Liouville quantum gravity and the Brownian map

Jason Miller and Scott Sheffield

Cambridge and MIT

October 30, 2015

Overview

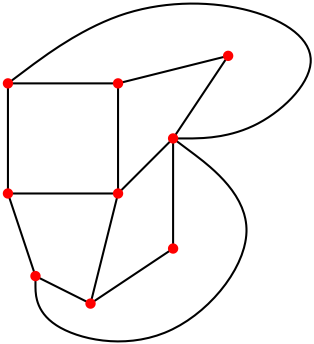
Part I: Introduction

Part II: An axiomatic characterization of the Brownian map

Part III: The $\mathrm{QLE}(8/3, 0)$ metric on $\sqrt{8/3}$-LQG

Part I: Introduction

Random planar maps



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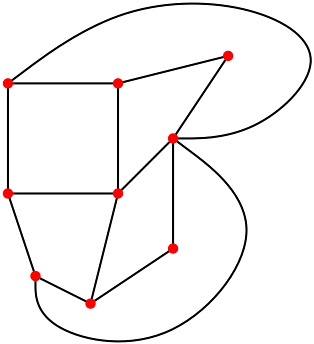

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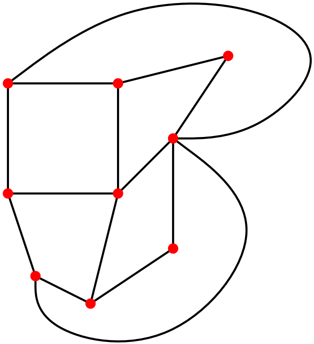

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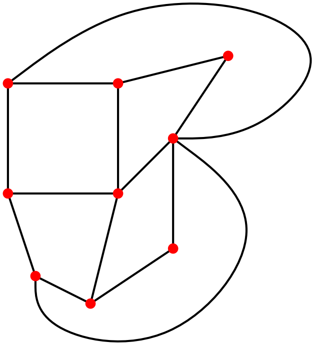

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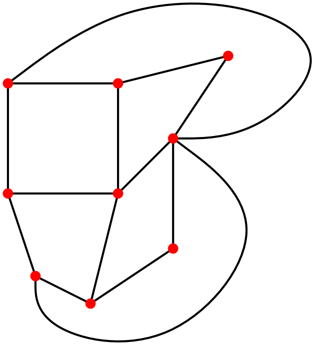

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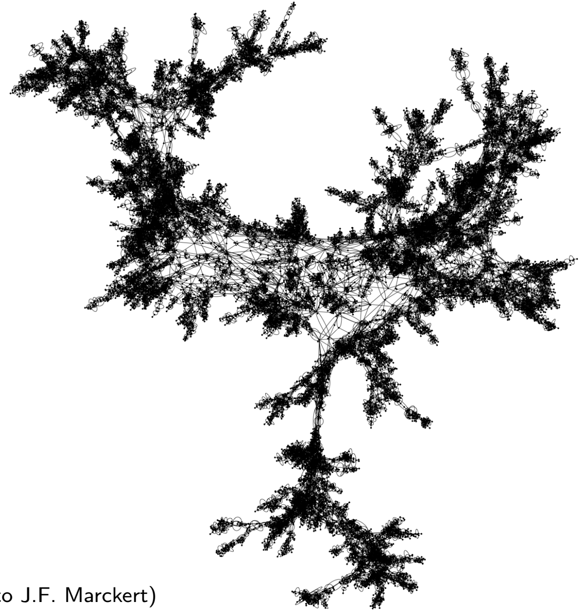
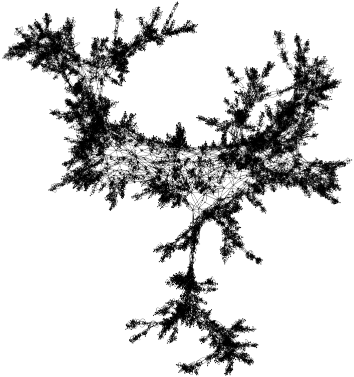
- Interested in **uniformly random quadrangulations** with n faces — **random planar map** (RPM)

Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Structure of large random planar maps



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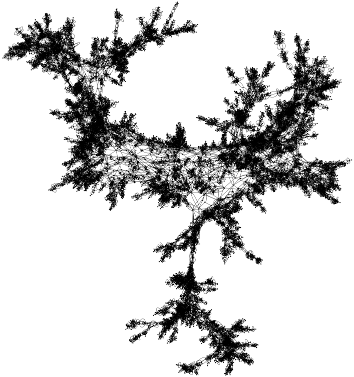

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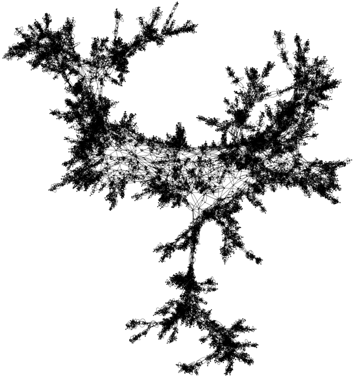

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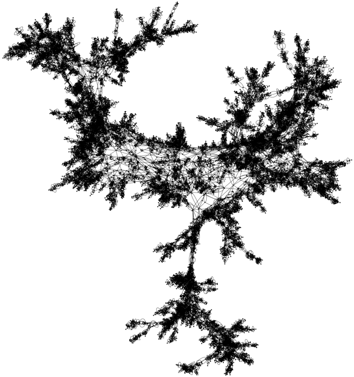

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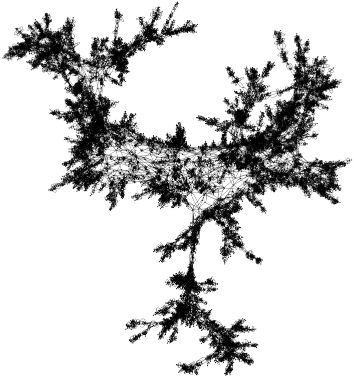

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- The Brownian map (TBM) comes equipped with an area measure which is the limit of the rescaled measure on RPM which assigns unit mass for each vertex

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This talk is about endowing each of these objects with the *other's* structure and showing they are equivalent.

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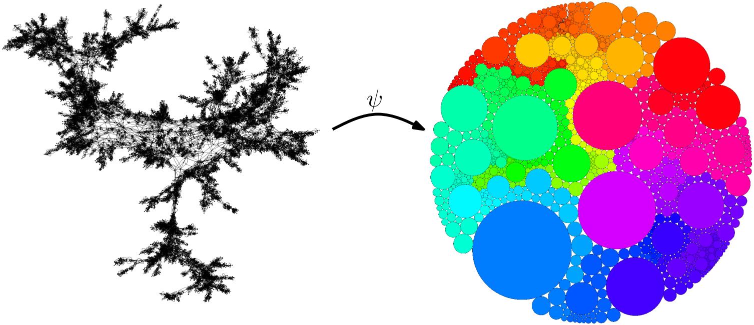

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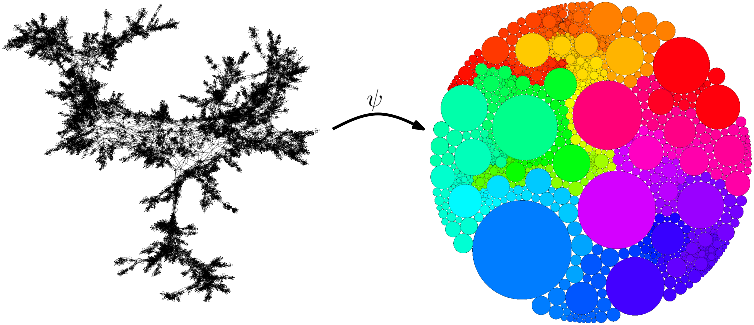

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Theorem (M., Sheffield)

Suppose that (M, d, μ) *is an instance of TBM. Then there exists a Hölder homeomorphism* $\varphi\colon (M, d) \to \mathbf{S}^2$ *such that the pushforward of* μ *by* φ *has the law of a* $\sqrt{8/3}$*-LQG sphere* $(\mathbf{S}^2, h)$*.*

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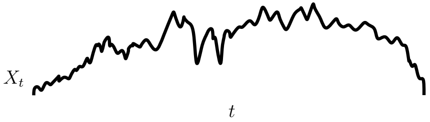
5. Metric construction is for the $\sqrt{8/3}$-LQG sphere. By absolute continuity, can construct a metric on any $\sqrt{8/3}$-LQG surface.

