Random graphs, forest fires and self-organised criticality

James Martin
Department of Statistics
University of Oxford

Joint work with Balázs Ráth (UBC)

University of Bristol,
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Erdős-Rényi random graph

Random graph $G(n, p)$ on n vertices: each edge is present independently with probability p.

Let C_k^n be the size of the kth largest component.

Phase transition: let $p = c/n$ where $c > 0$.

$$c < 1 : C_1^n \text{ on order } \log n$$
$$c = 1 : C_1^n \text{ on order } n^{2/3}$$
$$c > 1 : C_1^n \sim \theta(c)n \text{ for some } \theta(c) > 0.$$

Erdős-Rényi random graph process

Random graph process: start with empty graph on n vertices at time $t = 0$. Each absent edge arrives at rate $1/n$.

State at time t has distribution $G(n, p)$ where $p = 1 - e^{-t/n} \approx t/n$.

Let $C_k^n(t)$ be the size of the kth largest component at time t.

$$t < 1 : C_1^n(t) \text{ on order } \log n$$

$$t = 1 : C_1^n(t) \text{ on order } n^{2/3}$$

$$t > 1 : C_1^n(t) \sim \theta(t) n.$$

"multiplicative coalescence": blocks size a, b merge at rate $\propto ab$.

Define $v_k^n(t) = \dfrac{\#\text{ vertices in components of size } k \text{ at time } t}{n}$.

Then $v_k^n(t) \xrightarrow{d} v_k(t)$ as $n \to \infty$ for each k and each t, where

$$v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1}.$$

$t < 1$: $v_k(t)$ decays exponentially, $\sum_k v_k(t) = 1$.

$t = 1$: $v_k(t)$ on order $k^{-3/2}$ as $k \to \infty$.

$t > 1$: $v_k(t)$ decays exponentially, $\sum_k v_k(t) = 1 - \theta(t) < 1$.

(simple approximation by a branching process with Poisson(t) offspring distribution)

Mean-field forest-fire model (Ráth and Tóth 2009):

Start with empty graph on n vertices. Each absent edge arrives at rate $1/n$. In addition, each vertex is struck by lightning at rate $\lambda(n)$. When lightning strikes a vertex, remove all the edges in its component.

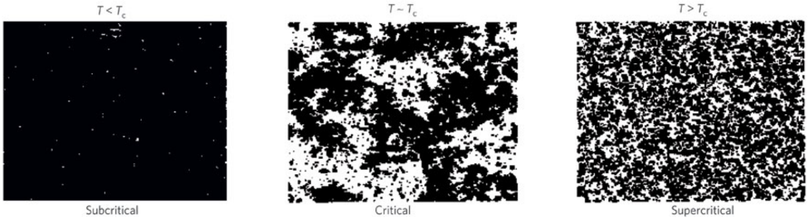
Mean-field "frozen percolation" (Ráth 2009):

Start with empty graph on n vertices. Each absent edge arrives at rate $1/n$. In addition, each vertex is struck by lightning at rate $\lambda(n)$. When lightning strikes a vertex, remove all the edges AND all the vertices in its component.

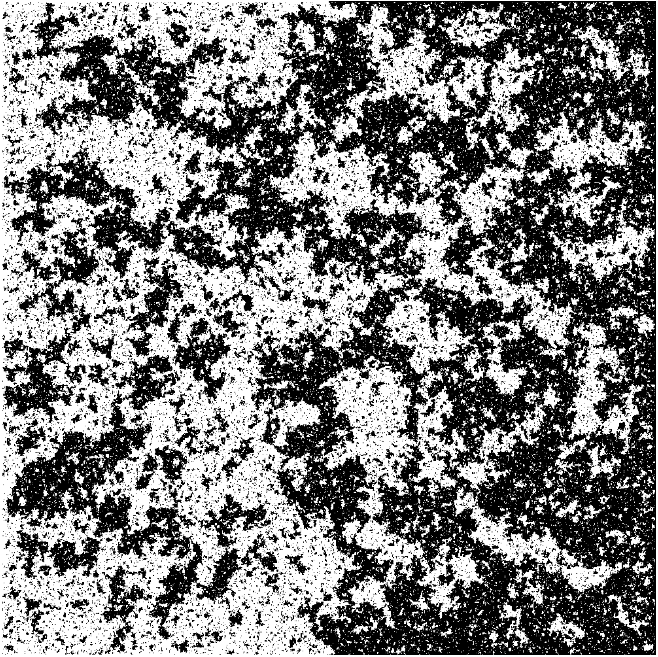
The frozen percolation model has a useful distributional property. At any time t, conditional on the graph having m remaining vertices, its distribution is that of $G\big(m, 1 - e^{-t/n}\big) \approx G(m, t/n)$.

Particularly interesting cases: when $1/n \ll \lambda(n) \ll 1$ as $n \to \infty$. Then we see "self-organised criticality".

