Tracy-Widom-beta distributions from the Stochastic Airy Operator point of view

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  - First for $\beta = 1, 2, 4$: Gaussian ensembles GOE, GUE, GSE (Tracy and Widom, 1994, 1996)
  - For every $\beta > 0$: β-ensembles (Ramírez, Rider, Virág, 2006).

Tracy-Widom and β-ensembles

The β-Tracy-Widom distribution is the limit distribution (when the dimension tends to infinity) of the largest eigenvalue of β-ensembles, introduced by Dumitriu, Edelman in 2002.

$$H_N^\beta := \frac{1}{\sqrt{\beta}} \begin{pmatrix} \sqrt{2}\, g_1 & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & \sqrt{2}\, g_2 & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \sqrt{2}\, g_{N-1} & \chi_\beta \\ & & & \chi_\beta & \sqrt{2}\, g_N \end{pmatrix}$$

g_k: independent $\mathcal{N}(0,1)$, $\chi_{k\beta}$: independent χ distributed.

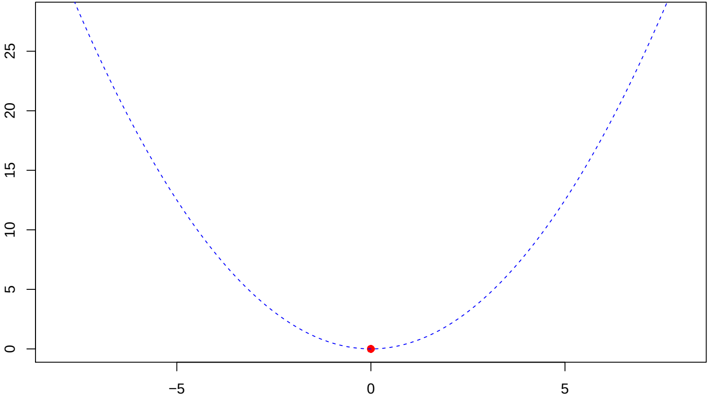
Eigenvalues of β-ensembles

Their eigenvalues represent charged particles in a one-dimensional Coulomb gas with electrostatic repulsion at temperature $T = 1/\beta$ for arbitrary $\beta > 0$.

$$P_\beta(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N^\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp(-\frac{1}{4}\beta \sum_{i=1}^N \lambda_i^2) \,.$$

For the special values $\beta = 1, 2, 4$, joint law of the eigenvalues of the classical Gaussian orthogonal/unitary/symplectic ensembles.

Dyson Brownian motion (dynamics of the Coulomb gas)



Behavior of the eigenvalues of β-ensembles

When $N \to \infty$, the empirical measures of the eigenvalues of β-ensembles renormalized by $\sqrt{N}$ converge to the celebrated Wigner semi-circle law.

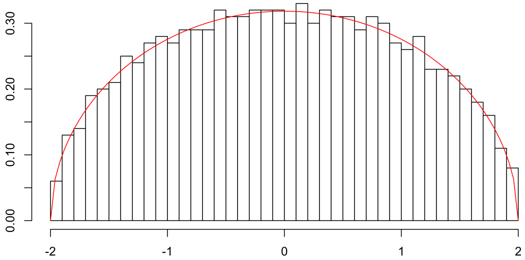


Figure : Wigner semi-circle and the histogram of eigenvalues of a simulation of a GOE of size $N = 1000$.

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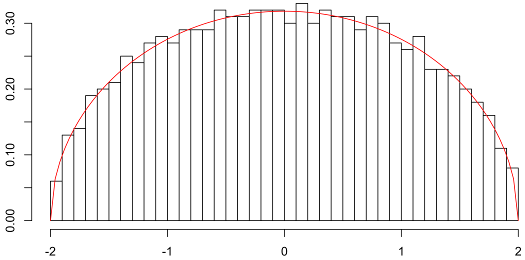

Figure : Wigner semi-circle and the histogram of eigenvalues of a simulation of a GOE of size $N = 1000$.

