Stochastic orders in stochastic networks

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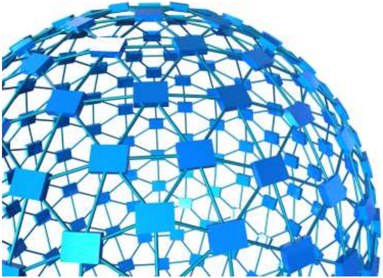
Stochastic dynamics on complex systems



$$\mathsf{E}\, f(X(t)) \;=\; ?$$

Analysis methods

- Stochastic simulation
- Scaling approximations and limit theorems
- Stochastic comparison and coupling

Outline

Stochastic comparison approach

$$\mathsf{E}\, f(X(t)) \;=\; ?$$

Find a reference model $Y(t)$ which

- Performs worse than $X(t)$
- Can be proven to do so analytically
- Is computationally tractable

$\rightsquigarrow$ *Computable & conservative performance estimates*

$\rightsquigarrow$ *Sufficient conditions for stochastic stability*

Stochastic ordering

How to define X less than Y for random variables?

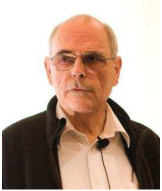
Strong order: $X \leq_{\mathrm{st}} Y$ if

$$\mathsf{E}\, f(X) \leq \mathsf{E}\, f(Y)$$

for all increasing test functions f

- This definition extends to random variables with values in a complete separable metric (=Polish) space with a closed partial order $(S, \leq)$

Strassen's coupling theorem



Theorem (Strassen 1965)

Two random variables on a complete separable metric space equipped with a closed partial order satisfy $X \leq_{\mathrm{st}} Y$ if and only if they admit a coupling $(\hat{X}, \hat{Y})$ such that $\hat{X} \leq \hat{Y}$ almost surely.

A coupling of random variables X and Y is a bivariate random variable $(\hat{X}, \hat{Y})$ such that:

- $\hat{X}$ has the same distribution as X
- $\hat{Y}$ has the same distribution as Y

Stochastic relations

Any meaningful distributional relation should have a coupling counterpart (Thorisson 2000).

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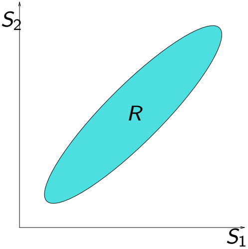


A relation is an arbitrary subset $R \subset S_1 \times S_2$

- Denote $x \sim y$ if $(x, y) \in R$
- Random variables X and Y are related by $X \sim_{\mathrm{st}} Y$ if they admit a coupling $(\hat{X}, \hat{Y})$ such that $\hat{X} \sim \hat{Y}$ almost surely.

$\rightsquigarrow$ *Coupling allows to define a randomized version an arbitrary relation*

Examples of stochastic relations

St. equality Let $=_{\mathrm{st}}$ be the stochastic relation generated by the equality $=$. Then $X =_{\mathrm{st}} Y$ if and only if X and Y have the same distribution.

St. order Let $\leq_{\mathrm{st}}$ be the stochastic relation generated by a partial order $\leq$. Then $X \leq_{\mathrm{st}} Y$ corresponds to the usual strong stochastic order.

St. ϵ-distance Define $x \approx y$ by $|x - y| \leq \epsilon$. Two real random variables satisfy $X \approx_{\mathrm{st}} Y$ if and only if for all x the corresponding c.d.f.'s satisfy
$F_Y(x - \epsilon) \leq F_X(x) \leq F_Y(x + \epsilon)$.

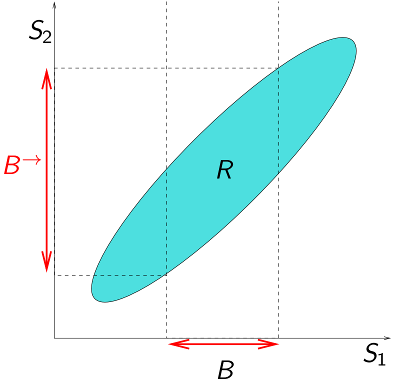
Functional characterization

Theorem

For any closed relation $\sim$ between complete separable metric spaces, $X \sim_{\mathrm{st}} Y$ is equivalent to both:

(i) $\mathrm{P}(X \in B) \leq \mathrm{P}(Y \in B^{\rightarrow})$ *for all compact* $B \subset S_1$

(ii) $\mathrm{E}\, f(X) \leq \mathrm{E}\, f^{\rightarrow}(Y)$ *for all upper semicontinuous compactly supported* $f : S_1 \rightarrow \mathbb{R}_+$



- $B^{\rightarrow} = \cup_{x_1 \in B} \{x_2 \in S_2 : x_1 \sim x_2\}$ is the set of points in S_2 related to a point in B
- $f^{\rightarrow}(x_2) = \sup_{x_1 : x_1 \sim x_2} f(x_1)$ is the supremum of f over points related to x_2

Outline

Stochastic orders and relations

Stochastic ordering of network populations

Stochastic ordering of network flows

Stochastic boundedness

Stochastic ordering of network populations

$\mathrm{E}\, f(X(t)) = ?$

Problem

Can we show that Markov processes X and Y satisfy

$$\lim_{t\to\infty} f(X(t)) \leq_{\mathrm{st}} \lim_{t\to\infty} f(Y(t))$$

without calculating the limiting distributions?

Stochastic ordering of network populations

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Assumptions and notation

- Countable state space S
- Continuous time
- $Q(x, y)$ is the rate of transition for $x \mapsto y$, and

$$Q(x, B) = \sum_{y \in B} Q(x, y)$$

is the aggregate rate of transitions from x into $B \subset S$

A sufficient condition

Theorem (Whitt 1986, Massey 1987)

The property $\lim_{t\to\infty} X_1(t) \le_{\mathrm{st}} \lim_{t\to\infty} X_2(t)$ *holds if the corresponding transition rate kernels satisfy for all* $x \le y$*:*

(i) $Q_1(x, B) \le Q_2(y, B)$ *for all upper sets* B *such that* $x, y \notin B$

(ii) $Q_1(x, B) \ge Q_2(y, B)$ *for all lower sets* B *such that* $x, y \notin B$

Notation

- A set is upper if its indicator function is increasing
- A set is lower if its indicator function is decreasing

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The above Whitt–Massey condition is not sharp in general
$\rightsquigarrow$ *Can we do any better?*

Markov coupling

A transition rate kernel Q on $S_1 \times S_2$ is a coupling of transition rate kernels Q_1 on S_1 and Q_2 on S_2 if

$$Q(x, B_1 \times S_2) = Q_1(x_1, B_1)$$
$$Q(x, S_1 \times B_2) = Q_2(x_2, B_2)$$

for all $x = (x_1, x_2)$, B_1 and B_2 such that $x_1 \notin B_1$ and $x_2 \notin B_2$

Andrei Markov (1856–1922)
St Petersburg University

Andrei Markov (1978–)
Montreal Canadiens

Markov coupling $\Longrightarrow$ path coupling

Theorem (Mu-Fa Chen 1986)

Let Q be a kernel that couples two nonexplosive kernels Q_1 and Q_2. Then Q is nonexplosive, and for all $x = (x_1, x_2) \in S$, the Markov process $X(x, \cdot)$ generated by Q couples the Markov processes $X_1(x_1, \cdot)$ and $X_2(x_2, \cdot)$ generated by Q_1 and Q_2.

- $X(x, \cdot)$ denotes the path of a Markov process started at x

Stochastic relations of Markov processes

A pair of Markov processes stochastically preserves a relation R if

$$x \sim y \quad \implies \quad X(x,t) \sim_{\mathrm{st}} Y(y,t) \ \text{ for all } t,$$

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$$x \le y \quad \implies \quad X(x,t) \le_{\mathrm{st}} X(y,t) \ \text{ for all } t.$$

- X is a stochastically distance-preserving if

$$x \approx y \quad \implies \quad X(x,t) \approx_{\mathrm{st}} X(y,t) \ \text{ for all } t.$$

Relation preservation

Theorem

For nonexplosive Markov jump processes, the following are equivalent:

(i) X_1 *and* X_2 *stochastically preserve the relation* R.

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(iii) *For all* $x_1 \sim x_2$, *the rate kernels* Q_1 *and* Q_2 *satisfy*

$$Q_1(x_1, B_1) \leq Q_2(x_2, B_1^{\rightarrow})$$

for all measurable B_1 *such that* $x_1 \notin B_1$ *and* $x_2 \notin B_1^{\rightarrow}$, *and*

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Open problem

Is it enough to look at compact B_1 and B_2?

Stochastic subrelations

Recall our starting point:

Problem

Can we show that Markov processes X_1 and X_2 satisfy

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- The Whitt–Massey condition requires that X_1 and X_2 stochastically preserve the order relation $R_{\leq} = \{(x,y) : x \leq y\}$.
- What about preserving a subrelation of $R_{\leq}$?

Less stringent sufficient condition

Theorem

If (irreducible, positive recurrent) Markov processes X_1 and X_2 stochastically preserve a nontrivial subrelation R of $R_{\leq}$, then $\lim_{t \to \infty} X_1(t) \leq_{\mathrm{st}} \lim_{t \to \infty} X_2(t)$.

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Proof.

- Fix $x = (x_1, x_2) \in R$, and let $\hat{X}(x, \cdot)$ be a Markov coupling of $X_1(x_1, \cdot)$ and $X_2(x_2, \cdot)$ for which R is invariant.