Part II:
An axiomatic characterization of the Brownian map

Brownian map review



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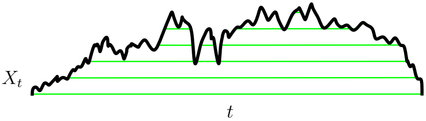

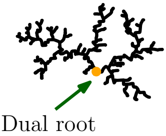

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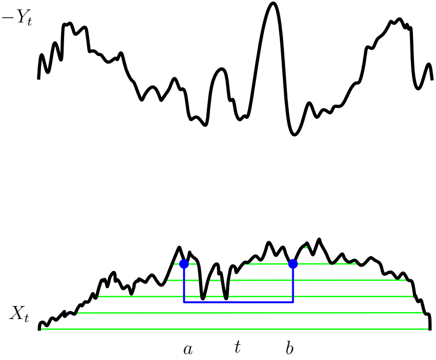

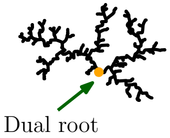

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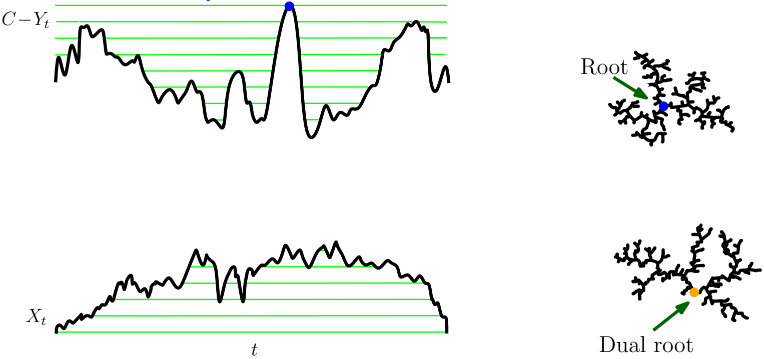

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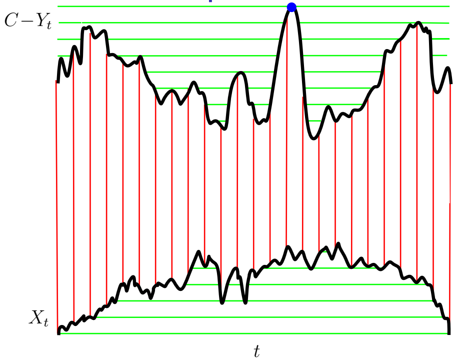

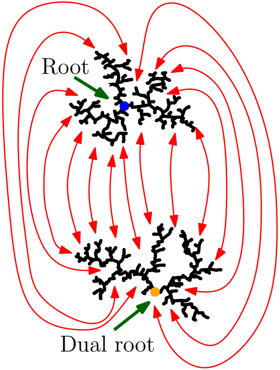

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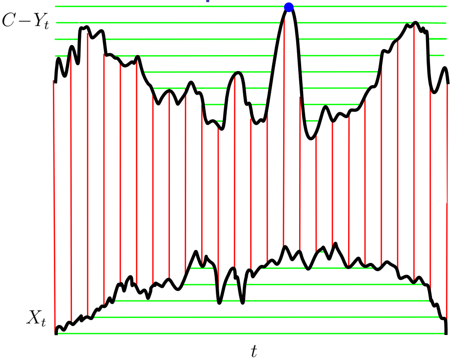

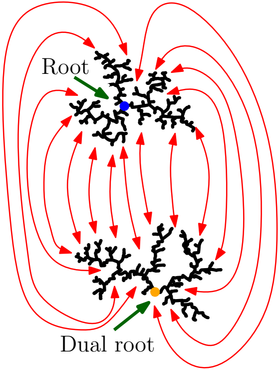

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- Glue together by declaring points on red and green lines to be equivalent. Metric quotient of $\mathcal{G}$ gives the metric for the Brownian map.

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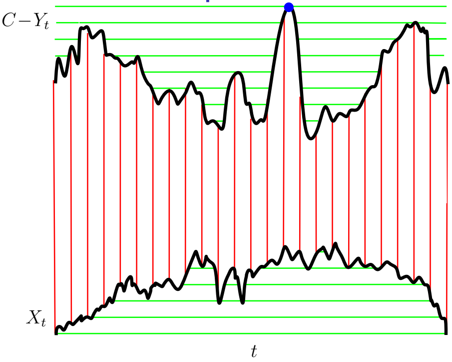
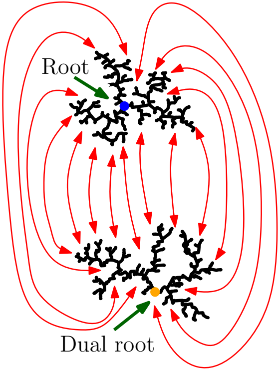
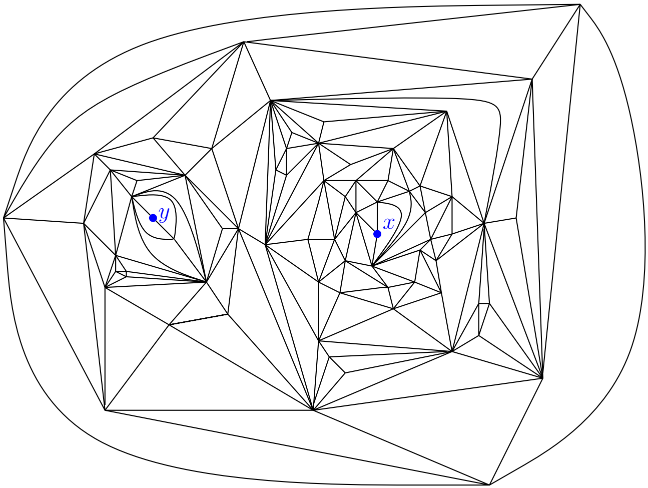




- X_t standard Brownian excursion on $[0,1]$ — encodes a CRT $\mathcal{T}$ (dual tree)
- Given X_t, Y_t Gaussian with covariance $\mathrm{cov}(Y_a, Y_b) = \inf\{X_r : r \in [a,b]\}$ (so Y_t is a Brownian motion on the branches of $\mathcal{T}$). Y_t encodes a tree $\mathcal{G}$ (geodesic tree).
- Glue together by declaring points on red and green lines to be equivalent. Metric quotient of $\mathcal{G}$ gives the metric for the Brownian map.
- Projection of Lebesgue measure on $[0,1]$ gives the measure μ

What resampling properties should TBM have?



