# Some McKean–Vlasov problems on the half-line

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November 6, 2015

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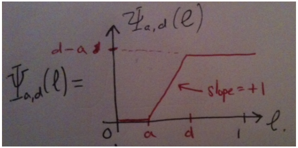
# Motivation

- Initial motivation: portfolio credit derivatives
- $N \geq 1$ defaultable assets, default times $\{\tau^{i,N}\}_{1 \leq i \leq N}$,
- Care about options on proportional loss process

$$L_t^N = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\tau^{i,N} \leq t}, \qquad \text{value} = \mathbf{E}\Psi((L^N)_{t \in [0,T]})$$

- Think of $\Psi$ of form




- Correlations matter: high $\rho \implies$ higher probability of defaulting together
- But also higher chance of surviving together

# Model

**Basic model — Bush, Hambly, Howarth, Jin, Reisinger (2011)**

$$dX_t^i = \mu dt + \rho dW_t + \sqrt{1-\rho^2} dW_t^i$$
$$\tau^i = \inf\{t > 0 : X_t^i \leq 0\}$$
$$X_0^i \sim \nu_0$$

- Take a limit as $N \to \infty$,
- Study the empirical processes

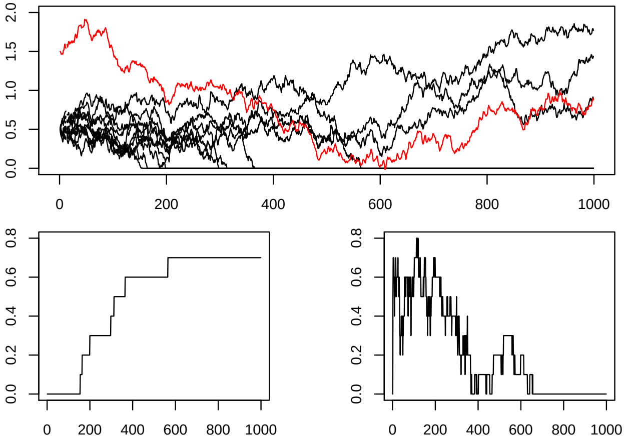
$$\nu_t^N = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{t<\tau^i} \delta_{X_t^i} \in \mathcal{M}$$

- Everything conditionally iid, so (marginal) convergence is easy

$$\nu_t^N(\phi) \to \nu_t(\phi) := \mathbf{E}[\phi(X_t^1)\mathbf{1}_{t<\tau^1}|W], \quad L_t^N \to \nu_t(\mathbf{1}_{(0,\infty)}) = \mathbf{P}(\tau \leq t|W)$$

- Price with $L_t := \mathbf{P}(\tau \leq t|W)$

# Basic model


0.0
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400
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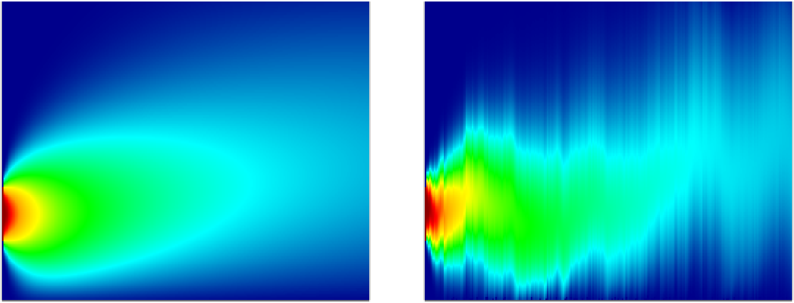
800
1000

# Basic model

## SPDE — Bush, Hambly, Howarth, Jin, Reisinger (2011)

$$d\nu_t(\phi) = \mu\nu_t(\partial_x\phi)dt + \frac{1}{2}\nu_t(\partial_{xx}\phi)dt + \rho\nu_t(\partial_x\phi)dW_t$$

$\phi(0) = 0.$



# Basic model

## Proof

Easy! Conditional independence, Linear evolution equation.