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- Then $\hat{X}_1(x, t) \sim \hat{X}_2(x, t)$ almost surely for all t, so that

$$\lim_{t\to\infty} X_1(t) =_{\mathrm{st}} \lim_{t\to\infty} \hat{X}_1(x, t) \sim \lim_{t\to\infty} \hat{X}_2(x, t) =_{\mathrm{st}} \lim_{t\to\infty} X_2(t).$$

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- $\implies \lim_{t\to\infty} X_1(t) \sim_{\mathrm{st}} \lim_{t\to\infty} X_2(t)$

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- $\implies \lim_{t\to\infty} X_1(t) \sim_{\mathrm{st}} \lim_{t\to\infty} X_2(t)$
- $\implies \lim_{t\to\infty} X_1(t) \leq_{\mathrm{st}} \lim_{t\to\infty} X_2(t)$ because $R \subset R_{\leq}$

□

Subrelation algorithm

How to find a good subrelation (does it exist)?

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Given a relation R and transition rate kernels Q_1 and Q_2, define a sequence of relations by $R^{(0)} = R$,

$$R^{(n+1)} = \left\{ (x, y) \in R^{(n)} : \ (Q_1(x, \cdot), Q_2(y, \cdot)) \in R_{\mathrm{st}}^{(n)} \right\},$$

where $(Q_1(x, \cdot), Q_2(y, \cdot)) \in R_{\mathrm{st}}^{(n)}$ means that (Q_1, Q_2) preserves the stochastic relation generated by $R^{(n)}$ locally at (x, y).

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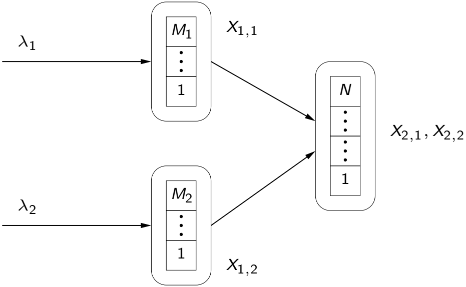
Theorem

The relation $R^* = \bigcap_{n=0}^{\infty} R^{(n)}$ is the maximal subrelation of R that is stochastically preserved by (Q_1, Q_2). Especially, the pair (Q_1, Q_2) preserves a nontrivial subrelation of R if and only if $R^* \neq \emptyset$.

Application: Call center

- M_1 English-speaking agents
- M_2 French-speaking agents
- N bilingual agents



Service rate (in calls/min) in state X equals $X_{1,1} + X_{1,2} + X_{2,1} + X_{2,2}$

Application: Call center

Does training improve performance?

Modified system $Y = (Y_{1,1}, Y_{1,2}; Y_{2,1}, Y_{2,2})$

- Replace one English-speaking agent by a bilingual agent
- Can we show that $\sum_{i,k} X_{i,k} \le_{\mathrm{st}} \sum_{i,k} Y_{i,k}$ in steady state?

Define the relation $x \sim y$ by $\sum_{i,k} x_{i,k} \le \sum_{i,k} y_{i,k}$.

- $\sim$ is not an order (different state spaces)
- X and Y do not preserve $\sim_{\mathrm{st}}$
- But maybe (X, Y) preserves some subrelation of $\sim_{\mathrm{st}}$?

Application: Call center

Numerical example

- Available call agents: 3 English, 2 French, 2 bilingual
- Calls arrive at rates 1 (English) and 2 (French) per min
- Mean call duration is 1 min

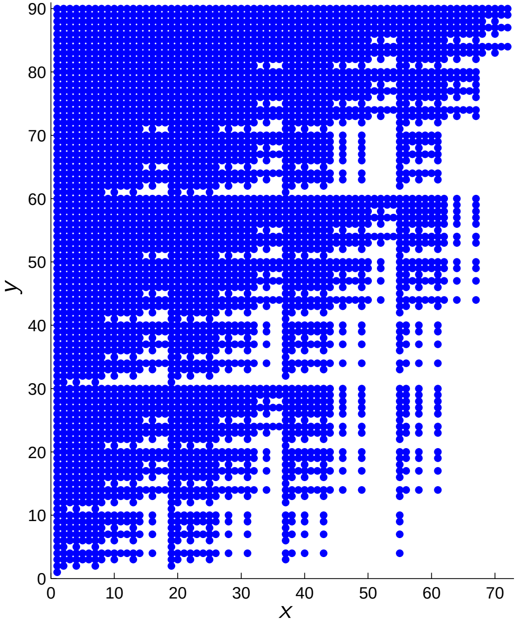
How many iterations do we need to compute R_∞?

- X has 72 possible states
- Y has 90 possible states

STOCHREL v1.0 – A Matlab stochastic relations package

http://www.iki.fi/lsl/software/stochrel/

Application: Call center



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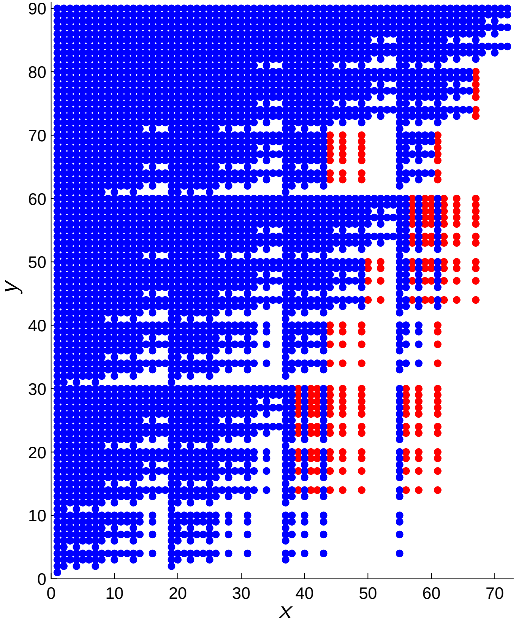

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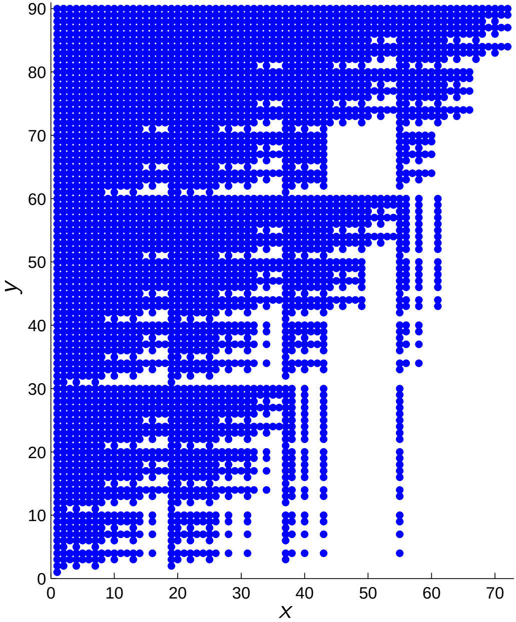

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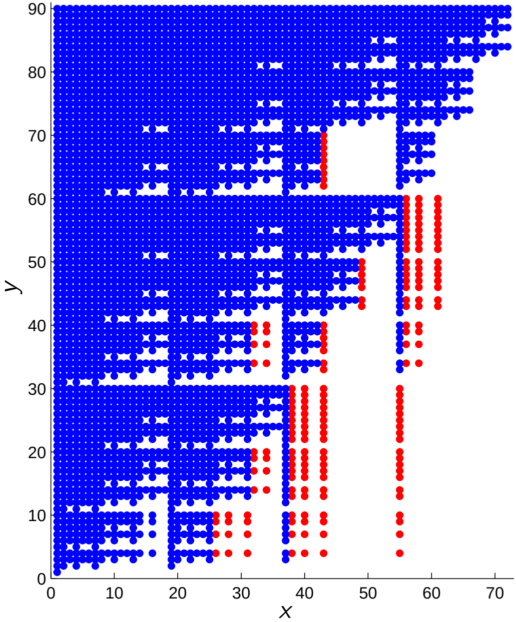

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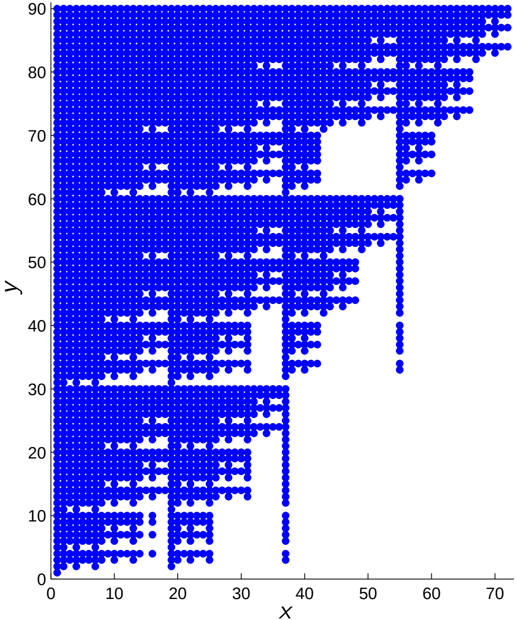