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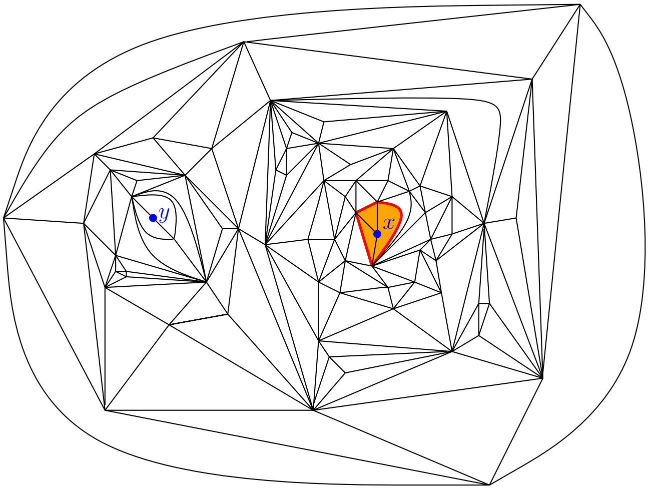

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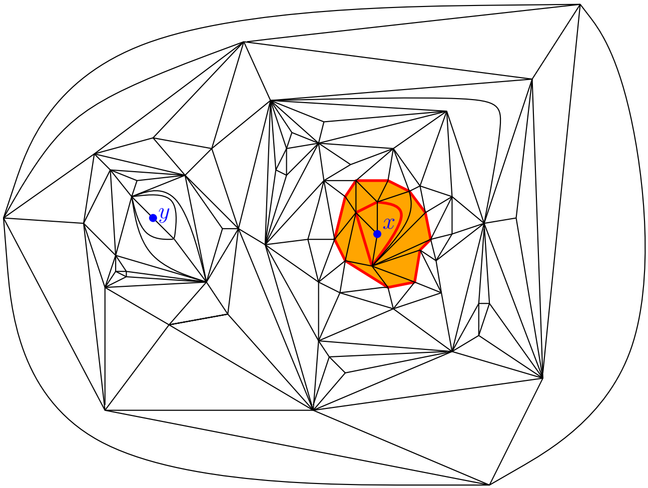

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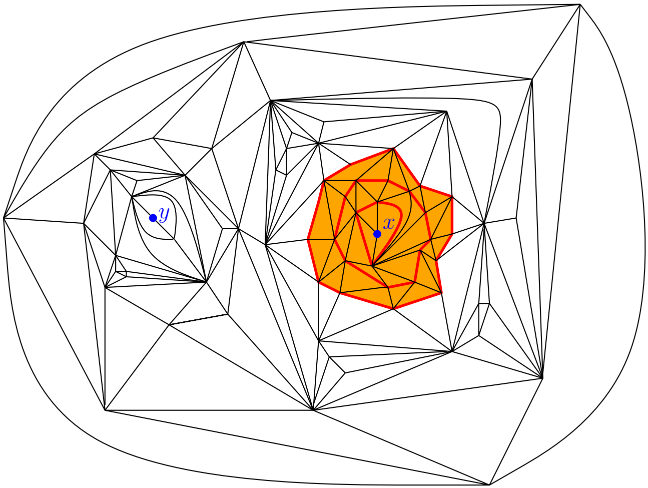

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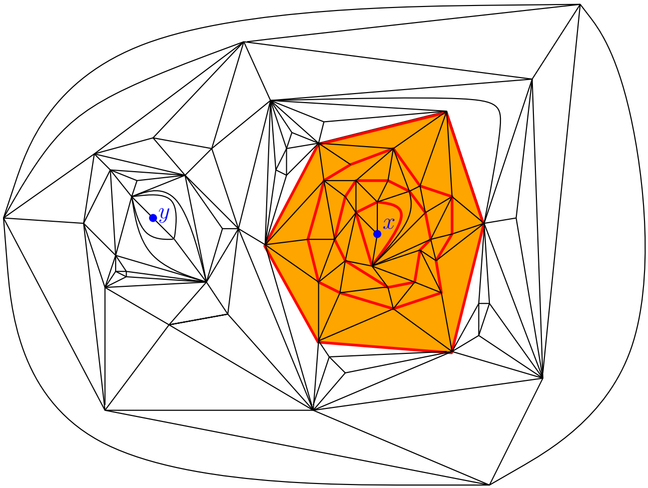

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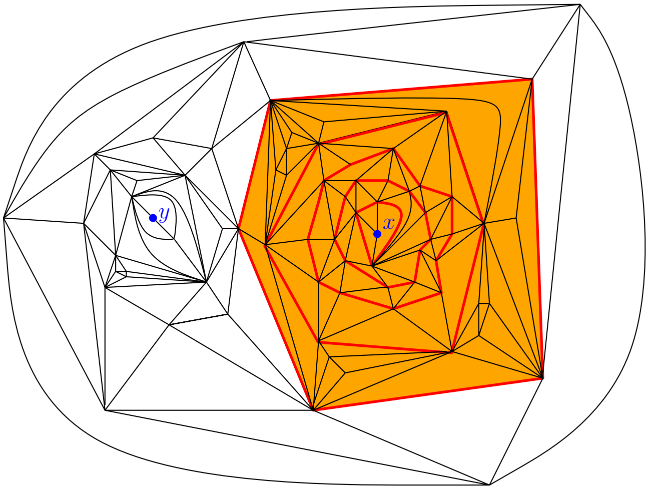

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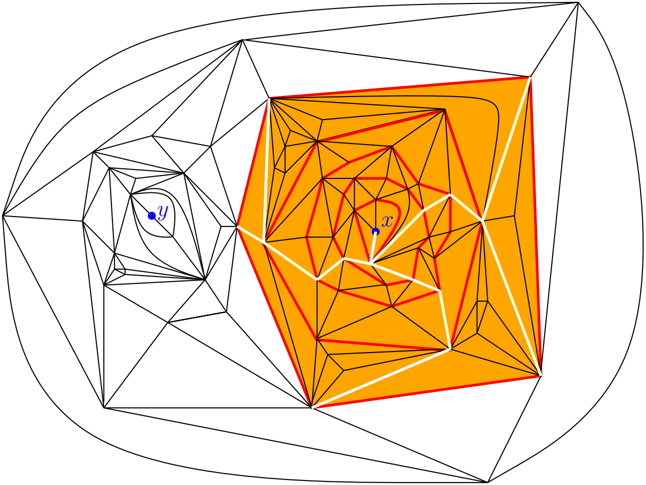

Axioms which characterize TBM

Theorem (M., Sheffield)

The Brownian map is the unique measure on measure-endowed, doubly marked, geodesic metric spheres (M, d, μ, x, y) *such that:*

1. *Its law is invariant under resampling* x *and* y *independently from* μ

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Comments: To be precise, one has to choose the σ-algebras for these random variables.

Axioms which characterize TBM

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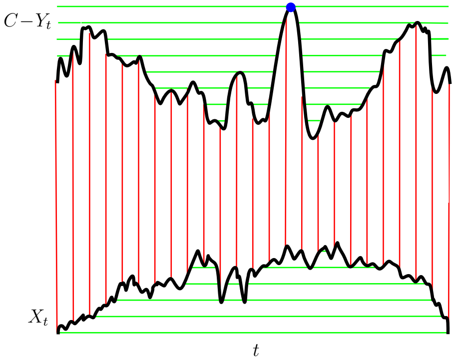
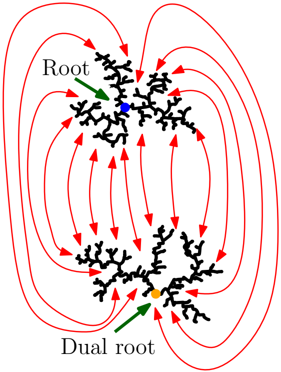
Comments: To be precise, one has to choose the σ-algebras for these random variables. Leads to interesting measurability questions, e.g., is the event that a metric space is geodesic and homeomorphic to $\mathbf{S}^2$ measurable wrt the Borel σ-algebra in the Gromov-Hausdorff topology?