*Existence*: Ito (drop $\mu$)

$$\phi(X_t^i) = \phi(X_0^i) + \frac{1}{2}\int_0^t \phi''(X_s^i)ds + \int_0^t \rho\phi'(X_s^i)dW_s + \int_0^t \sqrt{1-\rho^2}\phi'(X_s^i)dW_s^i$$

If $\phi(0) = 0$ stopping incorporates b.c.: $\phi(X_{t\wedge\tau^i}^i) = \phi(X_t^i)\mathbf{1}_{t<\tau^i}$.

Take average over $1 \leq i \leq N$

$$\mathrm{eqn}(\nu^N)_t = \frac{1}{N}\sum_{i=1}^N \int_0^t \sqrt{1-\rho^2}\phi'(X_s^i)dW_s^i = O_{\mathrm{m.s.}}(N^{-1})$$

In limit: $\mathrm{eqn}(\nu)_t = 0$.

# Basic model

## Proof

*Uniqueness*: Smooth with heat kernel

$$T_\varepsilon \nu_t(x) := \int_0^\infty \frac{1}{\sqrt{2\pi\varepsilon}} \Big( e^{-\frac{(x-x_0)^2}{2\varepsilon}} - e^{-\frac{(x+x_0)^2}{2\varepsilon}} \Big) \nu_t(dx_0) \in C^\infty$$

$\mathrm{eqn}(T_\varepsilon \nu) = \mathrm{rem}(\nu, \varepsilon)$. Manipulate classically using energy estimation

$$\mathbf{E}\|T_\varepsilon \nu_t\|_2^2 + \mathbf{E}\int_0^t \|\partial_x T_\varepsilon \nu_s\|_2^2 ds \leq \|T_\varepsilon \nu_0\|_2^2 + \mathbf{E}\|\mathrm{rem}(\nu, \varepsilon)\|_2^2$$

Control remainder with the quantity

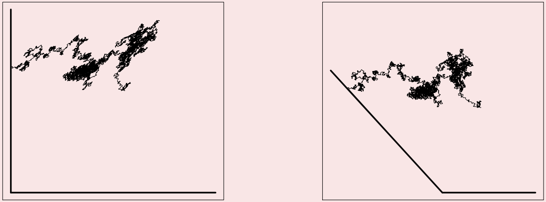
$$\mathbf{E}\nu_t(0, \varepsilon)^2 = \mathbf{P}(X_t^1, X_t^2 \in (0, \varepsilon), t < \tau^1 \wedge \tau^2)$$

need $o(\varepsilon^{3+\delta})$ control

# Basic model

## Proof

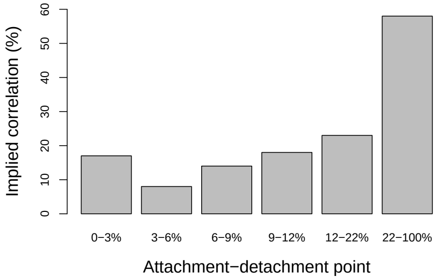
$$\mathbf{E}\nu_t(0,\varepsilon)^2 = \mathbf{P}(X_t^1, X_t^2 \in (0,\varepsilon), t < \tau^1 \wedge \tau^2)$$



Explicit formula, or note harmonic function with zero b.c. in wedge decays like $O(r^{1+\delta})$ □

# Extension

- Model is too simple, can't choose one $\rho$ to match all traded tranche spreads, *correlation skew* or *smile*,




- Why not make $\rho$ a function of the loss in the system?