We will look at the edge of the spectrum.

Random matrices and Stochastic Operators

Idea (Sutton (2005) and Edelman and Sutton (2006)):

Tridiagonal matrices are discrete differential operators. At the limit: it should become a random differential operator.

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Mathematically proved by Ramírez, Rider and Virág (2006).

Stochastic Airy Operator (SAO)

Introduce the Stochastic Airy Operator (SAO):

$$\mathcal{H}_\beta := -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} B'$$

where B' is a white noise on $\mathbb{R}_+$.

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where B' is a white noise on $\mathbb{R}_+$.

We expect that the renormalized operator

$$\tilde{H}_N^\beta := N^{2/3} \left(2 I_N - \frac{H_N^\beta}{\sqrt{N}} \right)$$

converges to the SAO $\mathcal{H}_\beta$.

Eigenvalues of the SAO

We will say that $(\phi, \lambda) \in H_1 \times \mathbb{R}$ is an eigenfunction/ eigenvalue pair for $\mathcal{H}_\beta$ if $\int_0^\infty (\phi')^2 + (1+t)\phi^2 < \infty$, if $\phi(0) = 0$ and if

$$\phi''(t) = (t - \lambda)\phi(t) + \frac{2}{\sqrt{\beta}}\,\phi(t)\,B'_t$$

holds for all $t \geq 0$ in the integration by part sense.

Convergence of the β-ensembles to the SAO

Theorem (Ramírez, Rider, Virág, 2006)

A.s., for each $k \geq 0$, the set of eigenvalues of $\mathcal{H}_\beta$ has a well-defined $(k+1)$-st lowest element Λ_k^β.

Moreover, let $\lambda_{\beta,1} \geq \lambda_{\beta,2} \geq \cdots$ denote the eigenvalues of the β-ensemble H_N^β. Then

$$N^{1/6}(2\sqrt{N} - \lambda_{\beta,l})_{l=1,\cdots,k}$$

converges in distribution as $N \to \infty$ to

$$(\Lambda_0^\beta, \Lambda_1^\beta, \cdots, \Lambda_{k-1}^\beta).$$

Convergence of the β-ensembles to the SAO

In particular, we have for the largest eigenvalue $\lambda_{\beta,1}$: When $N \to \infty$:

$$N^{1/6}\left(\lambda_{\beta,1} - 2\sqrt{N}\right) \Rightarrow \mathsf{TW}_\beta := -\Lambda_0^\beta$$

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For $\beta = 1, 2, 4$, it was already known: result of Tracy and Widom (1994, 1996). The limit law is moreover characterized in terms of the second Painlevé transcendent.

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For general β, Ramírez, Rider, Virág give a characterization in terms of a variational formula (thanks to the equivalence with the lowest eigenvalue of the SAO).

Riccati diffusion associated to the SAO

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The diffusion X_a may blow-up to $-\infty$ in a finite time. In this case: immediately restarted from $+\infty$. In this way, X_a is defined for all time.

Eigenvalues and the diffusion

Ramírez, Rider and Virág: Number of eigenvalues of the SAO $\mathcal{H}_\beta$ at most $-a$ = total number of explosions of the diffusion $(X_a(t))$ on $\mathbb{R}_+$.

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Immediate corollary: The marginal laws of the eigenvalues are characterized in a simple way. In particular:

$$\begin{aligned}\mathbb{P}\left[TW_\beta > a\right] &:= \mathbb{P}\left[\Lambda_0^\beta < -a\right] \\ &= \mathbb{P}\left[X_a(t) \text{ blows up to } -\infty \text{ in a finite time}\right].\end{aligned}$$