Breadth-first construction





- The usual construction of TBM is described in a depth-first manner

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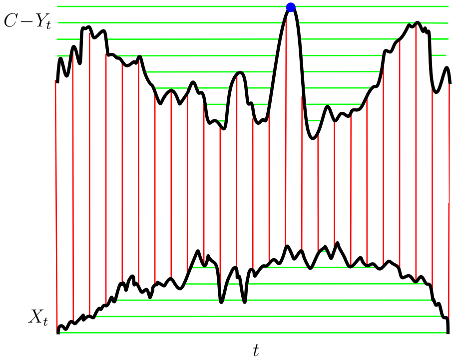

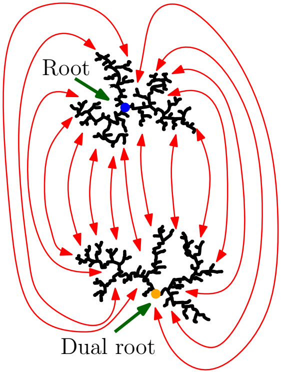

- The usual construction of TBM is described in a depth-first manner
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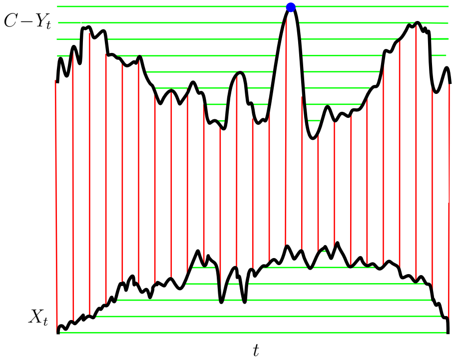
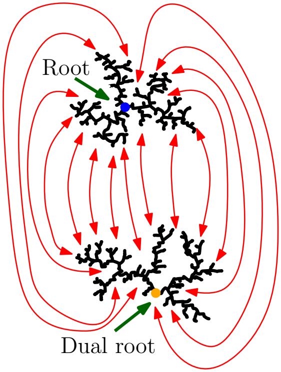
Breadth-first construction





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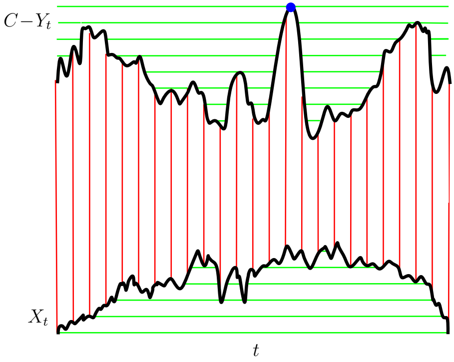

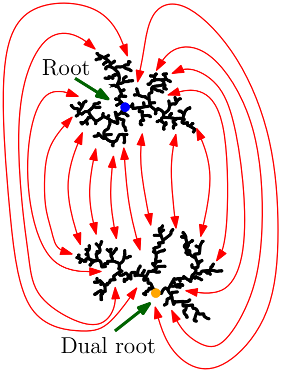

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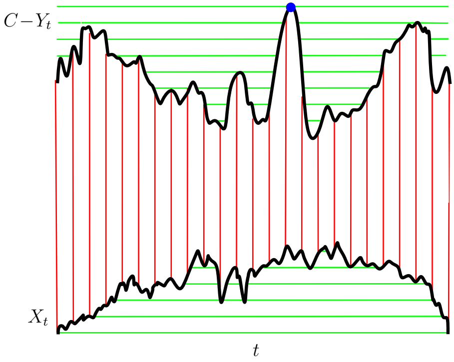

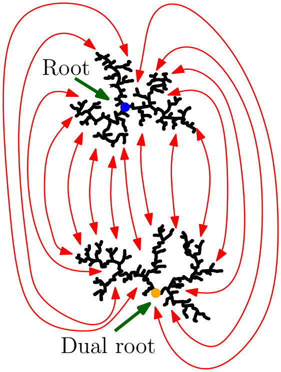

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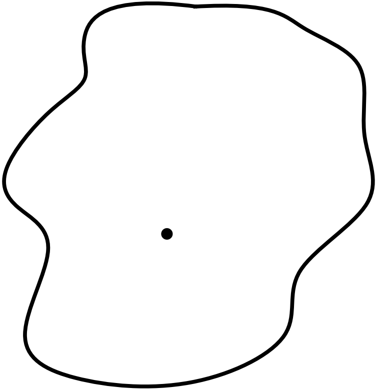
- To begin to prove the theorem, need to give a breadth-first description of TBM
- To do this, need to be able to:
  - Make sense of the "boundary length" measure for metric ball boundaries
  - Construct the law of a "Brownian disk" with given boundary length which describes the unexplored region in TBM when performing a metric exploration

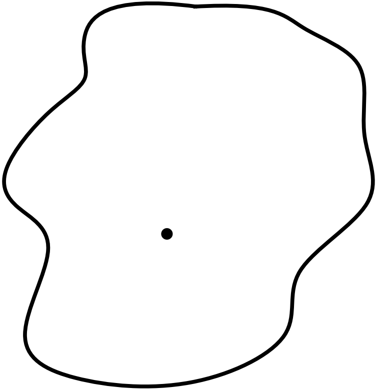
Consequence of slice independence

- Slice independence and scale invariance restrict the form of the geodesic tree from the boundary of a filled metric ball back to the root and the boundary length process L_r.



Consequence of slice independence

- Slice independence and scale invariance restrict the form of the geodesic tree from the boundary of a filled metric ball back to the root and the boundary length process L_r. Will see there is one parameter family of laws.



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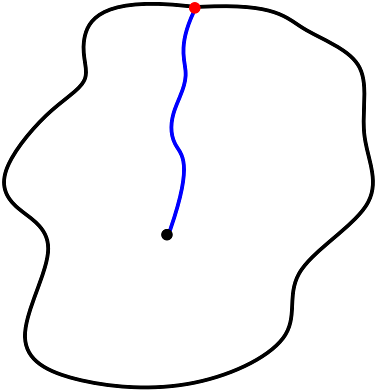

- Geodesic from a uniform point to root

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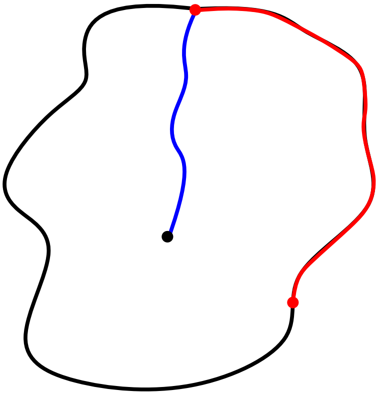

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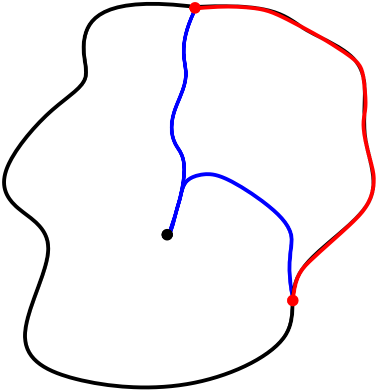

- Geodesic from a uniform point to root
- Second geodesic from 1 unit clockwise to right
- A is the merging time

Consequence of slice independence

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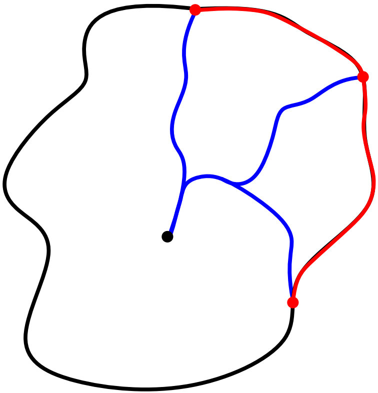

- Geodesic from a uniform point to root
- Second geodesic from 1 unit clockwise to right
- A is the merging time
- Add geodesic from midpoint
- A_i successive merging times (independent)

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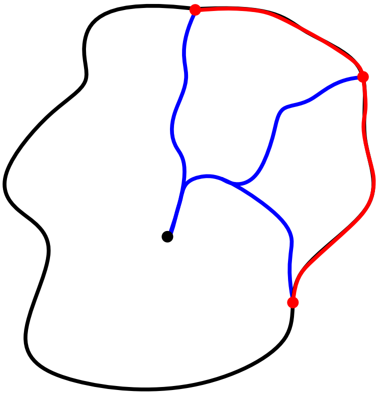

