Like a Needle in a Haystack: Rates and Algorithms for Group Testing

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Joint work with Oliver Johnson, Matthew Aldridge and Karen Gunderson

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Warm-up examples

- Molecular biology (DNA screening)
- Recommender systems

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- Molecular biology (DNA screening)
- Recommender systems

- Spectrum sensing
- High-throughput screening techniques
- Network tomography
- Cryptography and cyber security

DNA screening

Problem

Find a specific, rare sequence of nucleotides among many DNA samples

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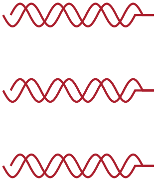

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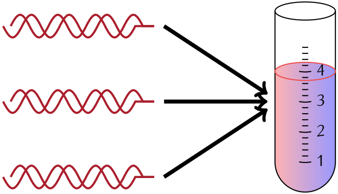

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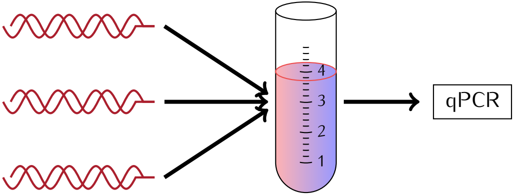

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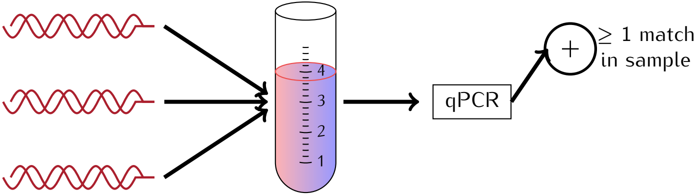

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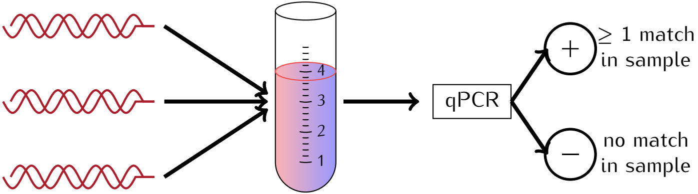

Recommender systems

Problem

Suggest songs to a user based on his past preferences.

Recommender systems

Problem

Suggest songs to a user based on his past preferences.

User selects
song

Recommender systems

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Suggest songs to a user based on his past preferences.

User selects song ⟶ RS proposes song

Recommender systems

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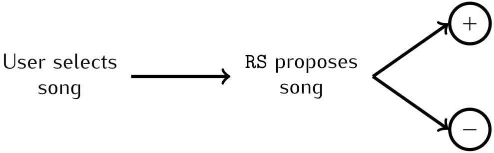

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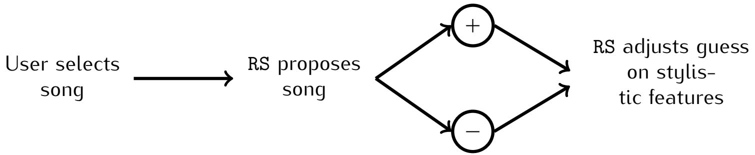

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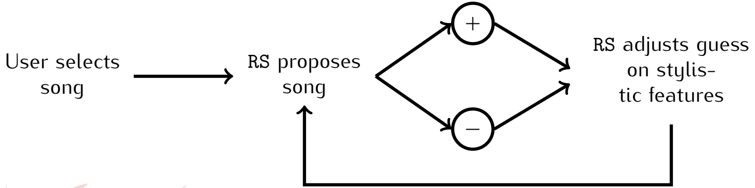

What do they have in common?

Common features

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COMMON FEATURES

- Search for *sparse* property

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- Property can be tested on groups

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Common features

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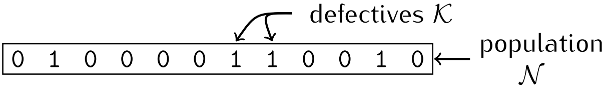
Group testing

Search methods to recover a sparse subset of items from a population that share a feature which can be detected on groups

Boolean group testing



Boolean group testing

defectives $\mathcal{K}$

population $\mathcal{N}$

0	1	0	0	0	0	1	1	0	0	1	0

0	0	0	0	0	0	1	1	0	0	1	0
0	0	0	0	1	0	1	0	1	1	0	0
0	1	0	1	0	0	0	0	0	0	1	0
1	0	1	0	0	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0	1	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1
1	0	0	1	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0

test design
X

Boolean group testing

defectives $\mathcal{K}$

population $\mathcal{N}$

0	1	0	0	0	0	1	1	0	0	1	0

test design X

0	0	0	0	0	0	1	1	0	0	1	0
0	0	0	0	1	0	1	0	1	1	0	0
0	1	0	1	0	0	0	0	0	0	1	0
1	0	1	0	0	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0	1	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1
1	0	0	1	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0

test oucomes $\mathbf{y}$

1
1
1
1
0
0
0
0
0
0
0

Boolean group testing

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0	1	0	0	0	0	1	1	0	0	1	0

test design X / test oucomes **y**

												y
0	0	0	0	0	0	1	1	0	0	1	0	1
0	0	0	0	1	0	1	0	1	1	0	0	1
0	1	0	1	0	0	0	0	0	0	1	0	1
1	0	1	0	0	0	1	1	0	0	0	0	1
0	0	1	1	0	0	0	0	1	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	0	0	0	0	0	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0