Criticality: Ising model



Critical Ising simulation (1000×1000):



$$1/n \ll \lambda(n) \ll 1.$$

$$v_k^n(t) = \frac{\#\text{ vertices in components of size } k \text{ at time } t}{n}.$$

Frozen percolation: $v_k^n(t) \xrightarrow{d} v_k(t)$ as $n \to \infty$ for each k and each t, where

$$v_k(t) = \begin{cases} \dfrac{k^{k-1}}{k!} e^{-kt} t^{k-1}, & t \leq 1 \\ \\ \dfrac{1}{t} v_k(1), & t > 1. \end{cases}$$

(Rath 2009)

$$1/n \ll \lambda(n) \ll 1.$$

$$v_k^n(t) = \frac{\#\text{ vertices in components of size } k \text{ at time } t}{n}.$$

Forest fire: $v_k^n(t) \xrightarrow{d} v_k(t)$ as $n \to \infty$ for each k and each t, where

$$t \leq 1: \; v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1}$$

$$t > 1: \; \sum_{l=k}^{\infty} v_l(t) \sim c(t) k^{-1/2}.$$

We expect that as $t \to \infty$,

$$v_k(t) \to v_k(\infty) = 2 \binom{2k-2}{k-1} \frac{4^{-k}}{k}.$$

$v_k(\infty)$ is of order $k^{-3/2}$ as $k \to \infty$, and corresponds to the distribution of the number of leaves of a critical binary branching process. (Rath and Toth 2009)

Erdős-Rényi process: scaling window

Let $C_k^n(t)$ be the size of the kth largest component at time t.

$t < 1 : C_1^n(t)$ on order $\log n$

$t = 1 : C_1^n(t), \ldots, C_k^n(t)$ all on order $n^{2/3}$

$t > 1 : C_1^n(t) \sim \theta(t)n$, $C_2^n(t)$ on order $\log t$

Scaling window of width order $n^{-1/3}$.

If $t = 1 + u_n n^{-1/3}$, then

if $u_n \to -\infty$: $\quad \frac{C_1^n(t)}{n^{2/3}} \xrightarrow{d} 0$

if $u_n \to u \in (-\infty, \infty)$: $\frac{C_1^n(t)}{n^{2/3}}$ converges in distribution to a non-trivial limit.

if $u_n \to \infty$: $\quad \frac{C_1^n(t)}{n^{2/3}} \xrightarrow{d} \infty.$

Scaling window: convergence of the process to the multiplicative coalescent

Under appropriate rescaling, the evolution of large components of the random graph in the "scaling window" around $t = 1$ converges as $n \to \infty$:

$$\left(n^{-2/3} C_1^n \left(1 + un^{-1/3}\right), n^{-2/3} C_2^n \left(1 + un^{-1/3}\right), \dots\right)_{u \in \mathbb{R}} \Longrightarrow \left(\mathcal{X}_1(u), \mathcal{X}_2(u), \dots\right)_{u \in \mathbb{R}}.$$

$\boldsymbol{\mathcal{X}}(u)$ is a version of the *multiplicative coalescent*, by which we mean a random process with state space

$$\ell_2^{\downarrow} = \{x_1, x_2, \dots : x_1 \geq x_2 \geq \dots \geq 0, \sum x_i^2 < \infty\}$$

in which any pair of blocks of size a and b merge at rate ab to form a block of size $a + b$.

This version is called the *standard multiplicative coalescent*.

Aldous's description of a state of the standard multiplicative coalescent

Let $B(s), s \geq 0$ be a standard Brownian motion. Define

$$W^u(s) = B(s) + us - \frac{s^2}{2},$$

a Brownian motion with drift $u - s$ at time s.

Theorem

Let $u \in \mathbb{R}$. The excursions of $W^u(s)$, $s \geq 0$ above its past minimum have the same distribution as the sizes of the blocks of $\boldsymbol{\mathcal{X}}(u)$, put into size-biased order.

Exploration process to find the components of a graph.

Choose v_1 uniformly at random in the vertex set V.

Let $v_2, \ldots, v_k$ be the neighbours of v_1.

Let $v_{k+1}, \ldots, v_{k+m}$ be the neighbours of v_2 in $V \setminus \{v_1, \ldots, v_k\}$.

And so on, continuing until we "run out of vertices" – we have finished exploring one whole component, of size r say. Then choose the next vertex v_{r+1} uniformly at random among those remaining, and continue as before.

Let $X_i = \#$ neighbours of v_i which are not neighbours of any vertex $v_1, \dots, v_{i-1}$.

Consider the random walk with step sizes $X_i - 1 \in \{-1, 0, 1, 2 \dots\}$.

- The size of the component of v_1 is the time until the random walk hits -1.
- More generally, the component sizes of the graph are the times between successive minima in the random walk.

The components appear in size-biased order in the exploration process.