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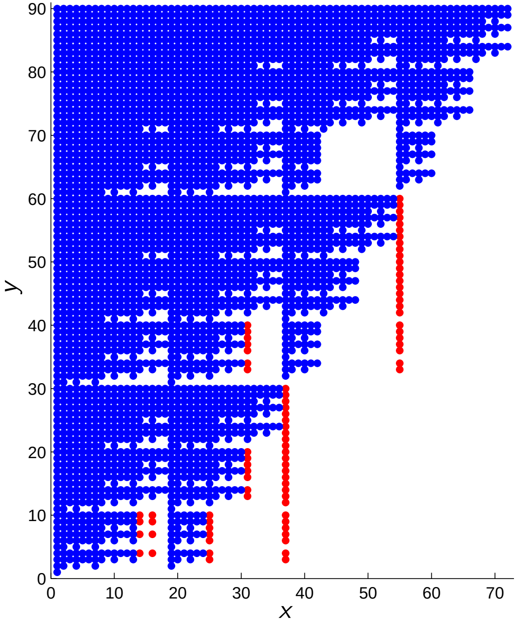

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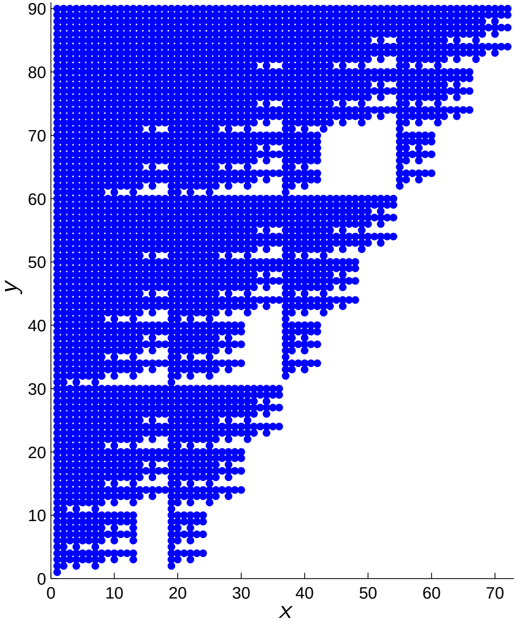

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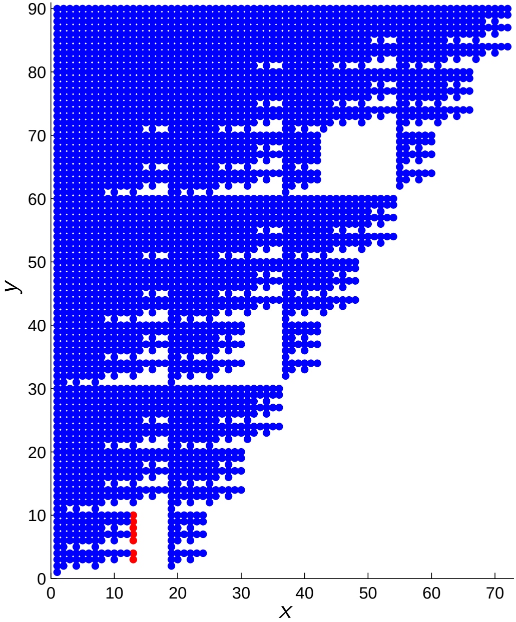

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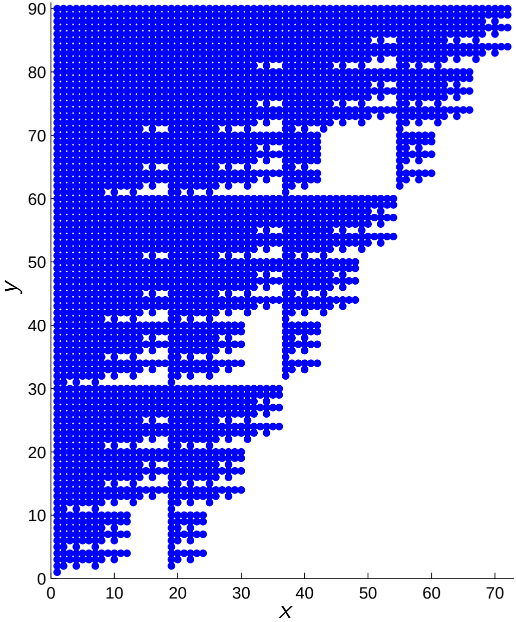


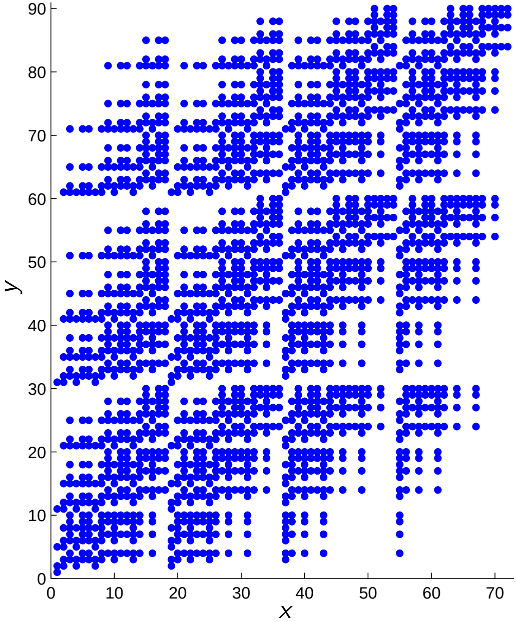
Application: Call center

What if we started with a stricter relation?

Redefine $x \sim y$ by

$$0 \leq \sum_{i,k} y_{i,k} - \sum_{i,k} x_{i,k} \leq 1$$

Application: Call center



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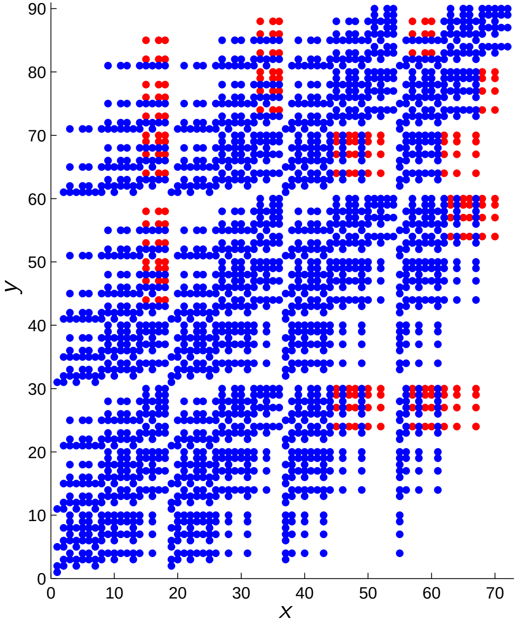

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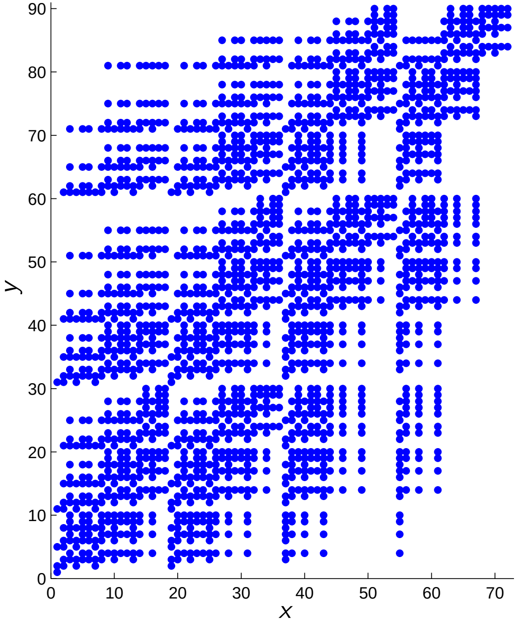

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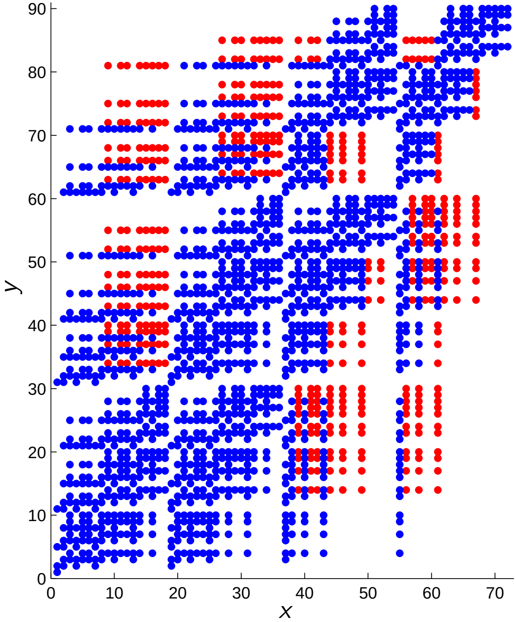

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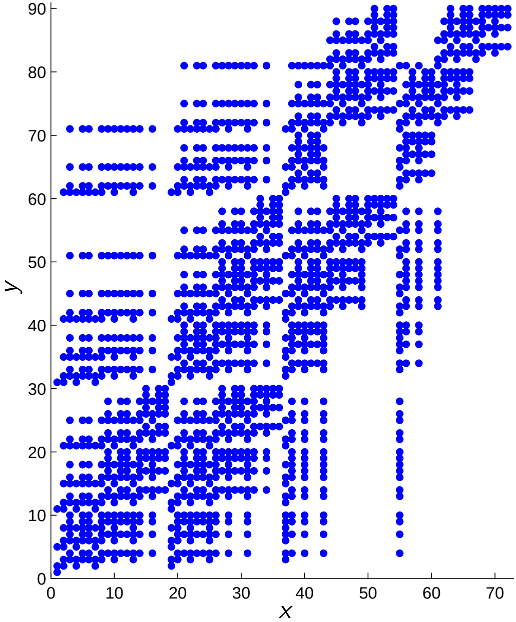