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Consequence of slice independence

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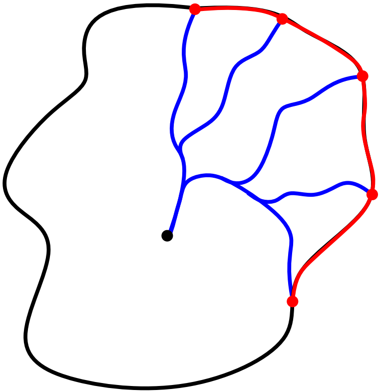

- Geodesic from a uniform point to root
- Second geodesic from 1 unit clockwise to right
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- Geodesics from four equally spaced points

Consequence of slice independence

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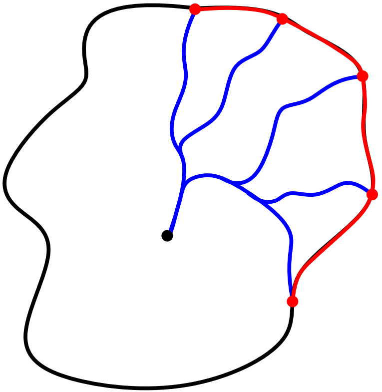

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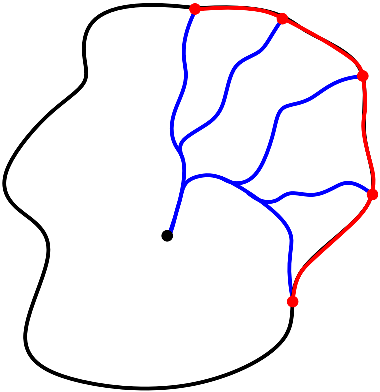

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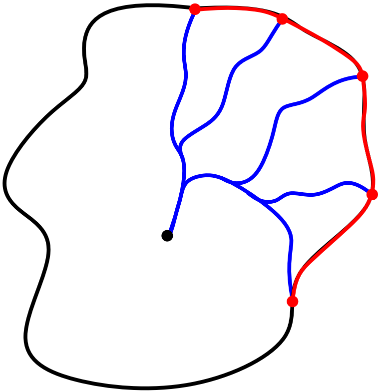

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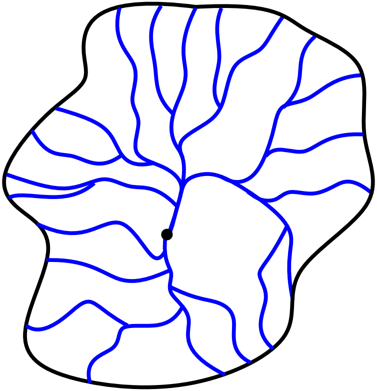

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- Determines the law of the geodesic tree from ball boundary

Consequence of slice independence

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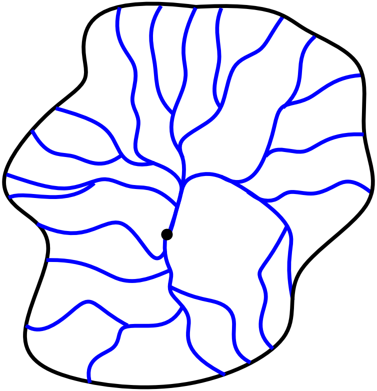

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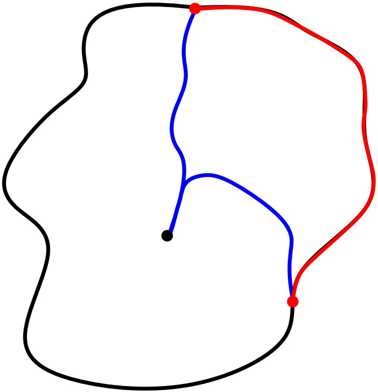
- Iterating this procedure determines the law of A
- Determines the law of the geodesic tree from ball boundary
- By varying radii and using inside-outside independence, determines law of geodesic tree

Consequence of slice independence

- Slice independence and scale invariance restrict the form of the geodesic tree from the boundary of a filled metric ball back to the root and the boundary length process L_r. Will see there is one parameter family of laws.
  - A merging time for geodesics 1 unit apart



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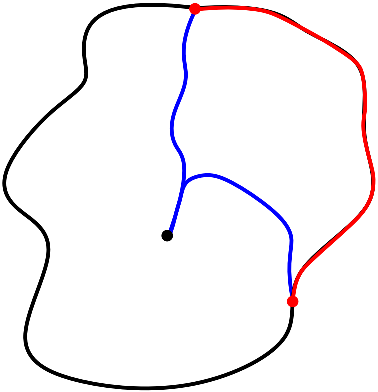

- A merging time for geodesics 1 unit apart
- Know $A = \max(A_1, \ldots, A_{2^n})$ for $A_i \stackrel{d}{=} 2^{-n\beta} A$ i.i.d.

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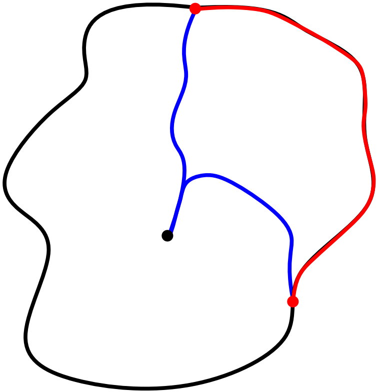

- A merging time for geodesics 1 unit apart
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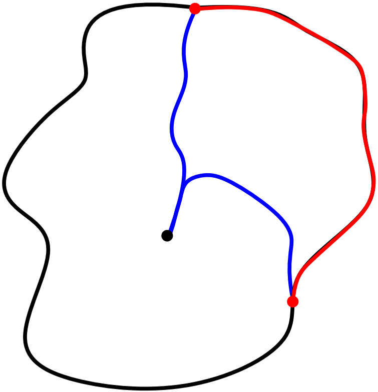

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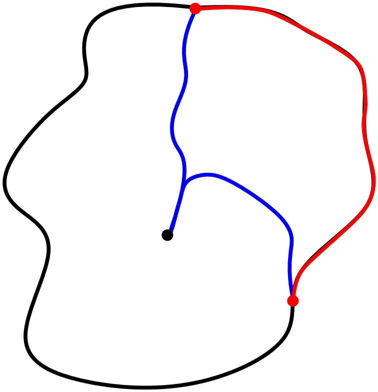

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- To finish coupling geodesic tree with TBM geodesic tree, need to show that theorem assumptions imply $\beta = 1/2$

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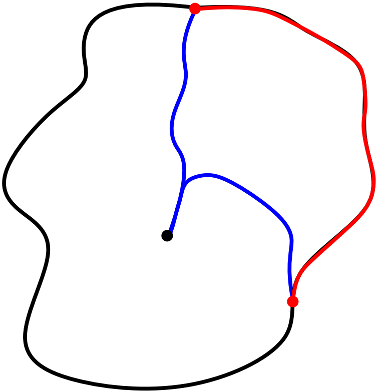

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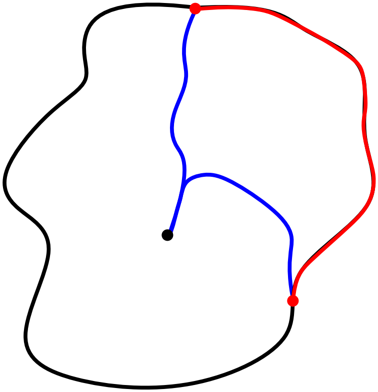
- Know $A = \max(A_1, \ldots, A_{2^n})$ for $A_i \stackrel{d}{=} 2^{-n\beta} A$ i.i.d.
- Implies $\mathbf{P}[A \le r] = q^{r^{-1/\beta}}$, some $q \in (0,1)$
- Same holds for TBM with $\beta = 1/2$
- To finish coupling geodesic tree with TBM geodesic tree, need to show that theorem assumptions imply $\beta = 1/2$
  1. use scale invariance to see that expected area in a disk given boundary length L is $L^{2\beta+1}$

Consequence of slice independence

- Slice independence and scale invariance restrict the form of the geodesic tree from the boundary of a filled metric ball back to the root and the boundary length process L_r. Will see there is one parameter family of laws.