$\#$ of eigenvalues at most $-a = \#$ of explosions of X_a

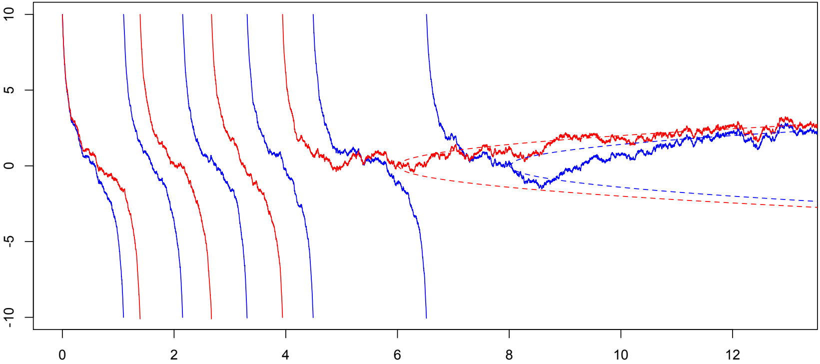

Figure : X_a and $X_{a'}$ (driven by the same Brownian motion) with $a = -8$ and $a' = -6$, for $\beta = 4$. On this event, we have $\Lambda_4^\beta < 8 < \Lambda_5^\beta$ and $\Lambda_2^\beta < 6 < \Lambda_3^\beta$.

First explosion for different values of a

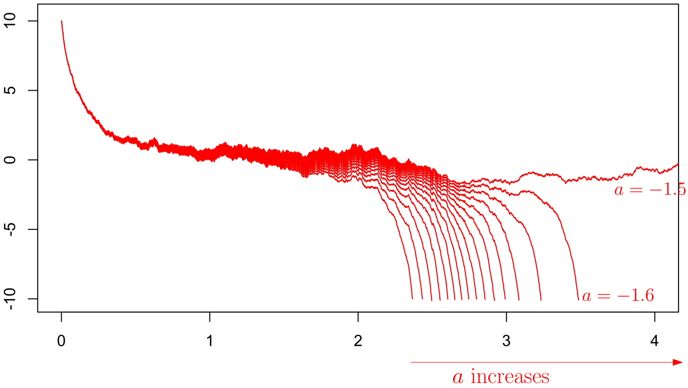

Figure : The first explosion of X_a for several values of a (driven by the same Brownian motion) for $\beta = 4$. Here, $1.5 < -TW(4) < 1.6$.

Trapping a diffusion in a stationary well

We consider a simpler (without the t-term in the drift) diffusion $(Y_a(t))_{t\geq 0}$ defined by

$$\begin{cases} dY_a(t) = (a - Y_a(t)^2)\,dt + dB(t) \quad \text{for} \quad t \geq 0\,, \\ Y_a(0) = y\,. \end{cases}$$

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The diffusion evolves in a potential $V(y) := -ay + \frac{y^3}{3}$. Potential barrier: $\Delta V = \frac{4}{3}\, a^{3/2}$.

Trapping a diffusion in a stationary well

Let

$$m(a) := \sqrt{2\pi} \int_0^\infty \frac{dv}{\sqrt{v}} \exp\left(2av - \frac{1}{6}v^3\right)$$
$$\sim_{a\to\infty} \frac{\pi}{a^{1/2}} \exp\left(\frac{8}{3}a^{3/2}\right).$$

Proposition (First exit time of Y_a starting in the well)

Let $f : \mathbb{R} \to \mathbb{R}$ such that $a^{1/4}(f(a) + a^{1/2}) \to_{a\to+\infty} +\infty$.
For any $y \geq f(a)$, denote by ζ the first blowup time to $-\infty$ of the diffusion $(Y_a(t))$ starting from position y. Then

$$\frac{\zeta}{m(a)} \Rightarrow_{a\to\infty} \mathcal{E}(1)$$

where $\mathcal{E}(1) \sim$ exponential of parameter 1.

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Possible proof: Convergence of the Laplace transform of the first exit time (it satisfies a boundary value problem).