**Loss-dependent model**

$$dX_t^{i,N} = \mu(L_t^N)dt + \rho(L_t^N)dW_t + \sqrt{1 - \rho^2(L_t^N)}dW_t^i$$

# Extension

- Can have diffusions, but drop for talk

**Loss-dependent model**

$$dX_t^{i,N} = \rho(L_t^N)dW_t + \sqrt{1 - \rho^2(L_t^N)}dW_t^i$$

- Piecewise constant $\rho$ across tranches desirable
- Allow at least finitely many discontinuities, piecewise Lipschitz $\rho$
- Need $0 \leq \rho(\ell) \leq \rho_{\max} < 1$, stop degeneracy
- Challenges: need to deal with boundary effects but correlation too complicated to do explicit calculations,
- For convergence, discontinuous $\rho$ bad, key to show limit points must have strictly increasing loss
- Associated SPDE is non-linear

# Results

## Tightness/Weak existence

The sequence of triples $(\nu^N, L^N, W)_{N\geq 1}$ are tight (with suitable topology).If $(\nu^*, L^*, W)$ realises a limiting law, then

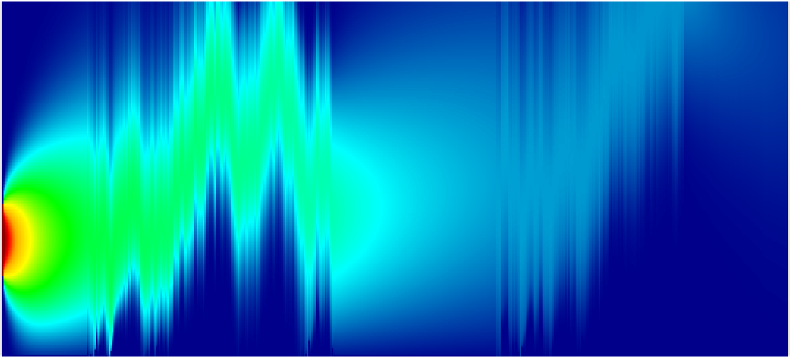
$$d\nu_t^*(\phi) = \frac{1}{2}\nu_t^*(\partial_{xx}\phi)dt + \rho(L_t)\nu_t^*(\partial_x\phi)dW_t$$
$$L_t^* = 1 - \nu_t^*(0,\infty).$$

[+ some regularity conditions.]

## Pathwise uniqueness/LLN

For a given $W$, the SPDE has a most one solution $\nu$. Hence there is a unique law of a solution $(\nu, L, W)$ and so the sequence converges to it.

# Results



# Results

- Weak existence + pathwise uniqueness gives strong solution (on rich enough space), so

## M–V problem

With $(\nu, L, W)$ be the unique soln. For any independent B.M. $W^\perp$ there exists a process $X$ satisfying

$$dX_t = \rho(L_t)dW_t + \sqrt{1-\rho(L_t)^2}\, dW_t^\perp$$

$$\tau := \inf\{t > 0 : X_t \leq 0\}$$

$$\nu_t(\phi) = \mathbf{E}[\phi(X_t)\mathbf{1}_{t<\tau} | W], \qquad L_t = \mathbf{P}(\tau \leq t | W).$$

The law of $(X, W)$ is unique.

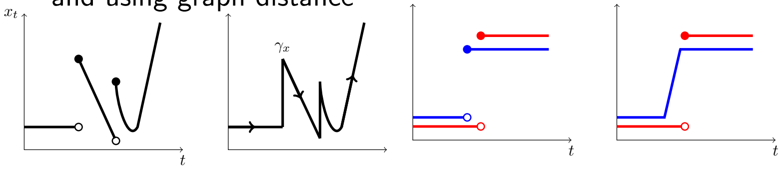
- Conditional/stochastic M–V problem
- Sznitman, *Topics in propagation of chaos* (1991)

# Methods

## Proof outline

Pick suitable topology $\to$ prove tightness $\to$ show limiting laws
solve SPDE $\to$ uniqueness for Lip. $\rho(\cdot)$ $\to$ extend to all $\rho(\cdot)$.