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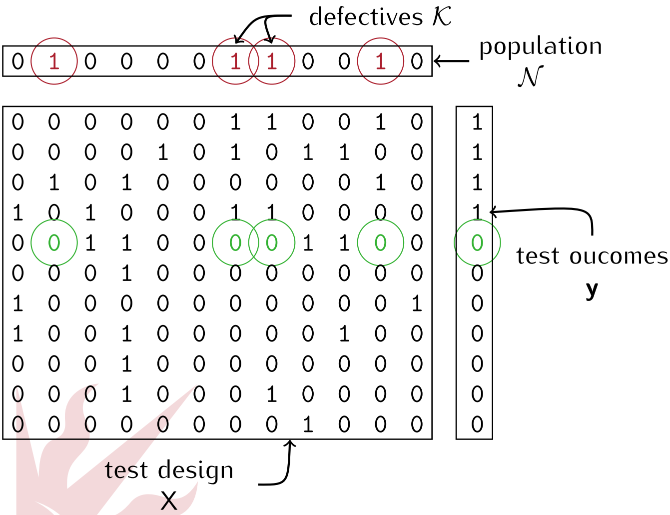

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0	1	0	0	0	0	1	1	0	0	1	0

test design X

0	0	0	0	0	0	1	1	0	0	1	0
0	0	0	0	1	0	1	0	1	1	0	0
0	1	0	1	0	0	0	0	0	0	1	0
1	0	1	0	0	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0	1	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1
1	0	0	1	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0

test oucomes **y**

1
1
1
1
0
0
0
0
0
0
0

$|\mathcal{N}| = N, |\mathcal{K}| = K$

$K = N^{1-\beta}, \ \beta > 0$

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0	1	0	0	0	0	1	1	0	0	1	0

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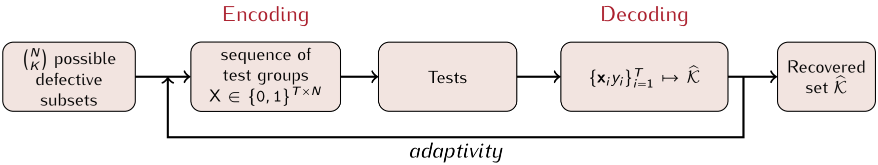
0	0	0	0	0	0	1	1	0	0	1	0
0	0	0	0	1	0	1	0	1	1	0	0
0	1	0	1	0	0	0	0	0	0	1	0
1	0	1	0	0	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0	1	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1
1	0	0	1	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0

test oucomes $\mathbf{y}$

1
1
1
1
0
0
0
0
0
0
0

particularly good for non-adaptive algorithms

Anatomy of an algorithm



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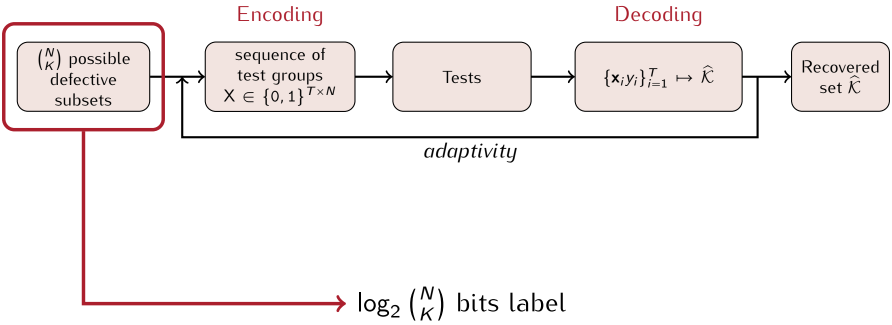

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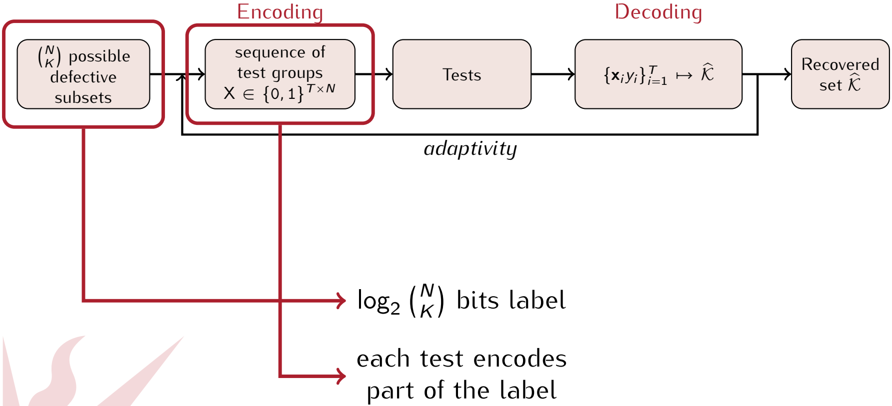