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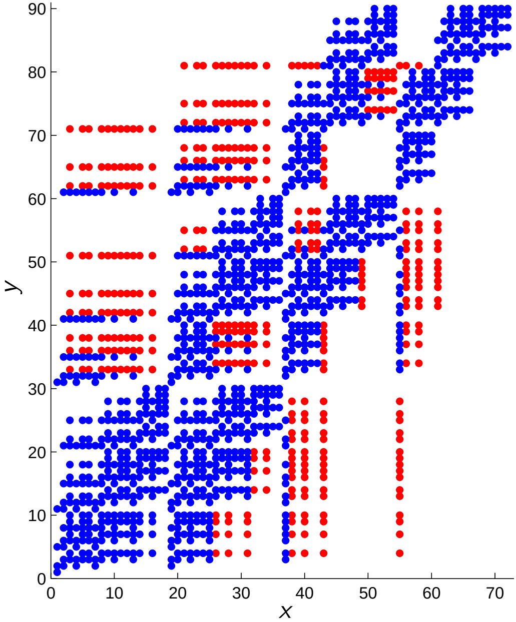

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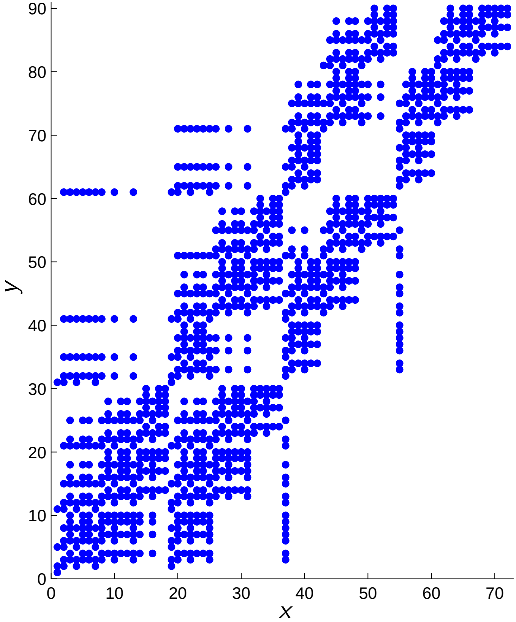

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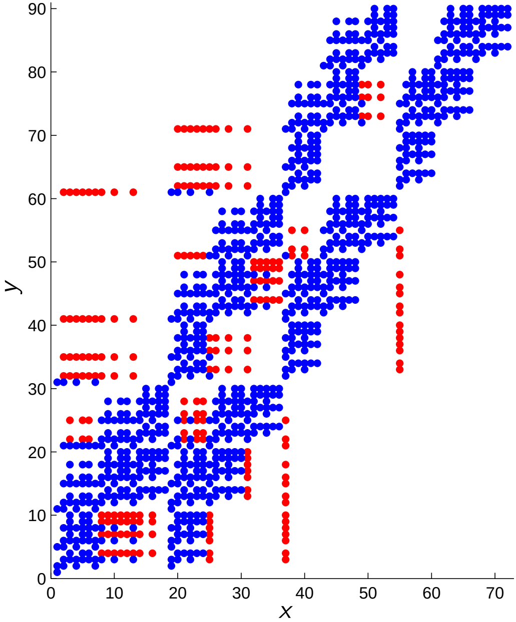

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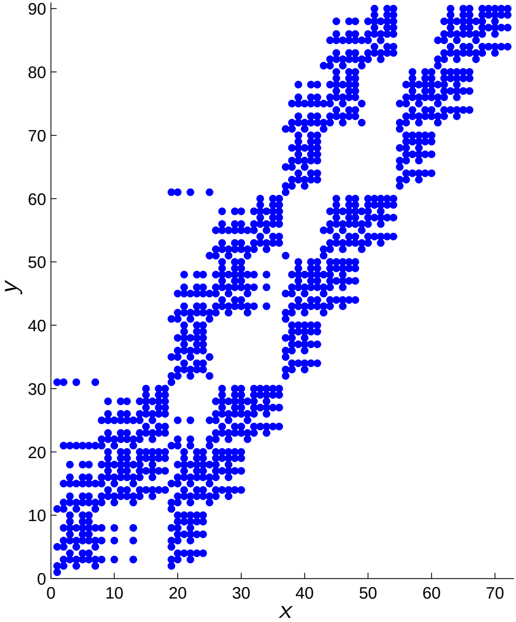

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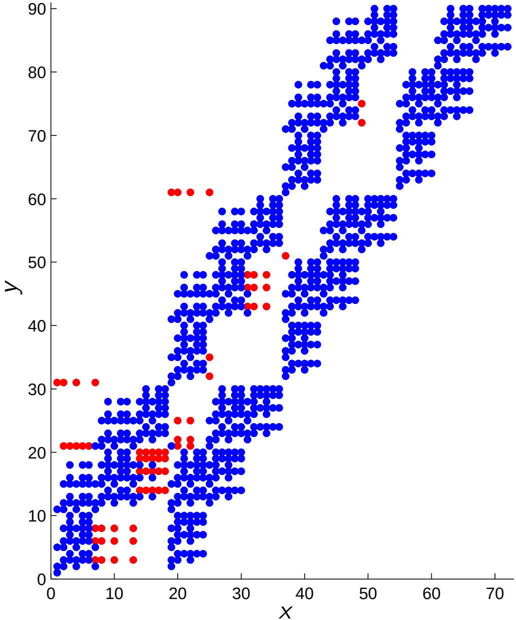

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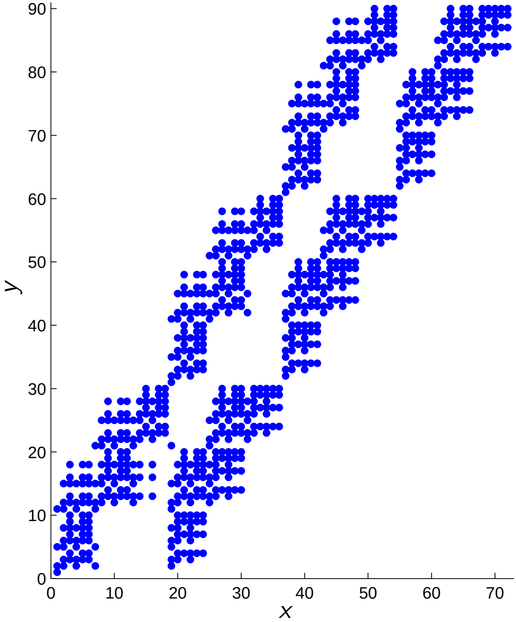
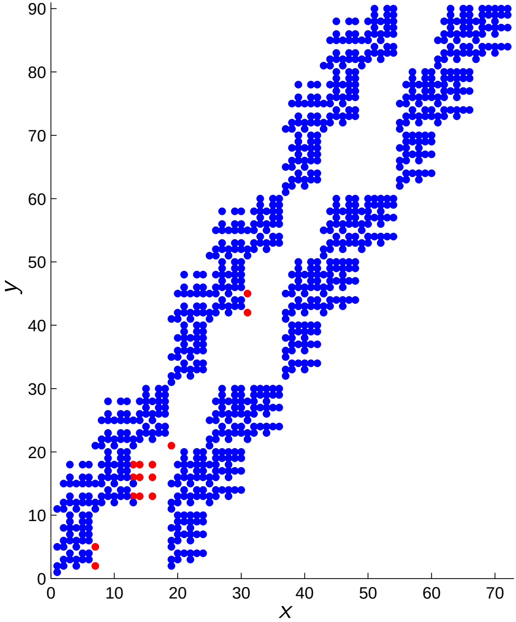
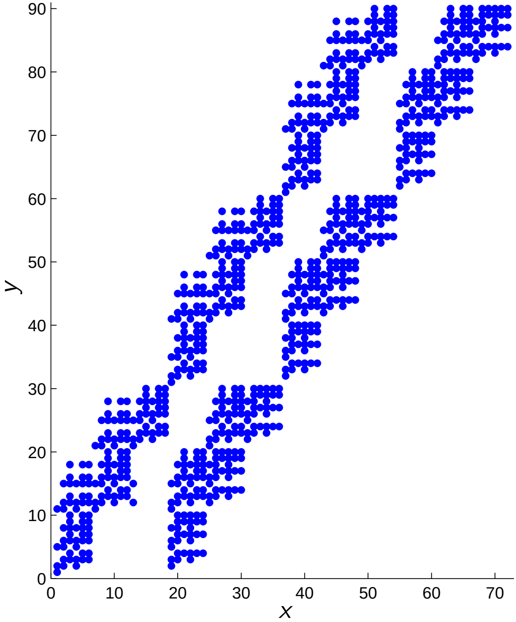