Trapping a diffusion in a stationary well

We introduce the empirical measure of the explosion times denoted $\zeta_1 < \zeta_2, \cdots$ for the process Y_a starting from $+\infty$ and restarted to $+\infty$ immediately after each explosion:

$$\mu_a = \sum_{i=1}^{\infty} \delta_{\zeta_i / m(a)}.$$

Corollary

The measures μ_a converges to a Poisson point process with intensity 1 in $\mathbb{R}_+$ when $a \to \infty$.

The right tail of the Tracy-Widom distribution

(joint work with Bálint Virág, 2010)

The right tail of Tracy-Widom

Theorem (D., Virág, 2010)

When $a \to \infty$, we have

$$\mathbb{P}(TW_\beta > a) = a^{-3\beta/4} \exp\left(-\frac{2}{3}\beta a^{3/2} + O(\sqrt{\ln(a)})\right)$$

Note that Borot and Nadal, using different heuristical methods, obtained in 2011 and 2012 the whole asymptotic expansion of Tracy-Widom for both the left and the right tail.

Idea of the proof: exit time problem

Recall that

$$\mathbb{P}\left[TW_\beta > a\right] = \mathbb{P}\left[X_a(t)\,\text{blows up to } -\infty \text{ in a finite time}\right].$$

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- Main cost: crossing the interior of the parabola (where the drift upwards is huge).
- The term $+t$ is crucial: competition between the time it would take for the analogous stationary diffusion to cross a large barrier, and the drift which increases throughout time.
- Conditionally on its existence, the blowing-up to $-\infty$ will happen very quickly (cost becomes much higher in a small amount of time).

Idea of the proof: exit time problem

Strategy of the proof:

1. Obtain a precise control of the time it takes to go down to the upper part of the parabola. Comparison with ODE (without the noise).
2. Analysis of the cost to cross the parabola: Girsanov formula. New diffusion: diffusion conditioned to blow up in a finite time. In view of the stationary case, choose diffusion Y with a reversed drift plus a correction term denoted by φ.

$$\begin{aligned} Y(0) &= +\infty \\ dY(t) &= \left(-t - a + Y(t)^2 + \varphi(Y_t)\right) \, dt \end{aligned}$$

3. Under the parabola: similar point 1.

From Tracy-Widom to Gumbel

(joint work with Romain Allez, 2013)

From Tracy-Widom to Gumbel

Recall the β-ensemble introduced before:

$$\frac{1}{\sqrt{\beta}} \begin{pmatrix} \sqrt{2}\, g_1 & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & \sqrt{2}\, g_2 & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \sqrt{2}\, g_{N-1} & \chi_{\beta} \\ & & & \chi_{\beta} & \sqrt{2}\, g_N \end{pmatrix}$$

From Tracy-Widom to Gumbel

When $\beta = 0$,

$$\begin{pmatrix} \sqrt{2}\, g_1 & & & & \\ & \sqrt{2}\, g_2 & & & \\ & & \ddots & & \\ & & & \sqrt{2}\, g_{N-1} & \\ & & & & \sqrt{2}\, g_N \end{pmatrix}$$

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Behaviour of the largest eigenvalue $g_{N,\max}$ when $N \to \infty$:

$$\sqrt{\ln(N)}\Big(g_{N,\max} - 2\sqrt{\ln(N)} - \frac{1}{8}\frac{\ln(\pi \ln(N))}{\sqrt{\ln(N)}}\Big) \Rightarrow \text{Gumbel} = e^{-x}e^{-e^{-x}}\,dx$$

From Tracy-Widom to Gumbel

When $\beta > 0$,

$$\frac{1}{\sqrt{\beta}} \begin{pmatrix} \sqrt{2}\, g_1 & \chi_{(N-1)\beta} & & & \\ \chi_{(N-1)\beta} & \sqrt{2}\, g_2 & \chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \sqrt{2}\, g_{N-1} & \chi_{\beta} \\ & & & \chi_{\beta} & \sqrt{2}\, g_N \end{pmatrix}$$

Recall we have: When $N \to \infty$:

$$N^{1/6}\left(\lambda_{\beta,1} - 2\sqrt{N}\right) \Rightarrow \mathrm{TW}_{\beta}$$

From Tracy-Widom to Gumbel

Morally, we have "Gumbel $= TW_0$" (mind the different scalings!).