Rate of group testing (algorithms)

$$R_{\mathtt{A}} = \frac{\log_2 \binom{N}{K}}{T_{\mathtt{A}}} \text{ bits per test}$$

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$R^*_{\mathtt{A}}(\beta)$ supremum of rates for algorithm A. (Yes, it's bounded).

Why do we like rates?

Universal upper bound for noiseless GT

Theorem (Aldridge, B., Johnson, 2013)

The probability of success of any algorithm $\mathcal{A}$ can be upper-bounded as

$$\mathbb{P}(success) \leq \frac{2^{T_{\mathcal{A}}}}{\binom{N}{K}} \ .$$

Two important consequences

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  - $R = \dfrac{\log_2 \binom{N}{K}}{T}$ "ranks" optimal algorithms
- Successful algorithms have $R^*_{\mathtt{A}}(\beta) \leq 1$

Capacity of GT

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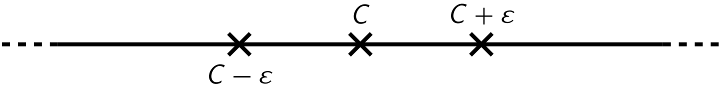


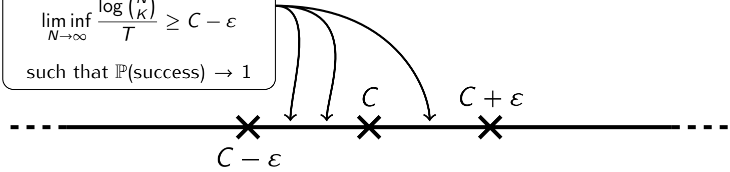
Capacity of GT

Direct part

There exists an algorithm with

$$\liminf_{N\to\infty} \frac{\log \binom{N}{K}}{T} \geq C - \varepsilon$$

such that $\mathbb{P}(\text{success}) \to 1$



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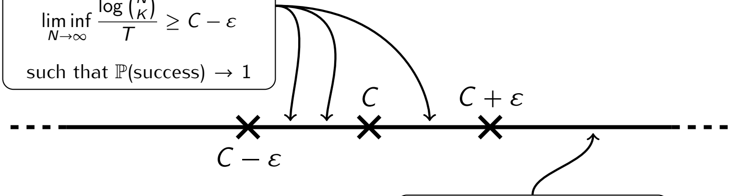

Any algorithm with

$$\liminf_{N\to\infty} \frac{\log \binom{N}{K}}{T} \geq C + \varepsilon$$

has $\mathbb{P}(\text{success}) < 1 - \eta$, $\eta > 0$

Converse part

Capacity of GT

Direct part

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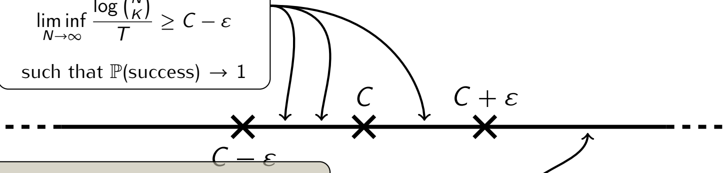

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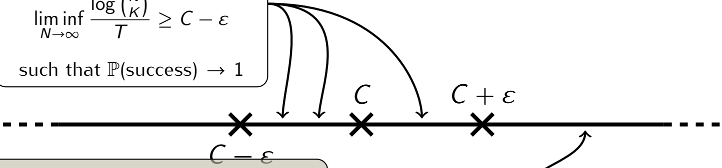

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- C measures the “hardness” of a GT problem

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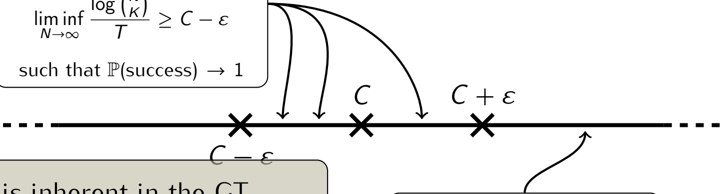

- C is inherent in the GT model
- C measures the "hardness" of a GT problem
- C quantifies an efficiency/effectiveness trade-off

Any algorithm with

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Converse part

Can we get to $R = 1$?