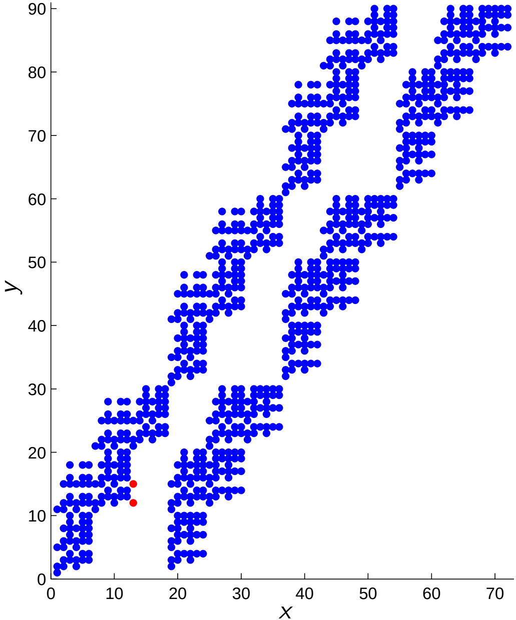
Application: Call center



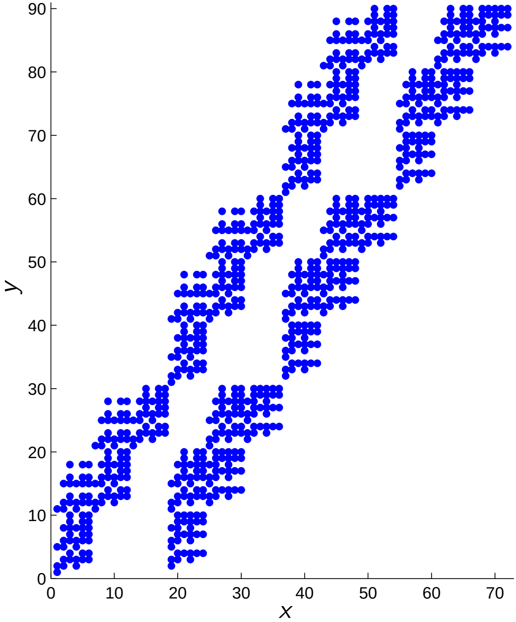
Application: Call center



Application: Call center



Application: Call center



Application: Call center

Theorem (Jonckheere Leskelä 2008)

The processes X and Y stochastically preserve the relation $R = \{(x, y) : |x - y| \in \Delta\}$, *where*

$$\Delta = \{0,\ e_2,\ e_2 - e_{1,1},\ 2e_2 - e_{1,1}\}.$$

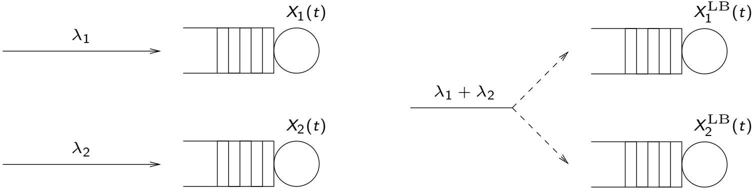
Especially, the stationary distributions of the processes satisfy

$$|Y| - 1 \le_{\mathrm{st}} |X| \le_{\mathrm{st}} |Y|,$$

and

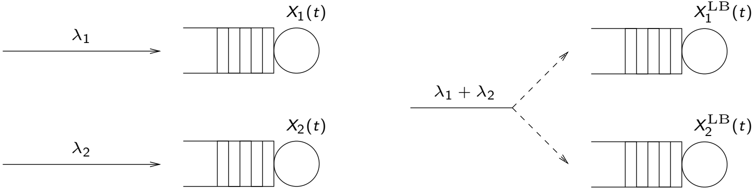
$$\begin{aligned} X_{1,1} &\ge_{\mathrm{st}} Y_{1,1}, \\ X_{1,k} &=_{\mathrm{st}} Y_{1,k} \quad \text{for all } k \ne 1, \\ \sum_k X_{2,k} &\le_{\mathrm{st}} \sum_k Y_{2,k}. \end{aligned}$$

Application: Load balancing



Common sense: $\mathsf{E}(X_1^{\mathrm{LB}}(t) + X_2^{\mathrm{LB}}(t)) \leq \mathsf{E}(X_1(t) + X_2(t))$

Application: Load balancing

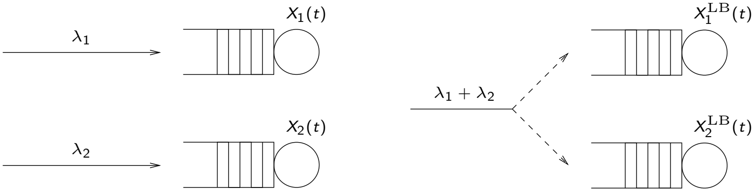


Common sense: $\mathsf{E}(X_1^{\mathrm{LB}}(t) + X_2^{\mathrm{LB}}(t)) \le \mathsf{E}(X_1(t) + X_2(t))$

Problem: (Q^{LB}, Q) does not stochastically preserve:

- $R^{\mathrm{nat}} = \{(x, y) : x_1 \le y_1,\ x_2 \le y_2\}$
- $R^{\mathrm{sum}} = \{(x, y) : |x| \le |y|\}$, where $|x| = x_1 + x_2$

Application: Load balancing



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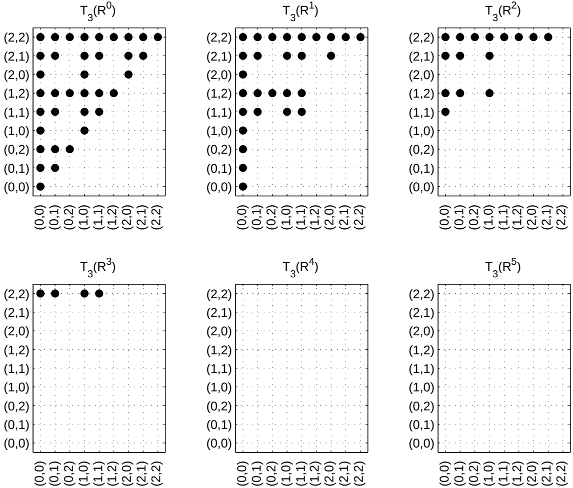
- $R^{\mathrm{sum}} = \{(x,y) : |x| \le |y|\}$, where $|x| = x_1 + x_2$

How about a subrelation of R^{nat} or R^{sum}?

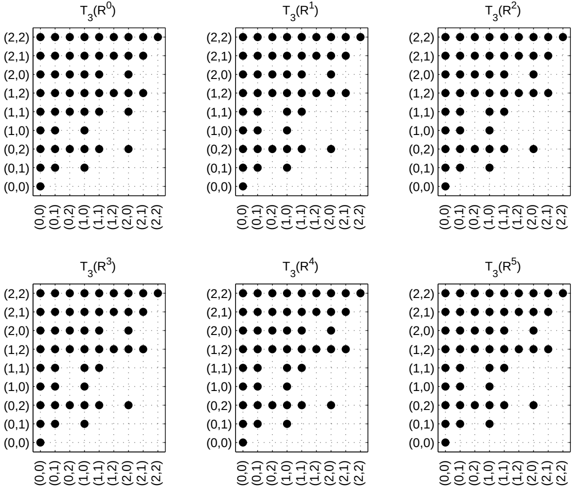
Application: Load balancing

Subrelation algorithm applied to $R^0 = R^{\mathrm{nat}}$



Application: Load balancing

Starting with R^{sum} instead of R^{nat}



Application: Load balancing

Theorem

The subrelation algorithm started from R^{sum} yields

$$R^{(n)} = \left\{(x,y) : |x| \le |y| \text{ and } x_1 \vee x_2 \le y_1 \vee y_2 + (y_1 \wedge y_2 - n)^+\right\}$$

$$\downarrow$$

$$R^{*} = \left\{(x,y) : |x| \le |y| \text{ and } x_1 \vee x_2 \le y_1 \vee y_2\right\}.$$

Especially, (Q^{LB}, Q) stochastically preserves the relation R^*.

Application: Load balancing

Theorem

The subrelation algorithm started from R^{sum} yields

$$R^{(n)} = \left\{(x,y) : |x| \le |y| \text{ and } x_1 \vee x_2 \le y_1 \vee y_2 + (y_1 \wedge y_2 - n)^+\right\}$$

$$\downarrow$$

$$R^* = \left\{(x,y) : |x| \le |y| \text{ and } x_1 \vee x_2 \le y_1 \vee y_2\right\}.$$

Especially, (Q^{LB}, Q) stochastically preserves the relation R^*.

Remark

- R^* is the weak majorization order on $\mathbb{Z}_+^2$
- $X \sim_{\text{st}}^* Y$ if and only if $\operatorname{E} f(X) \le \operatorname{E} f(Y)$ for all coordinatewise increasing Schur-convex functions f (Marshall Olkin 1979).