For $G(n, p)$, n large,

$$X_1, X_2, X_3, \dots \approx \text{ i.i.d. } \mathrm{Binomial}(n, p).$$

Note: there is some freedom in the order in which we "expand" the vertices in this exploration process (depth first? breadth first? something else?)

Aldous's description of a state of the standard multiplicative coalescent

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Aldous: how to describe the process $\boldsymbol{\mathcal{X}}(u), u \in \mathbb{R}$?

Theorem (Armendariz)

Aldous's construction, taken simultaneously for each u*, based on the same underlying Brownian motion* B*, corresponds to the whole process* $\boldsymbol{\mathcal{X}}(u), u \in \mathbb{R}$*.*

1D representation of random graph process

Let $v_1 =$ vertex chosen uniformly at random

$v_2 =$ first vertex to be joined by an edge to v_1

$v_3 =$ first vertex outisde $\{v_1, v_2\}$ to be joined by an edge to $\{v_1, v_2\}$

$\vdots$

$v_{k+1} =$ first vertex outside $\{v_1, \ldots, v_k\}$ to be joined by an edge to $\{v_1, \ldots, v_k\}$.

Then at every time t in the random graph process, each component consists of an interval of vertices of the form $\{v_a, v_{a+1}, \ldots, v_{a+m}\}$; all coalescences involve neighbouring blocks.

The components appear in size-biased order.

This coupling of the order of expansion of the vertices for the exploration processes at different times leads to the claimed limit for the process in the scaling window.

1D representation of the frozen percolation model

- only neighbouring blocks can coalesce
- only the leftmost block can be hit by lightning

To achieve this, we define

$$\begin{aligned} v_1 &= \text{first vertex to be hit by lightning} \\ v_2 &= \text{first vertex in } V \setminus \{v_1\} \text{ to be} \\ &\qquad \text{hit by lightning or joined to } v_1 \text{ by an edge} \\ &\vdots \\ v_{k+1} &= \text{first vertex in } V \setminus \{v_1, v_2, \ldots, v_k\} \text{ to be} \\ &\qquad \text{hit by lightning or joined to } \{v_1, \ldots, v_k\} \text{ by an edge.} \end{aligned}$$

Frozen percolation with $\lambda(n) = \lambda n^{-1/3}$ for some fixed $\lambda > 0$.

Theorem

Let $b_n = 1 + \frac{1}{3}\lambda \log(n) n^{-1/3}$.

$$\left(n^{-2/3} C_1^n\left(b_n + s n^{-1/3}\right), n^{-2/3} C_2^n\left(b_n + s n^{-1/3}\right), \ldots\right)_{s\in\mathbb{R}} \Longrightarrow \left(\mathcal{X}_1^\lambda(s), \mathcal{X}_2^\lambda(s), \ldots\right)_{s\in\mathbb{R}}.$$

Here $\boldsymbol{\mathcal{X}}^\lambda$ is a

"multiplicative coalescent with linear deletion"

which is an $\ell_2^{\downarrow}$-valued process such that

- any pair of blocks with size a and b merge at rate ab
- any block of size a is deleted at rate λa.

The multiplicative coalescent with linear deletion process $\boldsymbol{\mathcal{X}}^{\lambda}(s)$ can be described through its *window process* $U(s)$.

$U(s)$ is a Markov process which drifts up at rate 1 (representing coalescence) and jumps down (representing deletion).

$U(s)$ can be read off as a deterministic function of the state $\boldsymbol{\mathcal{X}}^{\lambda}(s)$. Conditional on $U(s) = u$, $\boldsymbol{\mathcal{X}}^{\lambda}(s)$ has the distribution of $\boldsymbol{\mathcal{X}}(u)$ (where $\boldsymbol{\mathcal{X}}(.)$ is the standard multiplicate coalescent, without deletion).

"Window process" because U describes the corresponding position in the scaling window of the original random graph process.

$U(s)$ converges to a stationary distribution as $s \to \infty$, and correspondingly so does the whole process $\boldsymbol{\mathcal{X}}^{\lambda}(s)$ itself.

Convergence results / conjectures

Fix $\lambda > 0$ and let $\lambda(n) = \lambda n^{-1/3}$.

(1) Already stated above: for frozen percolation

$$n^{-2/3}\mathbf{C}^n\left(1 + \frac{\lambda \log n}{3} n^{-1/3} + sn^{-1/3}\right)_{s\in\mathbb{R}} \Longrightarrow \boldsymbol{\mathcal{X}}^\lambda(s)_{s\in\mathbb{R}}$$

(2) Same for forest fire.

(3) For frozen percolation, for any $t > 1$,

$$n^{-2/3}\mathbf{C}^n\left(t + sn^{-1/3}\right)_{s\in\mathbb{R}}$$

converges to a stationary version of the MCLD.

(4) Conjecture: same for forest fire.

(5) Conjecture: same for forest fire in stationarity.