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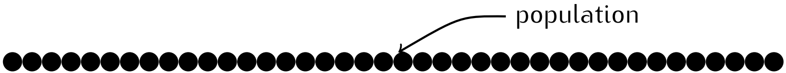

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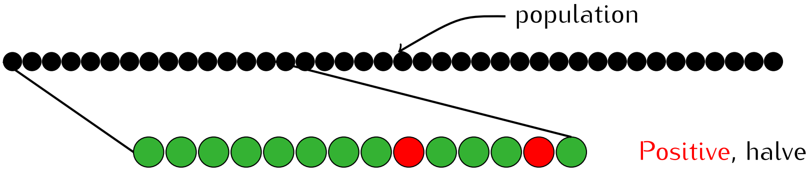

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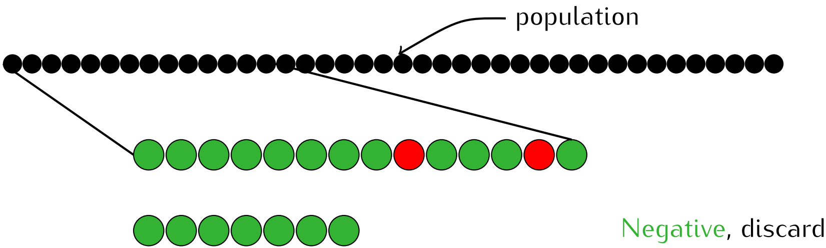

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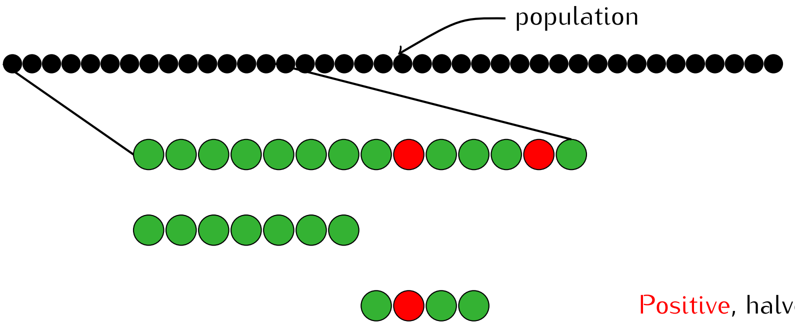
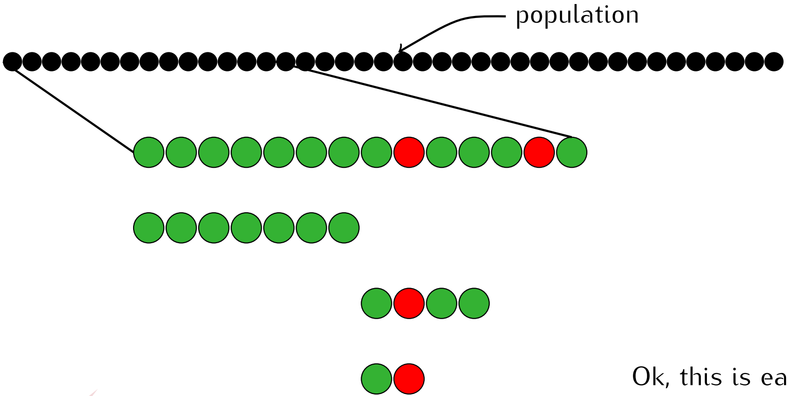


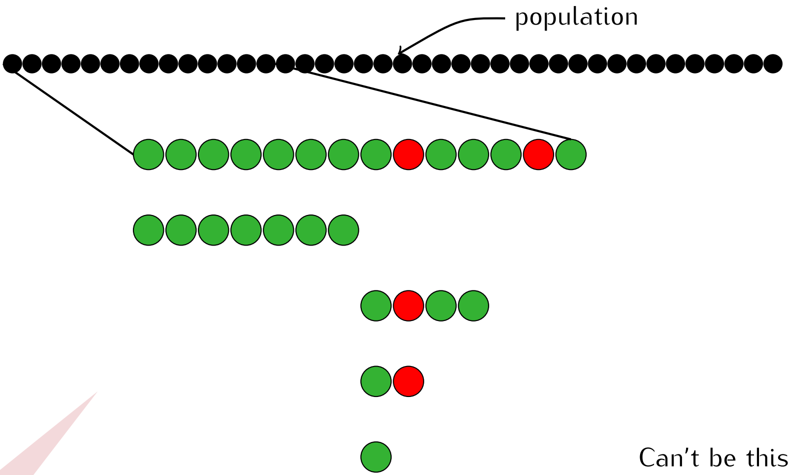
Positive, halve

Can we get to $R = 1$?