Outline

Stochastic orders and relations

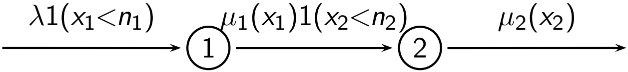
Stochastic ordering of network populations

Stochastic ordering of network flows

Stochastic boundedness

Two-node linear queueing network

Two queues with buffer capacities n_1 and n_2

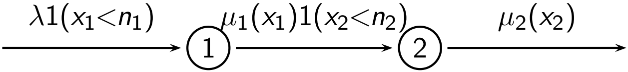


Blocking

- Arrivals blocked when $X_1(t) = n_1$
- 1st server halts when $X_2(t) = n_2$

Two-node linear queueing network

Two queues with buffer capacities n_1 and n_2



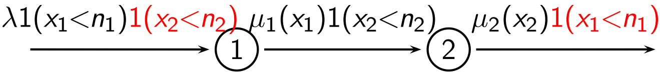
Blocking

- Arrivals blocked when $X_1(t) = n_1$
- 1st server halts when $X_2(t) = n_2$

Service station models

- Single-server: $\mu_i(x_i) = c_i 1(x_i > 0)$
- Multi-server: $\mu_i(x_i) = c_i x_i$
- Peer-to-peer: $\mu_i = \mu_i(x_1, x_2)$

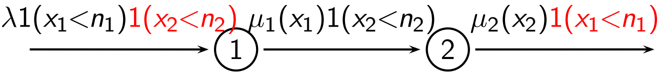
Balanced system modification



Balanced operation

- Arrivals blocked when $X_1(t) = n_1$ or $X_2(t) = n_2$
- 1st server halts when $X_2(t) = n_2$
- 2nd server halts when $X_1(t) = n_1$

Balanced system modification



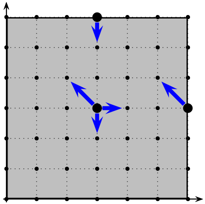
Balanced operation

- Arrivals blocked when $X_1(t) = n_1$ or $X_2(t) = n_2$
- 1st server halts when $X_2(t) = n_2$
- 2nd server halts when $X_1(t) = n_1$

Balanced system has a product-form equilibrium distribution (van der Wal & van Dijk 1989)

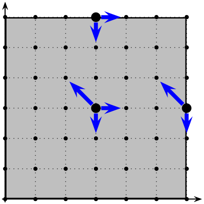
Balanced vs. original system

Balanced system



$B^{\mathrm{bal}} = \{x : x_1 = n_1 \text{ or } x_2 = n_2\}$

Original system



$B^{\mathrm{orig}} = \{x : x_1 = n_1\}$

Performance comparison

- Balanced system has more blocking states: $B^{\mathrm{bal}} \supset B^{\mathrm{orig}}$
- $\rightsquigarrow$ Balanced system should have a higher loss rate
- $\rightsquigarrow$ Conservative & computable performance bound

How to prove the comparison statement?

- Sample path comparison

Sample path comparison

Heuristic reasoning:

- Balanced system has more blocking states

Sample path comparison

Heuristic reasoning:

- Balanced system has more blocking states
- $\rightsquigarrow$ Blocks more jobs

Sample path comparison

Heuristic reasoning:

- Balanced system has more blocking states
- $\rightsquigarrow$ Blocks more jobs
- $\rightsquigarrow$ Has less jobs in the system

Sample path comparison

Heuristic reasoning:

- Balanced system has more blocking states
- $\rightsquigarrow$ Blocks more jobs
- $\rightsquigarrow$ Has less jobs in the system
- $\rightsquigarrow$ Spends less time in blocking states

Sample path comparison

Heuristic reasoning:

- Balanced system has more blocking states
- $\rightsquigarrow$ Blocks more jobs
- $\rightsquigarrow$ Has less jobs in the system
- $\rightsquigarrow$ Spends less time in blocking states
- $\rightsquigarrow$ Blocks less jobs?

How to prove the comparison statement?

- ~~Sample path comparison~~

How to prove the comparison statement?

- ~~Sample path comparison~~
- Order-preserving Markov coupling

How to prove the comparison statement?

- ~~Sample path comparison~~
- ~~Order-preserving Markov coupling~~

How to prove the comparison statement?

- ~~Sample path comparison~~
- ~~Order-preserving Markov coupling~~
- Relation-preserving Markov coupling

Relation-preserving Markov couplings

Find a relation $R \subset S \times S'$ such that

- $(x, x') \in R \implies 1_B(x) \leq 1_{B'}(x)$
- There exists an R-preserving Markov coupling of the systems.

Relation-preserving Markov couplings

Find a relation $R \subset S \times S'$ such that

- $(x, x') \in R \implies 1_B(x) \leq 1_{B'}(x)$
- There exists an R-preserving Markov coupling of the systems.

Does it exist? The existence of such a relation can be checked using the subrelation algorithm

- The answer is NO

How to prove the comparison statement?

- ~~Sample path comparison~~
- ~~Order-preserving Markov coupling~~
- ~~Relation-preserving Markov coupling~~
- Flow coupling

General Markov network

Network state: Markov process X on a subset of $\mathbb{Z}_+^n$ with transitions

$$x \mapsto x - e_i + e_j \text{ at rate } \alpha_{i,j}(x), \quad (i,j) \in E(G)$$

where e_i is the i-th unit vector in $\mathbb{Z}^n$ and $e_0 = 0$

- Network $G = (V, E)$ has n internal nodes $\{1, \ldots, n\}$ and one external node 0
- $\alpha_{0,j}(x)$ is the arrival rate to node j
- $\alpha_{i,0}(x)$ is the departure rate from node i

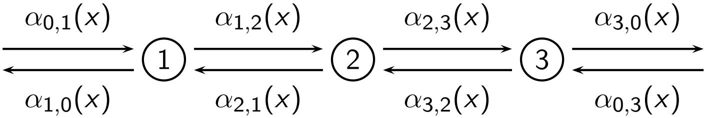
State–flow Markov process

Markov process (X, F) in $\mathbb{Z}_+^n \times \mathbb{Z}_+^{E(G)}$ with transitions

$$(x, f) \mapsto (x - e_i + e_j, f + e_{i,j}) \text{ at rate } \alpha_{i,j}(x), \quad (i,j) \in L$$

- $X_i(t)$ is the number of jobs in node i at time t
- $F_{i,j}(t) - F_{i,j}(0)$ is the number of transitions over link (i,j) during $(0, t]$

Netflow ordering



State–flow relation

- (x, f) has smaller netflow than (x', f') if

$$\begin{aligned} f_{i,i+1} - f_{i+1,i} &\le f'_{i,i+1} - f'_{i+1,i} \quad \text{for all } i = 0, 1, \ldots, n, \\ x_i - f_{\mathrm{in},i} + f_{i,\mathrm{out}} &= x'_i - f'_{\mathrm{in},i} + f'_{i,\mathrm{out}} \quad \text{for all } i = 1, \ldots, n, \end{aligned}$$

Flow coupling for linear networks

Theorem

Assume that

$$\begin{aligned} x_1 \ge x_1' &\implies \alpha_{0,1}(x) \le \alpha'_{0,1}(x') \text{ and } \alpha_{1,0}(x) \ge \alpha'_{1,0}(x'), \\ x_i \le x_i' \text{ and } x_{i+1} \ge x'_{i+1} &\implies \alpha_{i,i+1}(x) \le \alpha'_{i,i+1}(x') \text{ and } \alpha_{i+1,i}(x) \ge \alpha'_{i+1,i}(x'), \\ x_n \le x_n' &\implies \alpha_{n,0}(x) \le \alpha'_{n,0}(x') \text{ and } \alpha_{0,n}(x) \ge \alpha'_{0,n}(x'). \end{aligned}$$

Then there exists a Markov coupling of (X, F) *and* (X', F') *which preserves the netflow relation. Especially, the netflow counting processes are ordered by*

$$(F_{i,i+1}(t) - F_{i+1,i}(t))_{t\ge 0} \le_{\mathrm{st}} (F'_{i,i+1}(t) - F'_{i+1,i}(t))_{t\ge 0}$$

for all $i = 0, \dots, n$*, whenever* $X(0) =_{\mathrm{st}} X'(0)$.

Flow coupling for linear networks

Proof: Marching soldiers coupling.

Let $(\tilde{X}, \tilde{F}, \tilde{X}', \tilde{F}')$ be a Markov process on $(\mathbb{Z}_+^n \times \mathbb{Z}_+^{E(G)}) \times (\mathbb{Z}_+^n \times \mathbb{Z}_+^{E(G)})$ with transitions

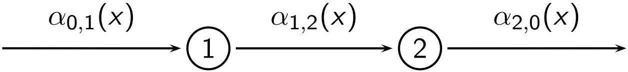
$$((x,f),(x',f')) \mapsto \begin{cases} (T_{i,j}(x,f), T_{i,j}(x',f')) & \text{at rate } \alpha_{i,j}(x) \wedge \alpha'_{i,j}(x'), \\ ((x,f), T_{i,j}(x',f')) & \text{at rate } (\alpha'_{i,j}(x') - \alpha_{i,j}(x))_+, \\ (T_{i,j}(x,f),(x,f)) & \text{at rate } (\alpha_{i,j}(x) - \alpha'_{i,j}(x'))_+, \end{cases}$$

where $T_{i,j}(x,f) = (x - e_i + e_j, f + e_{i,j})$

- This is the marching soldiers coupling of (X, F) and (X', F') (Mu-Fa Chen 2005).
- This coupling preserves the state–flow order relation

□

Balanced vs. original two-node network



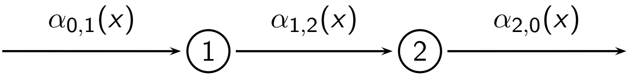
Balanced system

- $\alpha_{0,1}^{\mathrm{bal}}(x) = \lambda 1(x_1 < n_1) 1(x_2 < n_2)$
- $\alpha_{1,2}^{\mathrm{bal}}(x) = \mu_1(x_1) 1(x_2 < n_2)$
- $\alpha_{2,0}^{\mathrm{bal}}(x) = \mu_2(x_2) 1(x_1 < n_1)$

