\frac{N(a,b)}{N} - (b-a)\right| = \sup_x |F_N(x)-x|$$

$\swarrow$ # of terms of $x_1,\ldots,x_N$ in (a,b)

Ostrowski, Khinchin, Hardy & Littlewood (1921–1924) The asymptotic behavior of $D_N(\{n\alpha\})$ is closely connected to the continued fraction digits of α

Khinchin (1924)

$$D_N(\{k\alpha\}) = O\left(\frac{(\log N)^{1+\varepsilon}}{N}\right) \qquad \text{a.e. for } \varepsilon > 0$$

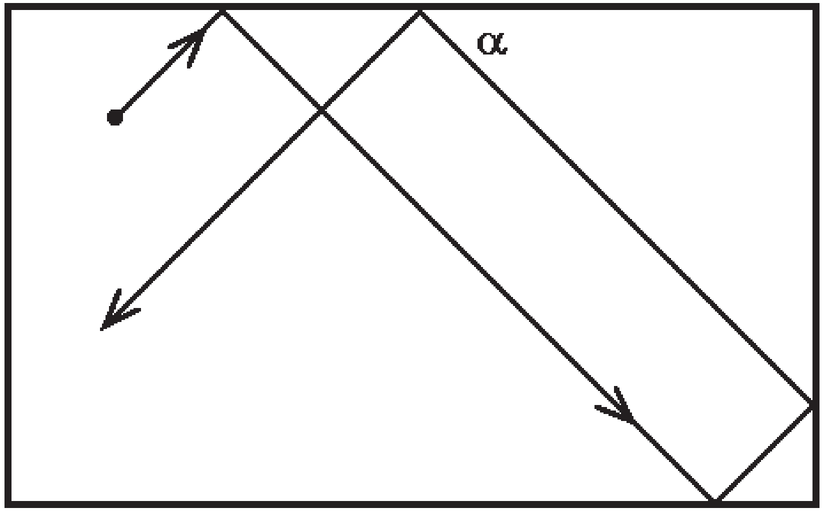
Kesten (1964)

$$D_N(\{k\alpha\}) \sim \frac{2}{\pi^2}\frac{\log N \log\log N}{N} \qquad \text{in measure}$$

Chung & Smirnov (1948, 1944) For i.i.d. sequences (ξ_k)

$$D_N(\{\xi_k\}) = O\left(\sqrt{\frac{\log\log N}{N}}\right) \qquad \text{a.s.}$$

Super-uniformity of the billiard path



Beck (2010) $\qquad A \subset [0,1]^2$ fix, speed =1

For $(1-\varepsilon)$-almost all starting positions

$$\left| \int_0^T I_A(X(t))dt - T\mu(A) \right| \le c_\varepsilon \sqrt{\log T}, \qquad T \ge T_0$$

Strong law of large numbers

$$f(x+1) = f(x), \quad \int_0^1 f(x)dx = 0$$

Khinchin conjecture (1923)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(kx) = 0 \qquad \text{a.e.}$$

Disproved by Marstrand (1970) Bounded counterexample exists

Koksma (1953) Let $f(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx}$ Fourier series

$$\sum_{k=1}^{\infty} |a_k|^2 w(k) < \infty \quad \text{suffices with} \quad w(k) = \sum_{d|k} 1/d = \sigma_{-1}(k)$$

Bourgain (1989) New counterexample via metric entropy criterion

Theorem. (B. & Weber 2014) Koksma's condition

$$\sum_{k=1}^{\infty} |a_k|^2 w(k) < \infty, \qquad w(k) = \sum_{d|k} 1/d$$

is optimal for the strong law.

Note: $w(k) = O(\log\log k)$, $\qquad \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} w(k) = \frac{6}{\pi^2}$

Three series criterion

The series $\sum_{k=1}^{\infty} c_k f(kx)$ converges a.e. if $\sum_{k=1}^{\infty} c_k^2 < \infty$

Valid if $f(x) = \sin 2\pi x$ Carleson (1966)
Fails if $f(x) = \operatorname{sgn} \sin 2\pi x$ Nikishin (1970)

No precise criteria exist even for stepfunctions f

Calculus for moments: GCD calculus

For $f \in \mathrm{BV}$

$$\int_0^1 \left(\sum_{k=1}^{N} f(n_k x) \right)^2 dx \le C \sum_{k,\ell=1}^{N} \frac{(n_k, n_\ell)^2}{n_k n_\ell} \qquad \textbf{GCD sum}$$

and for $f(x) = x - [x] - 1/2$ this is sharp (Franel, Landau 1924, Koksma 1951)