- Skorokhod M1 topology is useful
- $\mathrm{dist}(x, y)$ defined on $D_{\mathbf{R}}$ by joining up discontinuity points and using graph distance




- Tightness controlled by modulus of continuity:

$$w(x;\delta) = \sup_{t} \sup_{(t_1,t_2,t_3)\in \mathrm{Trip}_{t,\delta}} |x_{t_2} - [x_{t_1}, x_{t_3}]_{\mathbf{R}}|_{\mathbf{R}}$$

- Vanishes if $x$ is monotone

# Topology

- Useful for L.L.N. in queuing theory, Whitt, *Stochastic process limits* (2002)
- We extend to infinite-dim. range space,
- Work on $D_{\mathcal{S}'}$, space of distribution-valued càdlàg paths
- $\mathcal{M} \subseteq \mathcal{S}'$, recovering $f \in \mathcal{M}$ generally easy, allows for CLT (perhaps)

## Tightness characteristion

$(\nu^N)_{N\geq 1}$ tight on $(D_{\mathcal{S}'}, \mathrm{M1})$ iff $(\nu^N(\phi))_{N\geq 1}$ tight on $(D_{\mathbf{R}}, \mathrm{M1})$ for all $\phi \in C_0^\infty(\mathbf{R})$.

- Extends Mitoma (1983)
- Useful as

$$\nu_t^N(\phi) = \frac{1}{N}\sum_{i=1}^{N} \phi(X_t^{i,N})\mathbf{1}_{t<\tau^{i,N}} = \frac{1}{N}\sum_{i=1}^{N} \phi(X_{t\wedge\tau^{i,N}}^{i,N}) - \phi(0)L_t^N$$

# Loss-dependent SPDE

- Existence easy once tightness established, Care that topology strong enough to recover SPDE
- Also need to know $t \mapsto L_t^*$ strictly increasing, long estimates,
- For uniqueness estimates, work in space $H^{-1}$

$$\mathbf{E}\partial_x^{-1}\| T_\varepsilon \nu_t\|_2^2 + \mathbf{E}\int_0^t \| T_\varepsilon \nu_s\|_2^2 ds \leq \|\partial_x^{-1} T_\varepsilon \nu_0\|_2^2 + \mathbf{E}\|\partial_x^{-1}\mathrm{rem}(\nu, \varepsilon)\|_2^2$$

- Only required $o(\varepsilon^{1+\delta})$ control on remainder,
- Enough to use trivial bound

$$\mathbf{E}\nu_t(0,\varepsilon)^2 \leq \mathbf{E}\nu_t(0,\varepsilon) = \lim_{N\to\infty} \mathbf{P}(X_t^{1,N} \in (0,\varepsilon), t < \tau^{1,N})$$

- Eliminates correlation, much easier to work with
- Use a stopping argument for Lip. and uniqueness

# A neuroscience model

- Many neurons, look at typical behaviour
- Voltage level $t \mapsto X_t$
- Start at $X_0 > 0$, distributed as $\nu_0$
- Experiences own noise, $(B_t)_{t \geq 0}$, and spikes when hits level 0
- But receives a kick towards the origin of size $\alpha$ times the proportion of spiking particles:

**M–V problem (mean field)**

$$X_t = X_0 + B_t - \alpha L_t$$
$$\tau = \inf\{t > 0 : X_t \leq 0\}$$
$$L_t = \mathbf{P}(\tau \leq t)$$

- (Then restarts back on $[0, \infty)$ according to $\nu_0$ — but problem is hard enough)

# PDE and integral equation

## M–V problem (mean field)

$$X_t = X_0 + B_t - \alpha L_t$$
$$\tau = \inf\{t > 0 : X_t \le 0\}$$
$$L_t = \mathbf{P}(\tau \le t)$$

- What does it mean to solve? Find $L$
- Can rewrite as a PDE and IE
- Let $\nu_t(\phi) := \mathbf{E}[\phi(X_t)\mathbf{1}_{t<\tau}]$, so $L_t = 1 - \nu_t(\mathbf{1}_{(0,\infty)})$,

## PDE problem (large population distn.)

$$d\nu_t(\phi) = \frac{1}{2}\nu_t(\partial_{xx}\phi)dt - \alpha\nu_t(\partial_x\phi)dL_t$$
$$L_t = 1 - \nu_t(\mathbf{1}_{(0,\infty)})$$