Original system

- $\alpha_{0,1}^{\mathrm{orig}}(x) = \lambda 1(x_1 < n_1)$
- $\alpha_{1,2}^{\mathrm{orig}}(x) = \mu_1(x_1) 1(x_2 < n_2)$
- $\alpha_{2,0}^{\mathrm{orig}}(x) = \mu_2(x_2)$

Balanced vs. original two-node network



Balanced system

- $\alpha^{\mathrm{bal}}_{0,1}(x) = \lambda 1(x_1 < n_1) 1(x_2 < n_2)$
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- $\alpha^{\mathrm{bal}}_{2,0}(x) = \mu_2(x_2) 1(x_1 < n_1)$

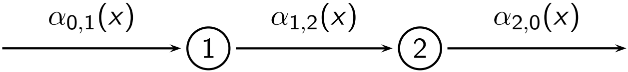
Original system

- $\alpha^{\mathrm{orig}}_{0,1}(x) = \lambda 1(x_1 < n_1)$
- $\alpha^{\mathrm{orig}}_{1,2}(x) = \mu_1(x_1) 1(x_2 < n_2)$
- $\alpha^{\mathrm{orig}}_{2,0}(x) = \mu_2(x_2)$

$(X^{\mathrm{bal}}, F^{\mathrm{bal}})$ has a stochastically smaller flow than $(X^{\mathrm{orig}}, F^{\mathrm{orig}})$ if

$$\begin{aligned}
x_1 \ge x_1' &\implies \alpha^{\mathrm{bal}}_{0,1}(x) \le \alpha^{\mathrm{orig}}_{0,1}(x') \\
x_1 \le x_1' \text{ and } x_2 \ge x_2' &\implies \alpha^{\mathrm{bal}}_{1,2}(x) \le \alpha^{\mathrm{orig}}_{1,2}(x') \\
x_2 \le x_2' &\implies \alpha^{\mathrm{bal}}_{2,0}(x) \le \alpha^{\mathrm{orig}}_{2,0}(x').
\end{aligned}$$

Balanced vs. original two-node network



Balanced system

- $\alpha^{\mathrm{bal}}_{0,1}(x) = \lambda 1(x_1{<}n_1)1(x_2{<}n_2)$
- $\alpha^{\mathrm{bal}}_{1,2}(x) = \mu_1(x_1)1(x_2{<}n_2)$
- $\alpha^{\mathrm{bal}}_{2,0}(x) = \mu_2(x_2)1(x_1{<}n_1)$

Original system

- $\alpha^{\mathrm{orig}}_{0,1}(x) = \lambda 1(x_1{<}n_1)$
- $\alpha^{\mathrm{orig}}_{1,2}(x) = \mu_1(x_1)1(x_2{<}n_2)$
- $\alpha^{\mathrm{orig}}_{2,0}(x) = \mu_2(x_2)$

$(X^{\mathrm{bal}}, F^{\mathrm{bal}})$ has a stochastically smaller flow than $(X^{\mathrm{orig}}, F^{\mathrm{orig}})$ if

$$\begin{aligned} x_1 \ge x_1' &\implies \lambda 1(x_1{<}n_1)1(x_2{<}n_2) \le \lambda 1(x_1'{<}n_1) \\ x_1 \le x_1' \text{ and } x_2 \ge x_2' &\implies \mu_1(x_1)1(x_2{<}n_2) \le \mu_1(x_1')1(x_2'{<}n_2) \\ x_2 \le x_2' &\implies \mu_2(x_2)1(x_1{<}n_1) \le \mu_2(x_2') \end{aligned}$$

The above conditions are valid when μ_1 and μ_2 are increasing.

How to prove the comparison statement?

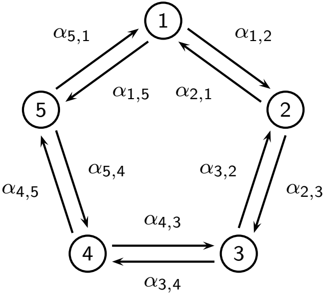
- ~~Sample path comparison~~
- ~~Order-preserving Markov coupling~~
- ~~Relation-preserving Markov coupling~~
- Flow coupling (OK for throughput distributions)

Generalizations

Other network structures?

- Closed cyclic networks
- Aggregate flows across linear partitions

Flow ordering in cyclic networks



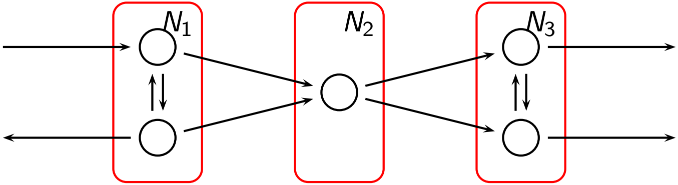
Theorem

Assume that for all i and for all x and x',

$$x_i \le x'_i \text{ and } x_{i+1} \ge x'_{i+1}$$
$$\Longrightarrow$$
$$\alpha_{i,i+1}(x) \le \alpha'_{i,i+1}(x') \text{ and } \alpha_{i+1,i}(x) \ge \alpha'_{i+1,i}(x').$$

Then (X, F) has stochastically smaller clockwise netflow than (X', F').

Aggregate flows through linear partitions



State-flow (x, f) has a smaller netflow through $N_1 \to N_2 \to N_3$ than (x', f') if

$$\begin{aligned} f_{N_r,N_{r+1}} - f_{N_{r+1},N_r} &\leq f'_{N_r,N_{r+1}} - f'_{N_{r+1},N_r} \quad \text{for all clusters } N_r, \\ x_i - f_{\mathrm{in},i} + f_{i,\mathrm{out}} &= x'_i - f'_{\mathrm{in},i} + f'_{i,\mathrm{out}} \quad \text{for all nodes } i, \end{aligned}$$

where

$$f_{N_r,N_s} = \sum_{i \in N_r,\ j \in N_s} f_{i,j}$$

Aggregate flows through linear partitions

Theorem

There exists a Markov coupling of state–flow processes (X, F) *and* (X', F') *which preserves the netflow ordering if and only if for all* $x, x' \in \mathbb{Z}_+^n$*:*

$$|x_{N_1}| \geq |x'_{N_1}| \implies \begin{cases} \alpha_{\{0\},N_1}(x) \leq \alpha'_{\{0\},N_1}(x') \\ \alpha_{N_1,\{0\}}(x) \geq \alpha'_{N_1,\{0\}}(x') \end{cases}$$

$$\left.\begin{matrix} |x_{N_k}| \leq |x'_{N_k}| \\ |x_{N_{k+1}}| \geq |x'_{N_{k+1}}| \end{matrix}\right\} \implies \begin{cases} \alpha_{N_k,N_{k+1}}(x) \leq \alpha'_{N_k,N_{k+1}}(x') \\ \alpha_{N_{k+1},N_k}(x) \geq \alpha'_{N_{k+1},N_k}(x') \end{cases}$$

$$|x_{N_m}| \leq |x'_{N_m}| \implies \begin{cases} \alpha_{N_m,\{0\}}(x) \leq \alpha'_{N_m,\{0\}}(x') \\ \alpha_{\{0\},N_m}(x) \geq \alpha'_{\{0\},N_m}(x'), \end{cases}$$

where $|x_I| := \sum_{i \in I} x_i$ *and* $\alpha_{N_r,N_s} := \sum_{i \in N_r, j \in N_s} \alpha_{i,j}$.

Outline

Stochastic orders and relations

Stochastic ordering of network populations

Stochastic ordering of network flows

Stochastic boundedness

Stochastic boundedness

When is a family of positive random variables (X_α) bounded

- in the strong order?

$$X_\alpha \le_{\mathrm{st}} Z \quad \text{if} \quad \mathsf{E}\,\phi(X_\alpha) \le \mathsf{E}\,\phi(Z) \text{ for } \phi \text{ increasing}$$

Stochastic boundedness

When is a family of positive random variables (X_α) bounded

- in the strong order?

$$X_\alpha \leq_{\mathrm{st}} Z \quad \text{if} \quad \mathsf{E}\,\phi(X_\alpha) \leq \mathsf{E}\,\phi(Z) \text{ for } \phi \text{ increasing}$$

- in the increasing convex order?