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    2. Lévy process argument gives that expected area in a disk as one explores towards the "center" is a martingale iff $\beta = 1/2$

Part III:

The $\mathrm{QLE}(8/3, 0)$ metric on $\sqrt{8/3}$-LQG

Overview of metric construction

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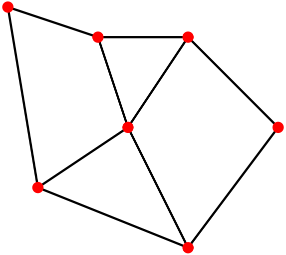
- We will extract the metric property by building on the reversibility of SLE_6

First passage percolation review

- Associate with a graph (V, E) i.i.d. $\exp(1)$ edge weights



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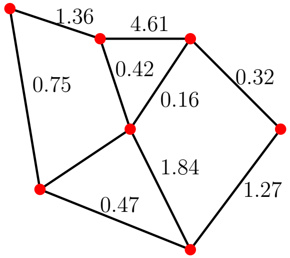

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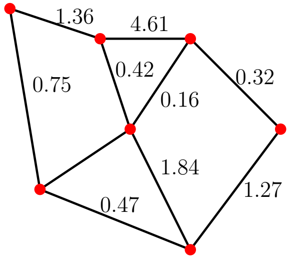

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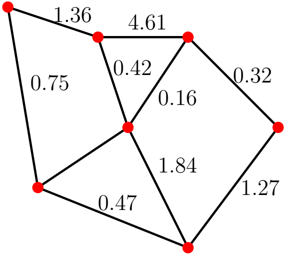

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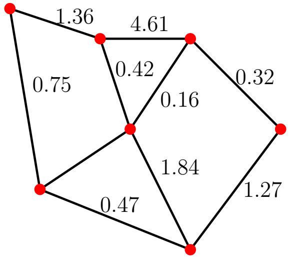

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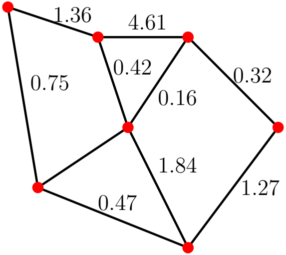
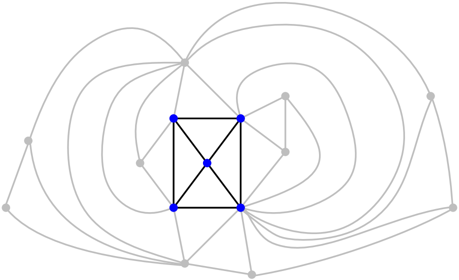
- There is a Markovian way of growing a metric ball in FPP: the Eden growth model



Eden model on random planar maps I

- RPM, random vertex x. Perform FPP from x (Angel's peeling process).



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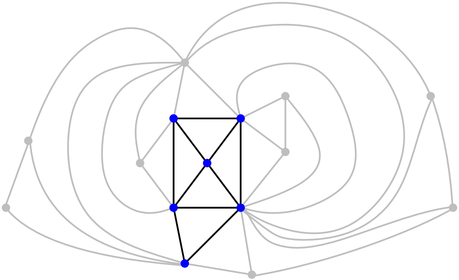

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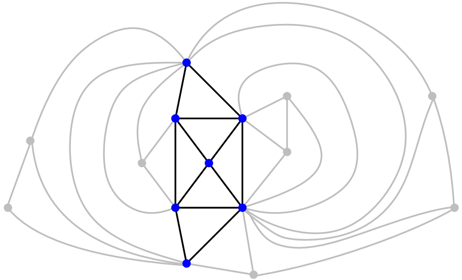

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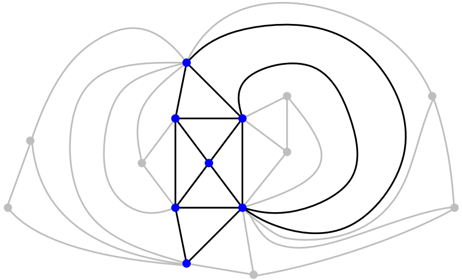

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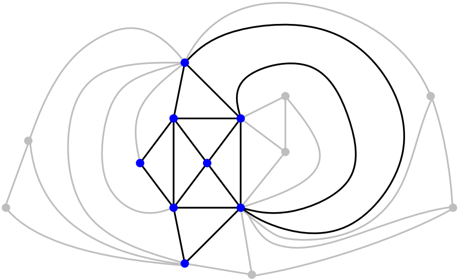

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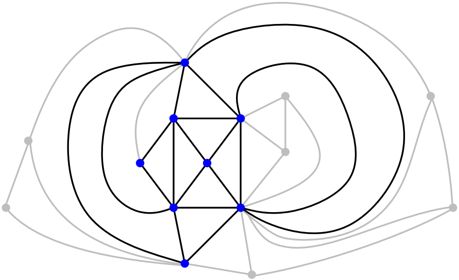

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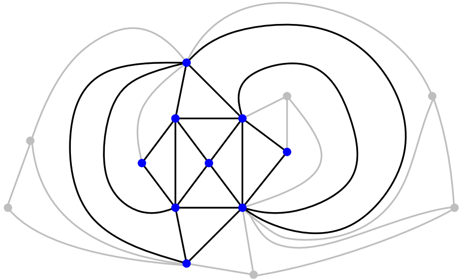

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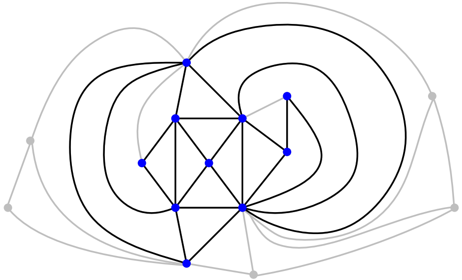

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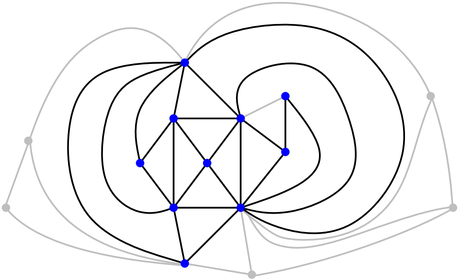

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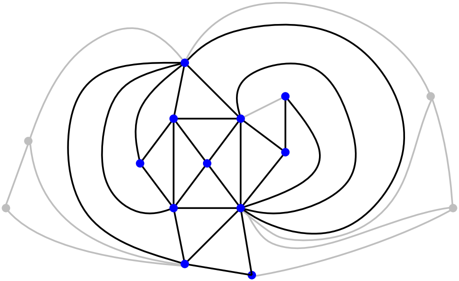

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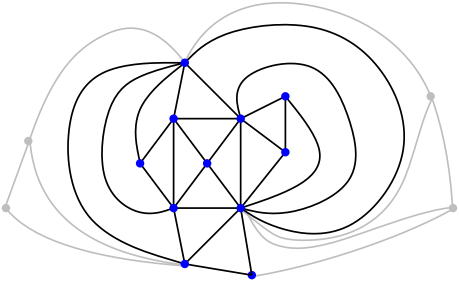

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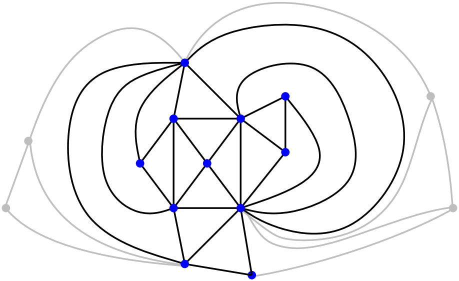