# PDE and integral equation

## PDE problem (large population distn.)

$$d\nu_t(\phi) = \frac{1}{2}\nu_t(\partial_{xx}\phi)dt - \alpha\nu_t(\partial_x\phi)dL_t$$
$$L_t = 1 - \nu_t(\mathbf{1}_{(0,\infty)})$$

- If solution is nice, $V_t$ density of $\nu_t$

## PDE problem (large population distn.)

$$\partial_t V_t(x) = \frac{1}{2}\partial_{xx} V_t(x)dt + \alpha\partial_t L_t \partial_x V_t(x)$$
$$L_t = 1 - \int_0^\infty V_t(x)dx$$
$$V_t(0) = 0, \qquad x \mapsto V_0(x) \text{ given}$$
$$[\quad \partial_t L_t = \partial_x V_t(0) \quad]$$

# PDE and integral equation

- Or, given Brownian motion with drift $t \mapsto \ell_t$

$$X_t^\ell = X_0 + B_t - \alpha \ell_t,$$

the function $\Gamma[\ell]_t := \mathbf{P}(\tau^\ell \leq t)$ satisfies

$$\int_0^\infty \Phi\Big(-\frac{x - \alpha\ell_t}{t^{1/2}}\Big) V_0(x)dx = \int_0^t \Phi\Big(\alpha\frac{\ell_t - \ell_s}{(t-s)^{1/2}}\Big) d\Gamma[\ell]_t$$

- Also: hitting law of Brownian motion on boundary $t \mapsto \alpha\ell_t$
- (Laplace transform on this law to get IE — Volterra IE of first kind)

**IE (fixed point)**

Solve $\Gamma[L] = L$

# Solutions

## Delarue, Inglis, Rubenthaler, Tanré (2015)

With $\nu_0 = \delta_{x_0}$, there exists $\alpha_0 > 0$ such that for all $\alpha < \alpha_0$ there exists a unique solution to (M–V) with $t \mapsto L_t \in C^1[0, T]$

## Cáceres, Carrillo, Perthame (2011)

For any initial condition, there exists $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ no continuous function $t \mapsto L_t$ solves (M–V)

## A simpler proof

Suppose a continuous solution exists. Set $\phi(x) = x$ into PDE.

$$0 < \nu_t(\phi) = \nu_0(\phi) - \alpha \int_0^t \nu_s(1) dL_s = \nu_0(\phi) - \alpha \int_0^t (1 - L_s) dL_s$$

- 

# Solutions

## ...A simpler proof

Can do this integral since continuous and increasing

$$\frac{\alpha}{2}(1-(1-L_t)^2) \leq \nu_0(\phi).$$

By ignoring drift, easy to see $L_t \to 1$ as $t \to \infty$. Therefore

$$\frac{\alpha}{2} \leq \nu_0(\phi).$$

So $\alpha_1 = 2\nu_0(\phi)$ will do. □

- Obviously not an optimal choice of $\alpha_1$, so ...

# Solutions

## Critical value?

Fix an initial condition (come back to this!) does there exist a critical value $\alpha_c > 0$ such that

$$\alpha < \alpha_c \implies \text{a continuous solution exists}$$
$$\alpha > \alpha_c \implies \text{a continuous solution cannot exist (blow-up)?}$$

- Obvious monotonicity arguments delicate: depends on rate, not absolute value, of $L$
- Location of blow-up tricky:

## Monotone?

Let $\bar{\alpha} > \alpha > \alpha_c$, does $L^{\bar{\alpha}}$ blow-up before $L^{\alpha}$?