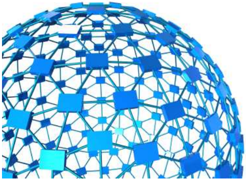
$$X_\alpha \leq_{\mathrm{icx}} Z \quad \text{if} \quad \mathsf{E}\,\phi(X_\alpha) \leq \mathsf{E}\,\phi(Z) \text{ for } \phi \text{ increasing and convex}$$

For any $p > 1$:

$\{\lvert X_\alpha\rvert^p\}$ is st-bounded by an integrable r.v.	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is st-bounded by a p-integrable r.v.	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is icx-bounded by a p-integrable r.v.
		$\Downarrow$		
$\{X_\alpha\}$ is uniformly p-integrable	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert^p\}$ is icx-bounded by an integrable r.v.	$\Leftrightarrow$	$\{\mu_\alpha\}$ is rel. compact in the p-Wasserstein metric
		$\Downarrow$		
		$\{X_\alpha\}$ is bounded in L^p	$\Leftrightarrow$	$\{\mu_\alpha\}$ is bounded in the p-Wasserstein metric
		$\Downarrow$		
		$\{\lvert X_\alpha\rvert\}$ is st-bounded by an integrable r.v.		
		$\Downarrow$		
$\{X_\alpha\}$ is uniformly integrable	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is icx-bounded by an integrable r.v.	$\Leftrightarrow$	$\{\mu_\alpha\}$ is rel. compact in the 1-Wasserstein metric
		$\Downarrow$		
		$\{X_\alpha\}$ is bounded in L^1	$\Leftrightarrow$	$\{\mu_\alpha\}$ is bounded in the 1-Wasserstein metric
		$\Downarrow$		
$\{X_\alpha\}$ is tight	$\Leftrightarrow$	$\{\lvert X_\alpha\rvert\}$ is st-bounded by a finite r.v.	$\Leftrightarrow$	$\{\mu_\alpha\}$ is rel. compact in the Prohorov metric

L Leskelä & M Vihola, Stat Probab Lett 2013, arXiv:1106.0607

Conclusions



You can compare things without ordering them.

Comparing populations

- Subrelation algorithm may help to reveal hidden monotone structure

Comparing flows

- Redundant state–flow model $\rightsquigarrow$ non-Markov couplings

L Leskelä, J Theor Probab 2010, arXiv:0806.3562

M Jonckheere & L Leskelä, Stoch Mod 2008, arXiv:0708.1927

L Leskelä & M Vihola, Stat Probab Lett 2013, arXiv:1106.0607

Lebesgue's dominated convergence theorem

Theorem

Assume that $X_n \to X$ *almost surely.* $\mathrm{E}\,|X_n - X| \to 0$ *if for some integrable* Y,

$$|X_n| \leq_{\mathrm{st}} Y \quad \text{for all } n.$$

Sharp dominated convergence theorem

Theorem

Assume that $X_n \to X$ almost surely. $\mathsf{E}\,|X_n - X| \to 0$ *if and only if for some integrable Y,*

$$|X_n| \leq_{\mathrm{icx}} Y \quad \text{for all } n.$$

Stochastic boundedness — Examples

Let U be a uniform r.v. in $(0,1)$ and

$$\phi_n = \begin{cases} n \text{ w.pr. } n^{-1}, \\ 0 \text{ else,} \end{cases} \qquad \psi_n = \begin{cases} n \text{ w.pr. } (n \log n)^{-1}, \\ 0 \text{ else.} \end{cases}$$

Then for any $p > 1$:

- $\{e^{1/U}\}$ is st-bounded by a finite r.v. (itself) but not bounded in L^ϵ for any $\epsilon > 0$.
- $\{\phi_n\}$ is bounded in L^1 but not uniformly integrable.
- $\{\psi_n\}$ is uniformly integrable but not st-bounded by an integrable r.v.
- $\{U^{-1/p}\}$ is st-bounded by an integrable r.v. but not bounded in L^p.
- $\{\phi_n^{1/p}\}$ is bounded in L^p but not uniformly p-integrable.
- $\{\psi_n^{1/p}\}$ is uniformly p-integrable but not st-bounded by a r.v. in L^p.

Prohorov metric

The Prohorov metric on the space M of probability measures on $\mathbb{R}^d$ is defined by

$$d_P(\mu, \nu) = \inf \{\epsilon > 0 : \mu(B) \leq \nu(B^\epsilon) + \epsilon \text{ and } \nu(B) \leq \mu(B^\epsilon) + \epsilon \text{ for all } B$$

where $B^\epsilon = \{x \in \mathbb{R}^d : |x - b| < \epsilon \text{ for some } b \in B\}$ denotes the ϵ-neighborhood of a Borel set B

- (M, d_P) is a complete separable metric space.
- Convergence in d_P is convergence in distribution

Wasserstein metric

For $p \geq 1$, denote by M_p the space of probability measures on $\mathbb{R}^d$ with a finite p-th moment. The p-Wasserstein metric on M_p is defined by

$$d_{W,p}(\mu, \nu) = \left(\inf_{\gamma \in K(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, \gamma(dx, dy) \right)^{1/p},$$

where $K(\mu, \nu)$ is the set of couplings of μ and ν.

- $(M_p, d_{W,p})$ is a complete separable metric space.
- A sequence converges in $d_{W,p}$ if and only if it is uniformly p-integrable and converges in distribution.

Open problems & discussion

Open problems

- Stochastic relations of diffusions
- Weak stochastic relations
- Structured Markov chains

Related work on non-Markov couplings

- Generalized semi-Markov processes (Glasserman & Yao 1994)
- Linear bandwidth-sharing networks (Verloop & Ayesta & Borst 2010)
- Chip-firing games (Eriksson 1996)
- Sleepy random walkers (Dickman & Rolla & Sidoravicius 2010)

Open problem: Coupling of diffusions

Assume that A_i are differential operators on $\mathbb{R}$ of the form

$$A_i f(x_i) = \frac{1}{2} a^{(i)}(x_i) f''(x_i) + b^{(i)}(x_i) f'(x_i),$$

and let A be a differential operator on $\mathbb{R}^2$ such that

$$Af(x) = \frac{1}{2} \sum_{i,j=1}^{2} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{i=1}^{2} b_i(x) \frac{\partial}{\partial x_i} f(x).$$

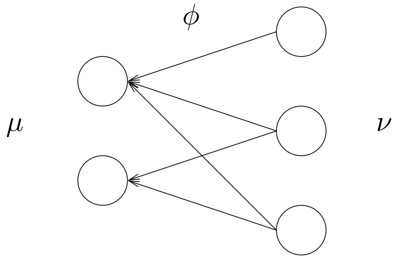
Then A couples A_1 and A_2 if

$$\frac{1}{2} a_{i,i}(x) f''(x_i) + b_i(x) f'(x_i) = \frac{1}{2} a^{(i)}(x_i) f''(x_i) + b^{(i)}(x_i) f'(x_i).$$

Discussion: Coupling vs. mass transportation

$$W_\phi(\mu,\nu) = \inf_{\lambda \in K(\mu,\nu)} \int_{S_1 \times S_2} \phi(x_1, x_2)\, \lambda(dx)$$

- $K(\mu,\nu)$ is the set of couplings of μ and ν



- W_ϕ is a Wasserstein metric, if ϕ is a metric.
- $\mu \sim_{\mathrm{st}} \nu$ if and only if $W_\phi(\mu,\nu) = 0$ for $\phi(x_1, x_2) = 1(x_1 \not\sim x_2)$.

(Monge 1781, Kantorovich 1942, Wasserstein 1969, Chen 2005)

Discussion: Subrelations vs. minimal bounding chains

Subrelation approach

- Given transition kernels P_1 and P_2, and a relation R, find a maximal subrelation of R stochastically preserved by (X_1, X_2)
- Intuitive bounding: P_2 needs to be a priori given

Minimal bounding chains
(Truffet 2000, Fourneau Lecoz Quessette 2004, Ben Mamoun Bušić Pekergin 2007)

- Given a transition matrix P_1 and an order relation R, find a minimal transition matrix P_2 (in a suitable class) such that X_1 and X_2 stochastically preserve R
- Computational bounding: P_2 found numerically

Questions and comments

- How to interpret minimal (when R is not a total order)?
- Can we combine the two approaches?

Truncated subrelation algorithm

- Assume Q_1 and Q_2 have locally bounded jumps
- Truncation operators $T_N : S_1 \times S_2 \to S_{1,N} \times S_{2,N}$
- Truncated subrelation algorithm can be computed in finite time and memory

Algorithm for computing $R^{(K)}$ truncated into $S_{1,N} \times S_{2,N}$:

$R' \leftarrow T_{N+K}(R)$
for $k = 1, \ldots, K$ **do**
    $n \leftarrow N + K + 1 - k$
    $Q_{1,n} \leftarrow$ truncation of Q_1 into $S_{1,n}$
    $Q_{2,n} \leftarrow$ truncation of Q_2 into $S_{2,n}$
    $R' \leftarrow T_n(R')$
    $R' \leftarrow$ subrelation algorithm applied to $(Q_{1,n}, Q_{2,n}, R')$
end for
$R' \leftarrow T_N(R')$

Operator coupling

Denote by π_i the projection map from $S_1 \times S_2$ to S_i. A linear operator A the space of bounded function on $S_1 \times S_2$ is a coupling of linear operators A_1 and A_2, if $f \circ \pi_i \in \mathcal{D}(A)$ and

$$A(f \circ \pi_i) = (A_i f) \circ \pi_i \quad \text{for all } f \in \mathcal{D}(A_i).$$

If A_1 and A_2 are the generators of Markov processes on S_i, then we say that A is a Markov coupling for A_1 and A_2 if A couples the linear operators A_1 and A_2, and the martingale problem for A is well-posed.

Operator coupling

Conjecture

Assume that $A_1 f(x) \leq A_2 g(y)$ *for all* $x \sim y$ *and* $f \sim g$. *Then there exists a coupling of* A_1 *and* A_2 *that preserves the relation* R.

- We denote $f \sim g$ if $f \in \mathcal{D}(A_1)$ and $g \in \mathcal{D}(A_2)$, and

$$x \sim y \implies f(x) \leq g(y).$$

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