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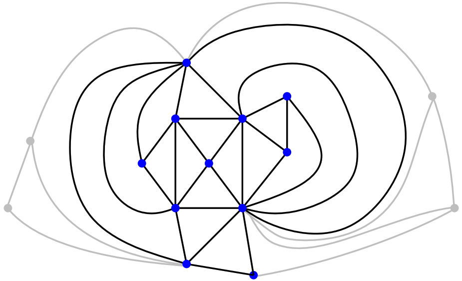

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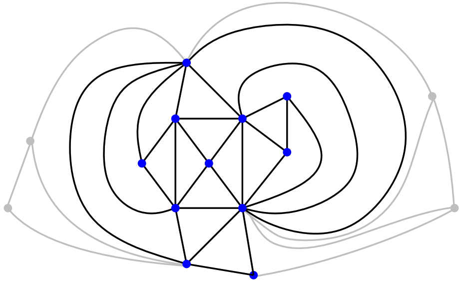

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Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric (now **proved** by Curien and Le Gall)

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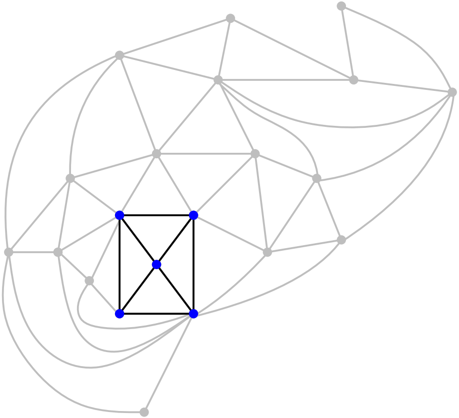
Variant:

- Pick two edges on outer boundary of cluster



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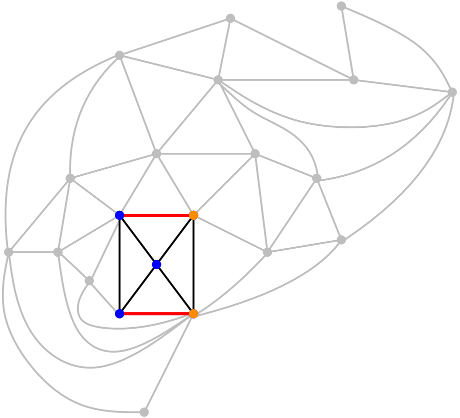

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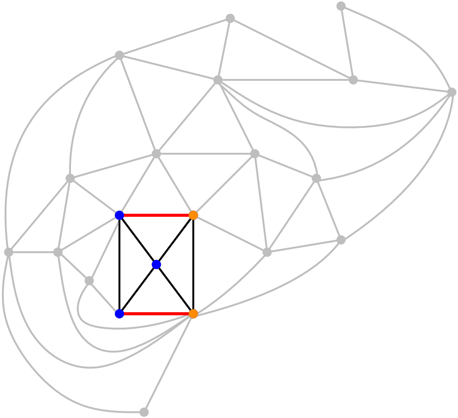

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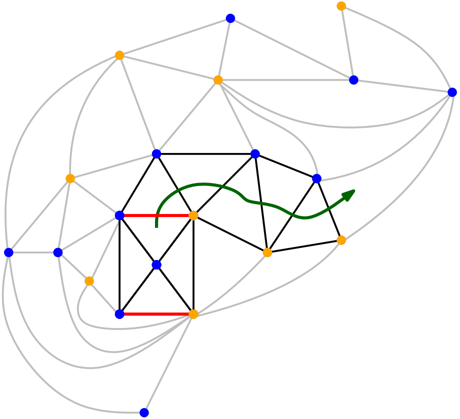

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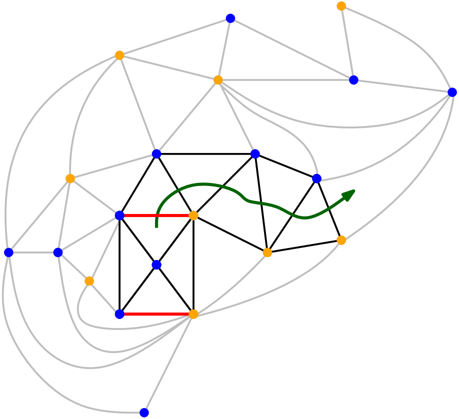

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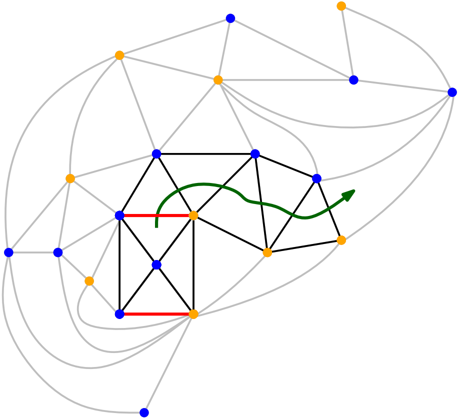

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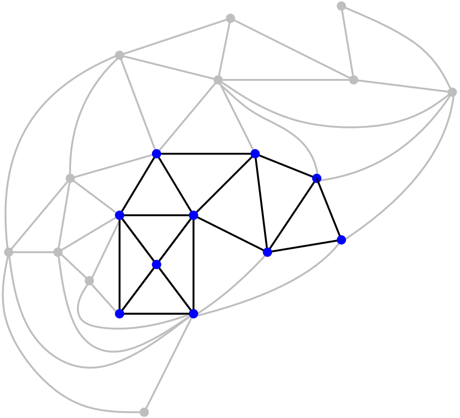

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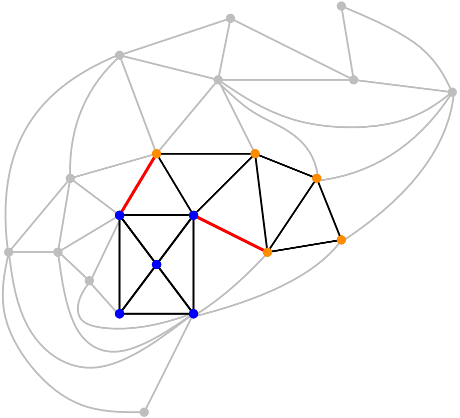

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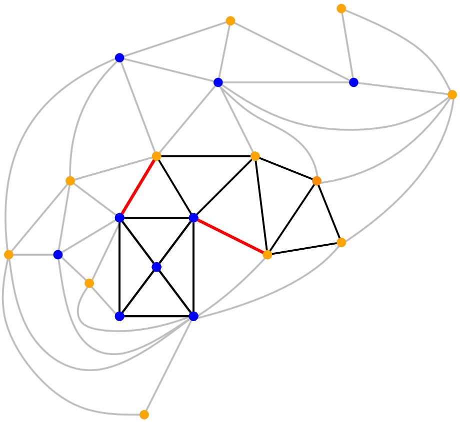

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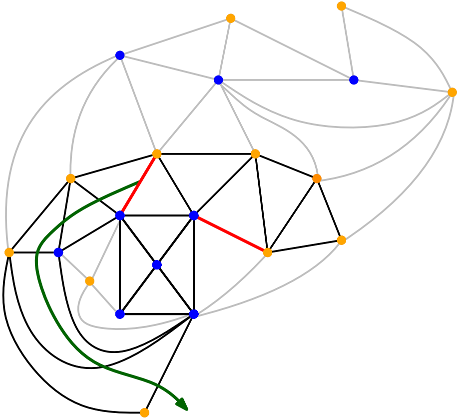