For higher moments the calculations become intractable, but substantial information on the tails can be drawn from estimates for the sums

$$I_N(\alpha) = \frac{1}{N} \sum_{k,l=1}^{N} \frac{(n_k, n_\ell)^{2\alpha}}{(n_k n_\ell)^{\alpha}}$$

History

$$I_N(\alpha) = \frac{1}{N} \sum_{k,l=1}^{N} \frac{(n_k, n_\ell)^{2\alpha}}{(n_k n_\ell)^\alpha}$$

Erdős (1940) $\quad I_N(1) \leq C \log N \quad \text{for all } (n_k)$

Gál (1949)

$$I_N(1) \leq C(\log\log N)^2 \quad \text{for all } (n_k) \text{ and this is precise}$$

Harman and Dyer (1986)

$$I_N(\alpha) \ll \begin{cases} \exp(c \log N / \log\log N) & \alpha = 1/2 \\ \exp\left((\log N)^{(4-4\alpha)/(3-2\alpha)}\right) & 1/2 < \alpha < 1 \end{cases}$$

Aistleitner, B. & Seip (2013)

$$I_N(\alpha) \ll \begin{cases} \exp\left(c_\alpha(\log N \log\log N)^{1/2}\right) & 0 < \alpha < 1/2 \\ \exp\left(c_\alpha(\log N)^{1-\alpha}(\log\log N)^{-\alpha}\right) & 1/2 < \alpha < 1 \end{cases}$$

Bondarenko & Seip (2014)

$$I_N(1/2) \ll \exp\left(c(\log N \log_3 N/\log\log N)^{1/2}\right)$$

Montgomery (1977)

$$\sup_{0\le t\le T} |\zeta(\alpha+it)| \ge \exp\left(c_\alpha \frac{(\log T)^{1-\alpha}}{(\log\log T)^{\alpha}}\right)$$

Hilberdink (2009), Aistleitner (2014) Lower bound via resonance method

Theorem 1. (Aistleitner, B. & Seip 2013) For $f \in \mathrm{BV}$ and any (n_k)

$$\sum c_k^2 (\log \log k)^\gamma < \infty$$

suffices for a.e. convergence of $\sum c_k f(n_k x)$ for $\gamma > 4$ but not for $\gamma < 2$.

Lewko and Radziwill (2014) Critical exponent is $\gamma = 2$.

Theorem 2. (B & Weber 2014) Let $f(x) = \sum a_k e^{2\pi i k x} \in L^2$,

$$g(r) = \sum_{k=1}^{\infty} |a_{rk}|^2, \qquad h(n) = \sum_{d|n} d g(d).$$

Then $\sum c_k f(kx)$ converges a.e. provided

$$\sum_{k=1}^{\infty} c_k^2 h(k) (\log k)^2 < \infty.$$

and this is optimal except the logarithmic factor.

For example, if $|a_k| = O(k^{-s})$, $s > 1/2$, then the condition reduces to

$$\sum_{k=1}^{\infty} c_k^2 \sigma_{1-2s}(k)(\log k)^2 < \infty, \qquad \sigma_{1-2s}(k) = \sum_{d|k} d^{1-2s}$$

Behaviour of $\sum_{k=1}^{N} f(n_k x)$ for "concrete" (n_k)

Erdős (1962) $N^{-1/2} \sum_{k=1}^{N} \sin n_k x \xrightarrow{d} N(0, 1/2)$ provided n_k grows faster than $e^{c\sqrt{k}}$ and this is sharp

Below this growth speed, the validity of CLT depends on the number theoretic properties of (n_k) and the question is usually intractable.

Conjecture: $n_k = e^{k^\alpha}$ satisfies CLT for any $\alpha > 0$.

Kaufman (1980) Valid for $n_k = e^{ck^\alpha}$ for almost all c

A classical "concrete" system: $\{k^2\alpha\}$

Diophantine approximation:

$$\sum_{k=1}^{N} e^{\pi i k^2 x} = \alpha(p,q) \int_0^N e^{\pi i t^2 \xi/q} dt + O(\sqrt{q})$$

provided

$$x = \frac{p}{q} + \frac{\xi}{q}, \quad |\xi| \le \frac{1}{4N}, \quad 0 < q \le 4N$$

Hardy and Littlewood (1914, 1923), Walfisz (1930), Fiedler, Jurkat & Körner (1977), Jurkat & Van Horne (1981, 1982, 1983), Marklof (1999, 2003)

Marklof (2003) Limit distribution of $N^{-1/2}\sum_{k=1}^{N} f(k^2 x)$ for a large class of f's

Lacunary sequences

Behavior of $\{n_k\alpha\}$ for exponentially growing n_k

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(2^k x) = 0 \qquad \text{a.e.}$$

Kac (1946) Under smoothness conditions

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(2^k x) \longrightarrow_d N(0,\sigma^2) \qquad \text{for some } \sigma^2 \geq 0$$

Erdős-Fortet (1949) Fails for $f((2^k - 1)x)$!