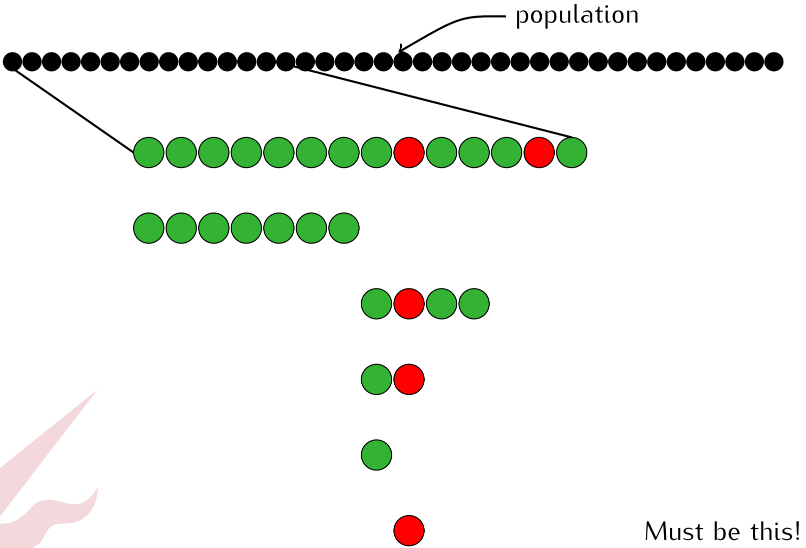
Ok, this is easy

Can we get to $R = 1$?



Can't be this...

Can we get to $R = 1$?



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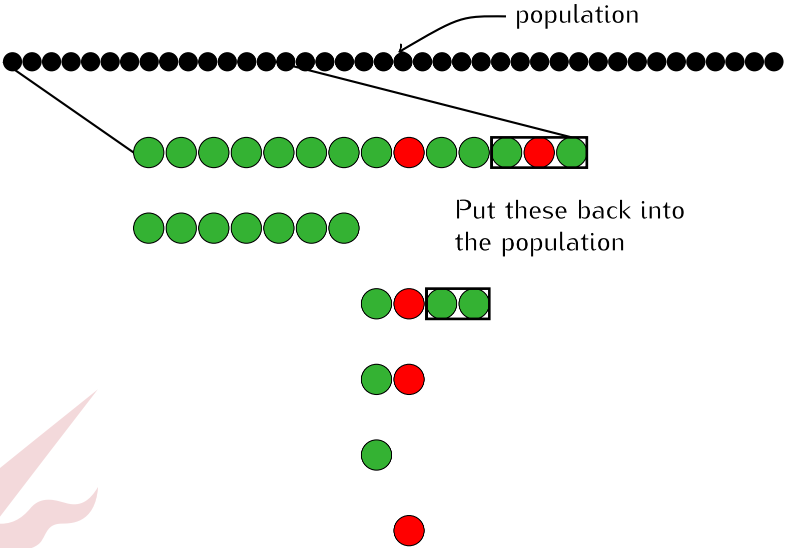

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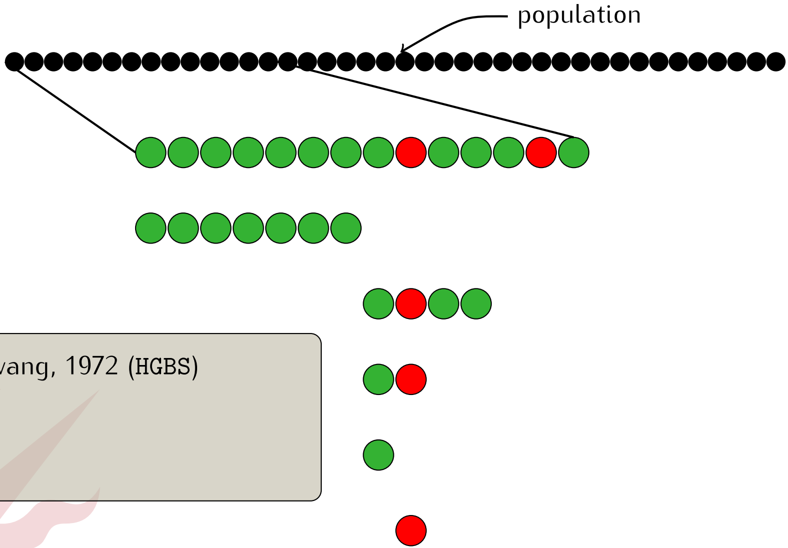
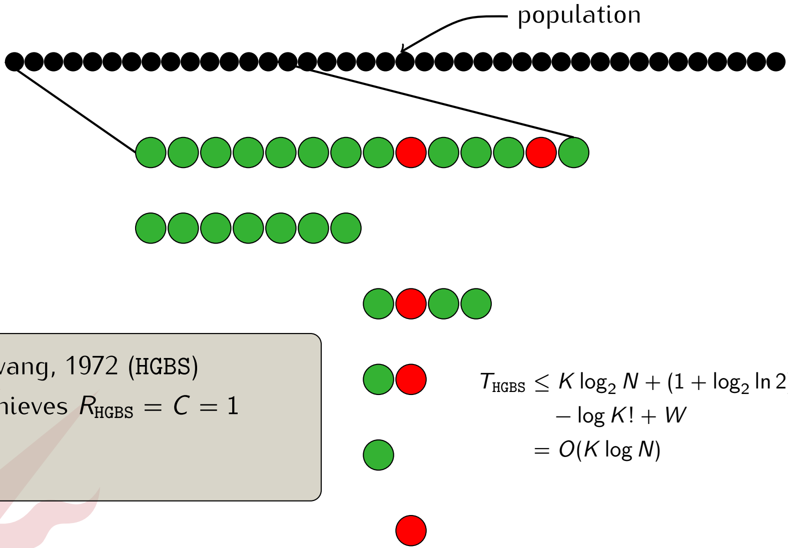