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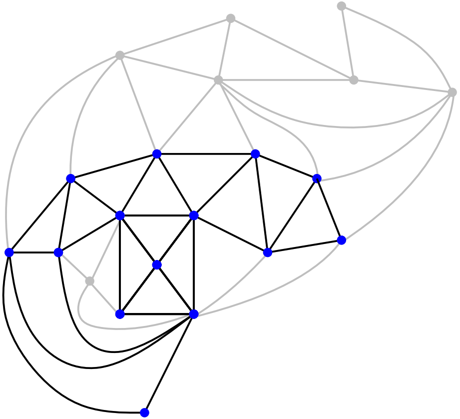

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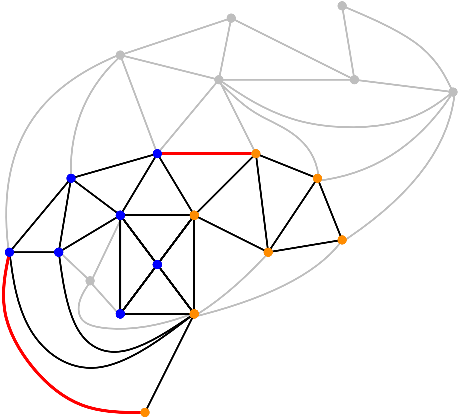

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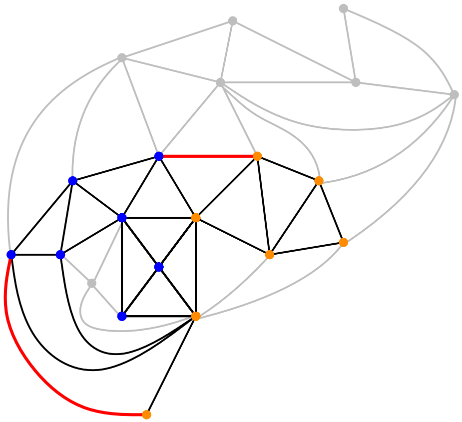

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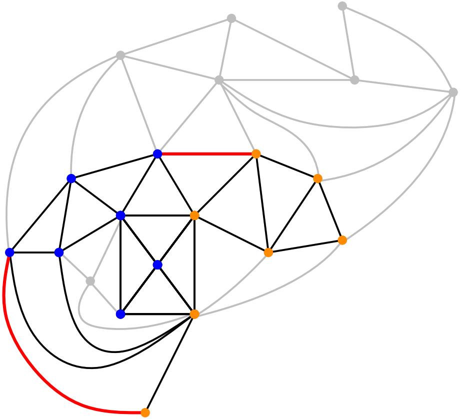

- *This exploration also respects the Markovian structure of the map.*
- Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball

Continuum analog of first passage percolation on LQG

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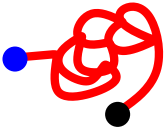

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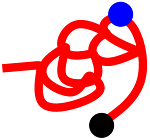

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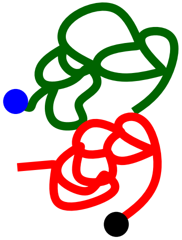

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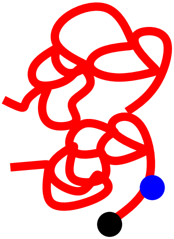

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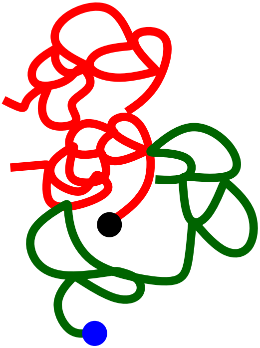

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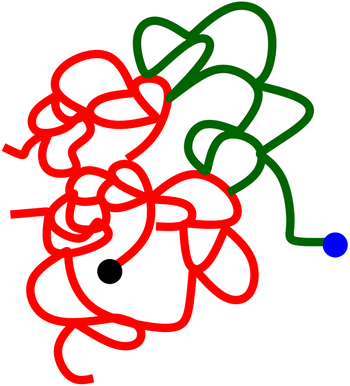

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$\mathrm{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion.

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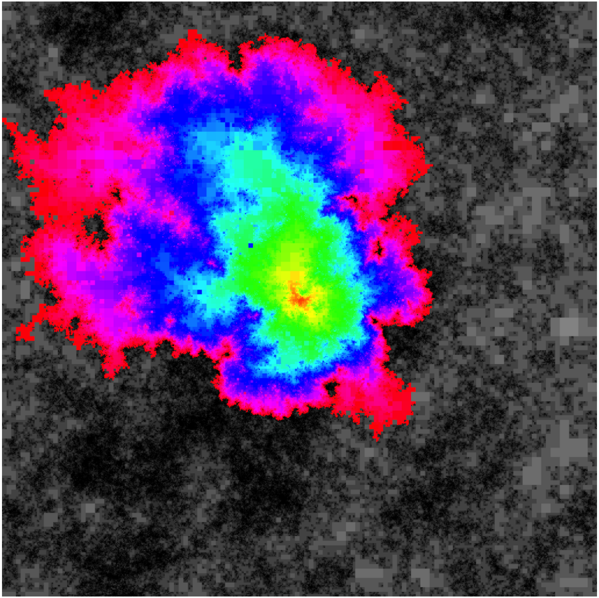
$\mathrm{QLE}(8/3, 0)$ is the limit as $\delta \to 0$ of this growth process. It is described in terms of a radial Loewner evolution which is driven by a measure valued diffusion.

$\mathrm{QLE}(8/3, 0)$ is SLE_6 with **tip re-randomization**.



Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG

QLE$(8/3,0)$ defines a metric on $\sqrt{8/3}$-LQG

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- Idea: use a strategy developed by Sheffield, Watson, Wu in the context of CLE_4
  - Gives (at a high level) conditions which imply that a family of growth processes (candidates for metric balls starting from a collection of points in the space) define a metric space.

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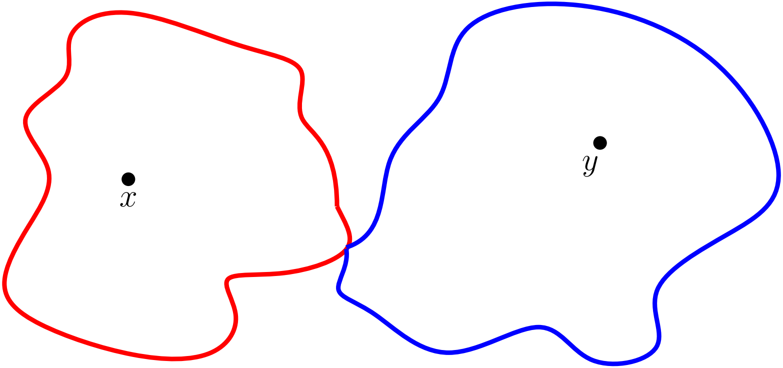

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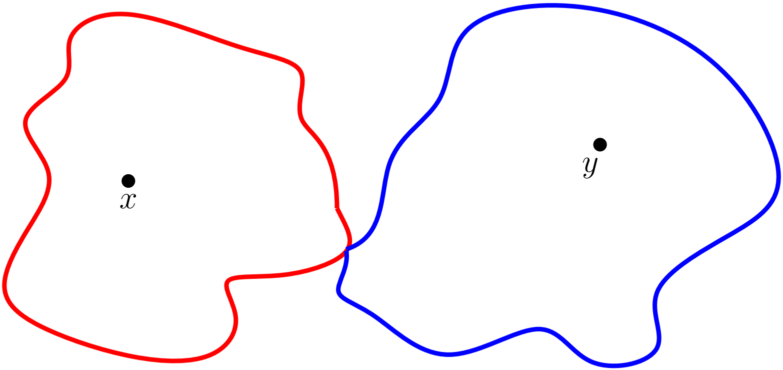

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- As $s = (1 - U)d(x, y) = Vd(x, y)$ for $V \in [0, 1]$ uniform, get the same picture if drawn in the opposite order

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- Show that the metric space structure of TBM determines the $\sqrt{8/3}$-LQG surface

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Thanks!