# How to build solutions

- Three ways
- First is fixed point problem:
- Set $\ell = 0$, put in $\Gamma[\ell]$, iterate $\Gamma[\Gamma[\ell]]$, ..., $\Gamma[\dots\Gamma[\Gamma[\ell]]\dots]$
- Done in [DIRT15] on subspace of $C^1$ with sup-norm ($W^{1,\infty}$)
- ...but requires assumptions from theorem
- Second approach: delay the equation:

$$X_t^\delta = X_0 + B_t - \alpha L_t^\delta, \qquad L_t^\delta = \mathbf{P}(\tau^\delta \le t - \delta)$$

- An easy problem: Initially no contribution (when $t < \delta$), so solve problem on $[0, \delta)$, then use that information to solve on $[\delta, 2\delta)$, etc...

# How to build solutions

- Delay prevents blow-up, so $L^\delta \in C^1[0, T]$
- Limit as $\delta \to 0$ shown to converge in [DIRT15a]
- Also in that paper: Third approach: The microscopic model
- This is important as it contains the physical meaning
- The problem is that we want blow-up to occur, models synchronisation of neurons
- Should have existence and uniqueness theory that incorporates blow-up — i.e. a way to restart solutions
- To solve the problem, let solutions $L$ have jumps (restrict to càdlàg)
- How?

# Global solutions

- Underspecified:

**M–V problem (mean field)**

$$X_t = X_0 + B_t - \alpha L_t$$
$$\tau = \inf\{t > 0 : X_t \leq 0\}$$
$$L_t = \mathbf{P}(\tau \leq t)$$

what if $L$ jumps? $\Delta L_t := L_t - L_{t-}$

- $L_t - L_s = \mathbf{P}(X_s + \inf_{s<u<t}\{B_{s,u} - \alpha L_{s,u}\} \leq 0, \tau > s)$, send $s \uparrow t$
- $\Delta L_t = \mathbf{P}(X_{t-} \in (0, \alpha \Delta L_t), \tau > t) = \nu_{t-}(0, \alpha \Delta L_t)$
- So jump at $t$ solves $J = \nu_{t-}(0, \alpha J)$, Still underspecified!

# Global solutions

- Obvious solution: look for càdlàg solutions with jumps as small as they need to be at each time

**Defn: Minimal-jump solution**

If $L$ solves (M–V) and is càdlàg, then $L$ is a *minimal-jump solution* if whenever $\bar{L}$ is another such solution agreeing with $L$ on $[0, t)$ then

$$\Delta L_t \le \Delta \bar{L}_t.$$

**Characterisation**

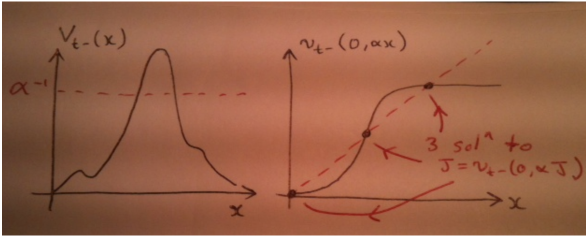
$L$ is a minimal-jump solution iff $L$ solves (M–V) and

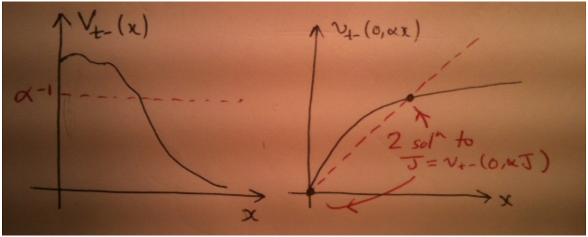
$$\Delta L_t = \inf\{x > 0 : \nu_{t-}(0, \alpha x) < x\}, \quad \text{for all } t$$

- Existence of minimal-jump solutions [DIRT2015a]

# Global solutions

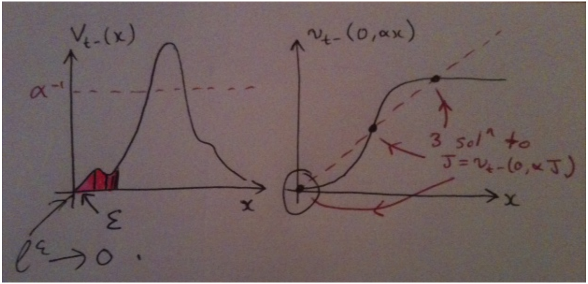
- $\Delta L_t = \inf\{x > 0 : \nu_{t-}(0, \alpha x) < x\}$,