- Hwang, 1972 (`HGBS`)

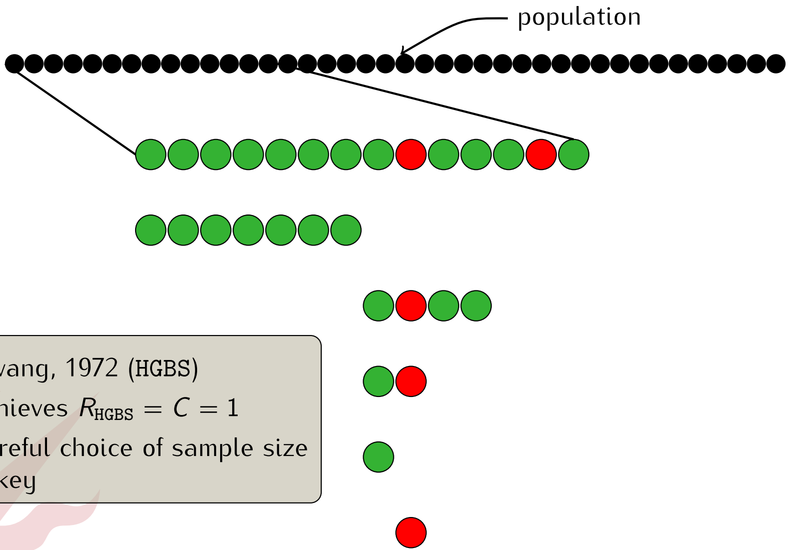
Can we get to $R = 1$?



- Hwang, 1972 (HGBS)
- Achieves $R_{\text{HGBS}} = C = 1$

$$\begin{aligned} T_{\text{HGBS}} &\leq K \log_2 N + (1 + \log_2 \ln 2)K \\ &\quad - \log K! + W \\ &= O(K \log N) \end{aligned}$$

Can we get to $R = 1$?



- Hwang, 1972 (HGBS)
- Achieves $R_{\texttt{HGBS}} = C = 1$
- Careful choice of sample size is key

Can we get there...non-adaptively?

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- Maybe

Can we get there...non-adaptively?

- Maybe
- What we did:

Can we get there...non-adaptively?

- Maybe
- What we did:
  - Introduced and studied `DD` (Definite Defectives) and `SSS` (Smallest Satisfying Set)
  - ...hence Bernoulli sampling, $x_{ij} \sim \text{Bern}(p)$, $p = \frac{1}{K+1}$
  - Showed limitation of Bernoulli-based algorithms

DD: Definite Defectives

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0	1	0	0	0	0	1	1	0	0	1	0

0	0	1	0	0	0	1	1	0	0	1	0
0	0	0	0	1	0	1	0	1	1	0	0
1	1	0	1	0	1	0	0	0	0	1	0
1	0	1	0	0	0	1	0	0	0	0	0
0	0	1	1	0	0	0	0	1	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1
1	0	0	1	0	0	0	0	0	1	0	0
0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0
0	0	0	0	0	1	0	0	1	0	0	0

1
1
1
1
0
0
0
0
0
0
0

1. X Bernoulli design.

DD: Definite Defectives

0	1	0	0	0	0	1	1	0	0	1	0

0	0	1	0	0	0	1	1	0	0	1	0	1
0	0	0	0	1	0	1	0	1	1	0	0	1
1	1	0	1	0	1	0	0	0	0	1	0	1
1	0	1	0	0	0	1	0	0	0	0	0	1
0	0	1	1	0	0	0	0	1	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	0	0	0	0	0	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	1	0	0	0	0

1. X Bernoulli design.
2. Look at negative tests...

DD: Definite Defectives

0	1	0	0	0	0	1	1	0	0	1	0

0	0	1	0	0	0	1	1	0	0	1	0	1
0	0	0	0	1	0	1	0	1	1	0	0	1
1	1	0	1	0	1	0	0	0	0	1	0	1
1	0	1	0	0	0	1	0	0	0	0	0	1
0	0	1	1	0	0	0	0	1	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	0	0	0	0	0	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	1	0	0	0	0