Theorem (Aistleitner & B. 2010) The sequence $f(n_k x)$ satisfies the CLT for all "nice" f iff the number of solutions of the equation

$$an_k + bn_l = c, \qquad 1 \le k, l \le N$$

is $o(N)$ for any fixed $a, b \neq 0$, uniformly in c.

Empirical process

Philipp (1975) If $n_{k+1}/n_k \ge q > 1$, then

$$0 < \limsup_{N\to\infty} \sqrt{\frac{N}{\log\log N}} D_N(\{n_k x\}) < \infty \qquad \text{a.e.}$$

For i.i.d. sequences

$$\limsup_{N\to\infty} \sqrt{\frac{N}{\log\log N}} D_N(\{\xi_k\}) = \frac{1}{2} \qquad \text{a.s.}$$

Open problems:

(a) Value of limsup?
(b) Is the limsup constant a.e.?
(c) What is the limit distribution of $\sqrt{N} D_N(n_k x)$?

Fukuyama (2008) $n_k = a^k$

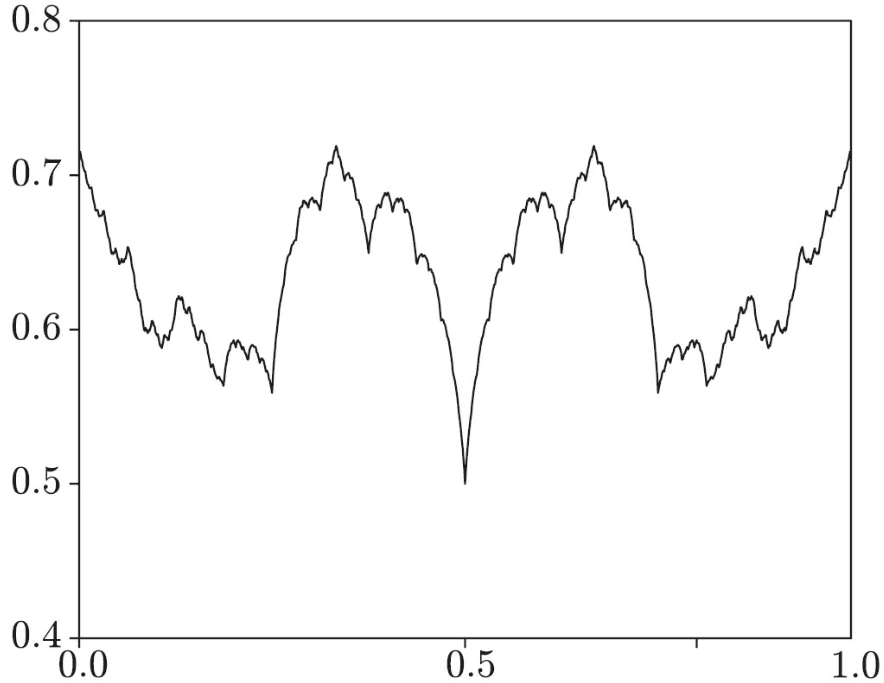
$$\Sigma_a = \sqrt{42}/9 \qquad \text{if } a = 2$$

$$\Sigma_a = \frac{\sqrt{(a+1)a(a-2)}}{2\sqrt{(a-1)^3}} \qquad \text{if } a \geq 4 \text{ is an even integer,}$$

$$\Sigma_a = \frac{\sqrt{a+1}}{2\sqrt{a-1}} \qquad \text{if } a \geq 3 \text{ is an odd integer}$$

First values: $\sqrt{42}/9,\ 1/\sqrt{2},\ \sqrt{10/27},\ \sqrt{6}/4,\ \sqrt{42}/10$

Fukuyama, Aistleitner (2010) There exists a lacunary (n_k) with nonconstant limsup



Limsup function for $n_k = 2^k - 1$

Aistleitner and Berkes (2013): Limit distribution of $\sqrt{N} D_N(a^k x)$
Gaussian process with "fractal" covariance