Is there an interpolation between Tracy-Widom-beta distribution and Gumbel distribution? How does the transition occur?

From Tracy-Widom to Gumbel

Theorem (Allez, D., 2013)

When properly rescaled and centered, the Tracy-Widom-β law converges weakly to the Gumbel law. More precisely, when $\beta \to 0$, the random variable

$$2 \cdot 3^{1/3} \cdot \left(\ln \frac{1}{\beta} \right)^{1/3} \left[\left(\frac{\beta}{4} \right)^{2/3} \mathbf{TW}(\beta) - \left(\frac{3}{8} \right)^{2/3} \left(\ln \frac{1}{\beta \pi} \right)^{2/3} \right]$$

converges in law towards the Gumbel distribution.

Sketch of the proof

First let us make a convenient change of variable: we will study instead of X_a

$$Z_a(0) = +\infty$$
$$Z_a(t) = \left(a + \frac{\beta}{4}t - Z_a^2(t)\right)dt + dB(t)$$

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We have now the equality:

$$\mathbb{P}\left(\left(\frac{\beta}{4}\right)^{2/3} TW_\beta > a\right) = \mathbb{P}(\text{there exists at least one explosion for } Z_a)$$

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If $\beta \to 0$ while a is fixed: Z_a blows-up to $-\infty$ infinitely often a.s.

We need to find the correct speed $a := a_\beta$ such that the limit is non trivial.We should have: $a_\beta \to \infty$ when $\beta \to 0$.

Sketch of the proof

Fix $x \in \mathbb{R}$ and let us choose

$$a := a_\beta(x) := \frac{3}{8}\ln\left(\frac{1}{\pi\beta}\right) - \frac{1}{2}\frac{1}{3^{1/3}}\left(\ln\left(\frac{1}{\beta}\right)\right)^{-1/3} x$$

Notice that

$$m(a_\beta(x)) \sim_{\beta\to 0} \beta^{-1}\left(\frac{3}{8}\ln\left(\frac{1}{\beta}\right)\right)^{-1/3} e^{-x}$$

Suppose that the time-term in the drift is negligible. After a time of the order $s := \beta^{-1}(3/8\ln(1/\beta))^{-1/3}\, t$, the diffusion has a positive probability to explode and the drift becomes:

$$a_\beta(x) + \frac{1}{\beta}\Big(\frac{3}{8}\ln(\frac{1}{\beta})\Big)^{-1/3} t \times \frac{\beta}{4} - Z_a^2(s) \sim a_\beta(x-t) - Z_a^2(s)$$

Consider the point process defined by the explosion times of Z_{a_β}, renormalized by $\beta^{-1}\left(\frac{3}{8}\ln\left(\frac{1}{\beta}\right)\right)^{-1/3}$ i.e.

$$\nu_\beta := \sum_{i=1}^{\infty} \delta_{\beta\left(\frac{3}{8}\ln\left(\frac{1}{\beta}\right)\right)^{1/3}\zeta_i}$$

Proposition (Allez, D.)

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It is then straightforward to deduce the convergence result for the TW_β distribution (and we can extend this proof to deduce the convergence result for the k-th largest eigenvalues).

Some related open questions

- Limiting joint law of the 1st/2nd/3rd/$\cdots$ largest eigenvalues? We proved that the marginal density converges to the marginals of the inhomonegeous PPP. But open question for the joint law (it should be Poissonian).
- What happens for the initial random matrix model? We conjecture that in the double limit $N \to \infty$ and $\beta_N \to 0$ such that $N\beta_N \gg \ln(N)$, the largest eigenvalue properly rescaled converges to the Gumbel distribution (difficulty here: the limit object is no longer a stochastic operator).

THANK YOU !