1. X Bernoulli design.
2. Look at negative tests…
3. … classify items therein as non-def. …

DD: Definite Defectives

0	1	0	0	0	0	1	1	0	0	1	0

0	0		0	0	0	1	1	0	0	1	0
0	0	0	0	1	0	1	0			0	0
	1	0		0		0	0	0	0	1	0
	0		0	0	0	1	0	0	0	0	0

1
1
1
1
0
0
0
0
0
0
0

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0	0		0	0	0	1	1	0	0	1	0	1
0	0	0	0	1	0	1	0			0	0	1
	1	0		0		0	0	0	0	1	0	1
	0		0	0	0	1	0	0	0	0	0	1
												0
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	1	0		0		0	0	0	0	1	0	1
	0		0	0	0	1	0	0	0	0	0	1
												0
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	1	0		0		0	0	0	0	1	0	1
	0		0	0	0	1	0	0	0	0	0	1
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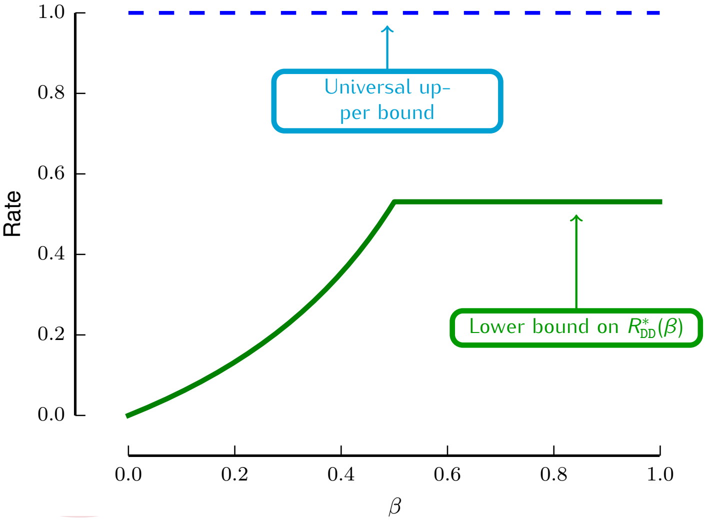
$$T_{\mathrm{DD}} \leq \max\{\beta, 1-\beta\}\, \mathrm{e} K \ln N$$

$$R^*_{\mathrm{DD}}(\beta) \geq \frac{1}{\mathrm{e}\ln 2} \min\left\{1, \frac{\beta}{1-\beta}\right\}$$

What $R^*_{\mathtt{DD}}(\beta)$ looks like



Computing $R^*_{\mathtt{DD}}(\beta)$: core of the proof - construction

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there's a sum in here, too!

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SSS: Smallest Satisfying Set

Group testing as LP:

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\text{subject to} \quad & \mathbf{x}_t \cdot \mathbf{z} = 0 \quad \text{for } t \text{ with } y_t = 0 \\
& \mathbf{x}_t \cdot \mathbf{z} \geq 1 \quad \text{for } t \text{ with } y_t = 1 \\
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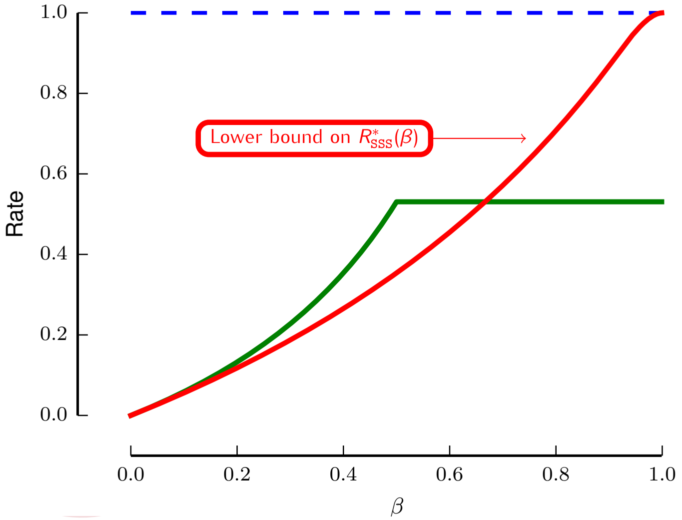
$$R^*_{\text{SSS}}(\beta) \geq \frac{1}{\ln 2} \max_{\alpha \in [\ln 2, 1]} \min \left\{ 2\alpha e^{-\alpha} \frac{\beta}{2-\beta}, -\ln\left(1 - 2e^{-\alpha} + 2e^{-2\alpha}\right) \right\} .$$

What $R^*_{\text{SSS}}(\beta)$ looks like



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Definition

X is K-separable if, for any K-subsets $\mathcal{L}, \mathcal{M} \subset \{1, \ldots, N\}$, it is

$$\bigvee_{i \in \mathcal{M}} \mathbf{x}^{(i)} \neq \bigvee_{j \in \mathcal{L}} \mathbf{x}^{(j)}$$

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Notice that $\mathbf{y} = \bigvee_{\mathcal{K}} \mathbf{x}^{(i)}$.