# Global solutions

- Eat $\ell_0^\varepsilon = \varepsilon$ mass,
- Solution takes $\ell_1^\varepsilon = \varepsilon + \nu_{t-}(0, \alpha\ell_0^\varepsilon)$
- Then takes $\ell_2^\varepsilon = \varepsilon + \nu_{t-}(0, \alpha\ell_1^\varepsilon)$, ...
- $\ell^\varepsilon = \varepsilon + \nu_{t-}(0, \alpha\ell^\varepsilon)$, decreasing in $\varepsilon$
- $\to \ell = \nu_{t-}(0, \alpha\ell)$
- Easy to see $\ell = \inf\{x > 0 : \nu_{t-}(0, \alpha x) < x\}$



# Global solutions

- Eat $\ell_0^\varepsilon = \varepsilon$ mass,
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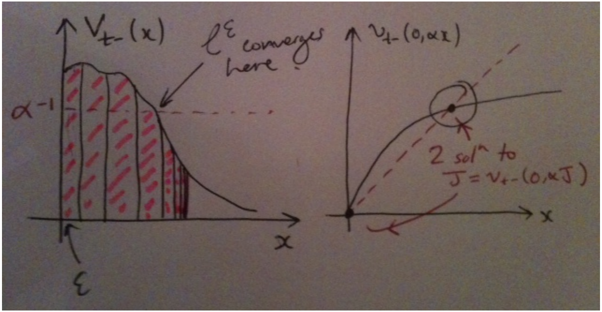
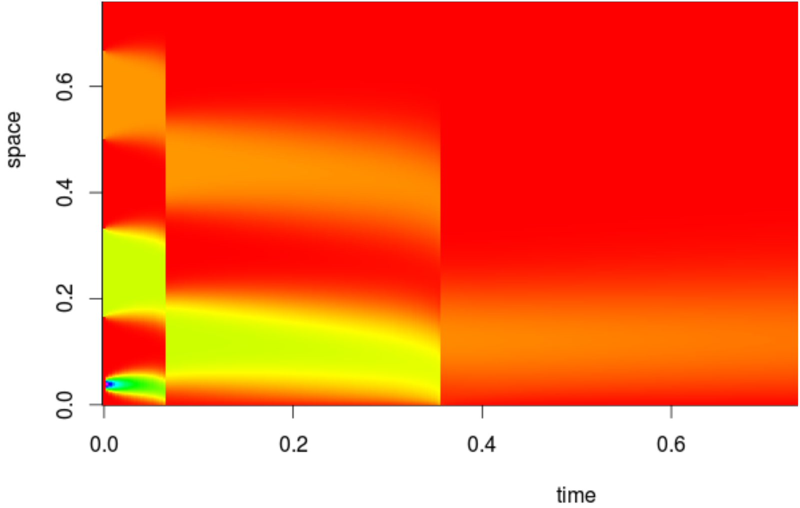


# Global solutions



# Global solutions

## Uniqueness?

Is there a unique minimal-jump solution for a given initial condition?

- All current uniqueness arguments exploit smoothness and initial conditions that are too well-behaved— after blow-up $\nu_t(0,x) = O(x)$, $\sqrt{t}$ singularity

## P/w $C^1$?

Does the solution behave as $C^1$ function between jumps? Weighted space?

# Global solutions

- Solution density decays in max value quicker than the heat equation, $t^{-1/2}$,
- So there is a last possible time for a jump to occur,

**Frequency?**

Is there an upper bound on the number of jumps that can occur for a given $\alpha$ over all possible initial conditions?

**Common noise**

What about adding a rough (deterministic) noise $t \mapsto z_t$

$$X_t = X_0 + B_t + z_t - \alpha L_t, \qquad L_t = \mathbf{P}_B(\tau \leq t)?$$

What if not Holder$-1/2$?