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Definition

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- Almost-separable matrices exist
- Need $T = O(K \log N)$ tests to get one
- A Bernoulli test design is almost-separable w.h.p. (via concentration)

Family-wise upper bound via SSS

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Theorem (Aldridge, B., Johnson, 2014)

Consider SSS *using* T_{SSS} *tests. Then,*

$$\mathbb{P}(\text{success}) \to 1 \Rightarrow T_{\text{SSS}} > \frac{(1-\beta)\mathrm{e}\ln 2}{\beta} \log_2 \binom{N}{K} ,$$

and, if the necessary condition is violated,

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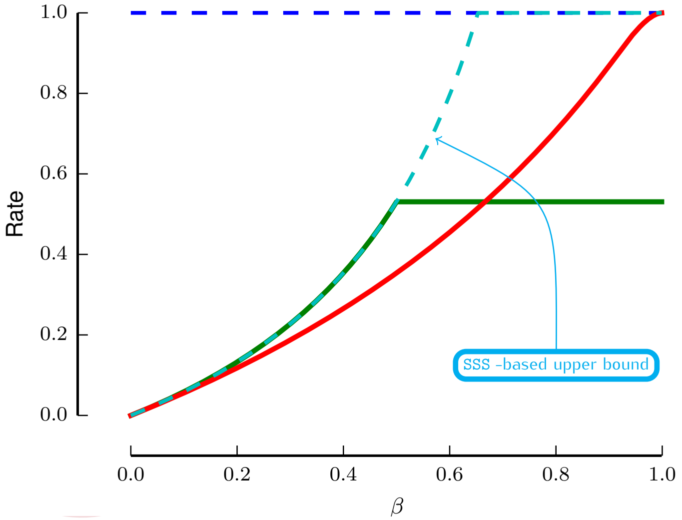
Adaptivity gap



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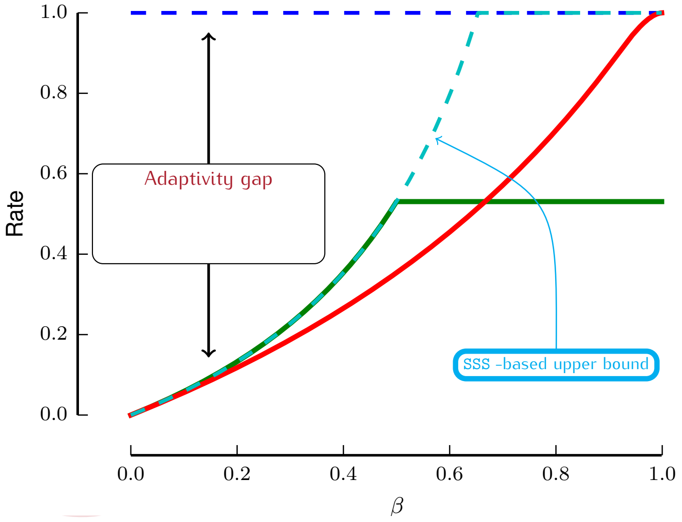

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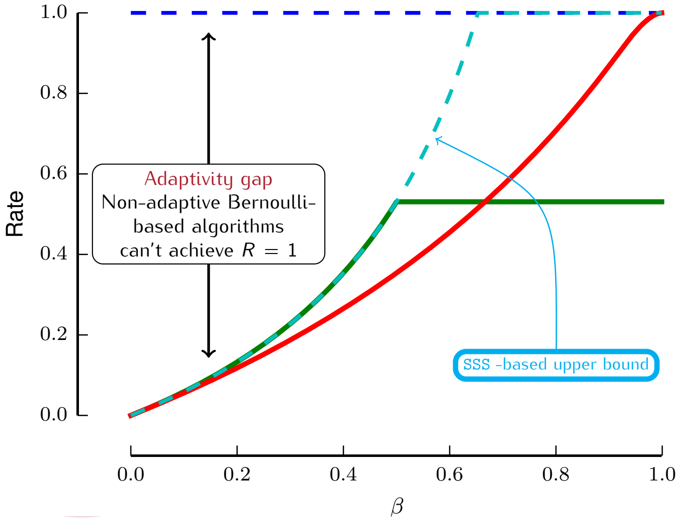

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- Future work: noise models, applications, non-identical GT, non-independent GT, collateral open questions...