Heat kernel estimates and local CLT for random walk among random conductances with a power-law tail near zero

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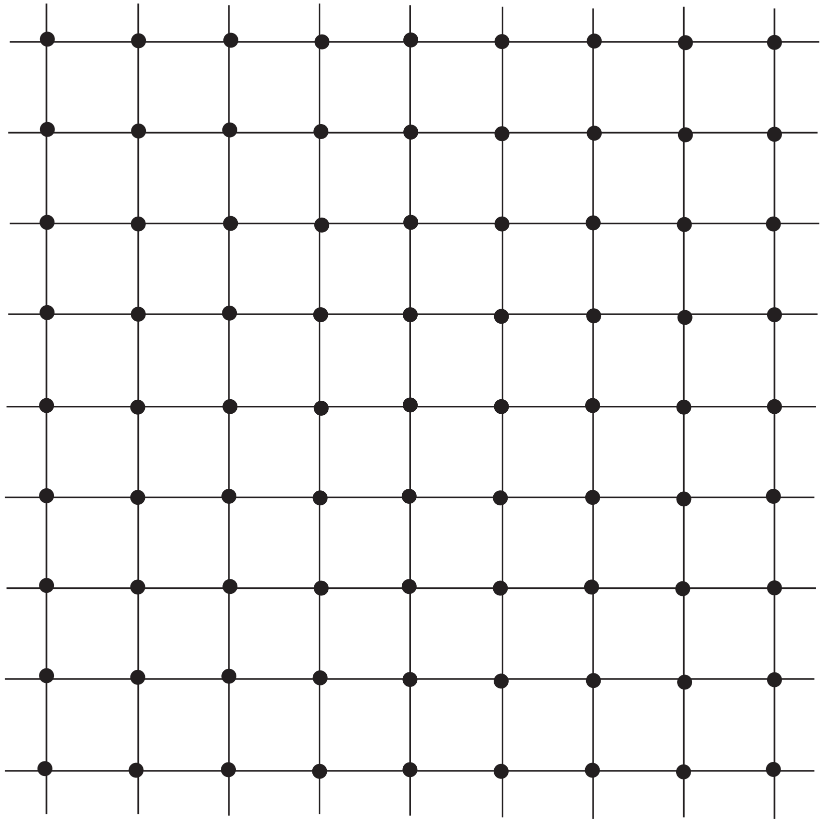
1 Random walk on Random conductance model - Brief Survey

- Random conductance model (symmetric (reversible) RWRE)



Consider $(\mathbb{Z}^d, E_d)$, $d \geq 2$ where E_d is the set of non-oriented n.n. bonds.

Let the conductance $\{\mu_e : e \in E_d\}$ be i.i.d. (more generally stat. ergo.) on $(\Omega, \mathcal{F}, \mathbb{P})$.

Two natural MCs Trans. prob. $P(x,y)=\mu_{xy}/\mu_x$ $\;(\mu_x := \sum_{y\sim x}\mu_{xy})$.

1. Constant speed random walk (CSRW): holding time is $exp(1)$ for each point
2. Variable speed random walk (VSRW): holding time at x is exp. distri. with mean μ_x^{-1}

The corresponding discrete Laplace operators are

$$\mathcal{L}_C f(x) = \frac{1}{\mu_x}\sum_y (f(y)-f(x))\mu_{xy}, \quad \mathcal{L}_V f(x) = \sum_y (f(y)-f(x))\mu_{xy}.$$

Let ν be s.t. $\nu(x)=1$, $\forall x\in\mathbb{Z}^d$. Then, for each finite supported f,g,

$$\begin{aligned}\mathcal{E}(f,g) &= -(\mathcal{L}_V f,g)_\nu = -(\mathcal{L}_C f,g)_\mu \\ &= \frac{1}{2}\sum_{x,y}(f(x)-f(y))(g(x)-g(y))\mu_{xy}.\end{aligned}$$

RW on supercrit. perco. is a special case ($\mathbb{P}(\mu_e=1)=p$, $\mathbb{P}(\mu_e=0)=1-p$, $p>p_c(\mathbb{Z}^d)$)

Assume $\mathbb{P}(\mu_e > 0) > p_c(\mathbb{Z}^d)$. Then $\exists 1 \mathcal{C}$ infinite cluster. We consider $\mathbb{P}(\cdot | 0 \in \mathcal{C})$.

Let $(\{Y_t\}_{t\geq 0}, \{P_\omega^x\}_{x\in\mathbb{Z}^d})$ be either the CSRW or VSRW and define

$$p_t^\omega(x,y) = P_\omega^x(Y_t = y)/\theta_y$$

be the heat kernel of $\{Y_t\}_{t\geq 0}$ where θ is ν for VSRW and μ for CSRW.

Fundamental questions

(Q1) Behavior of the heat kernel? (Q2) Invariance principle?

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Fundamental questions

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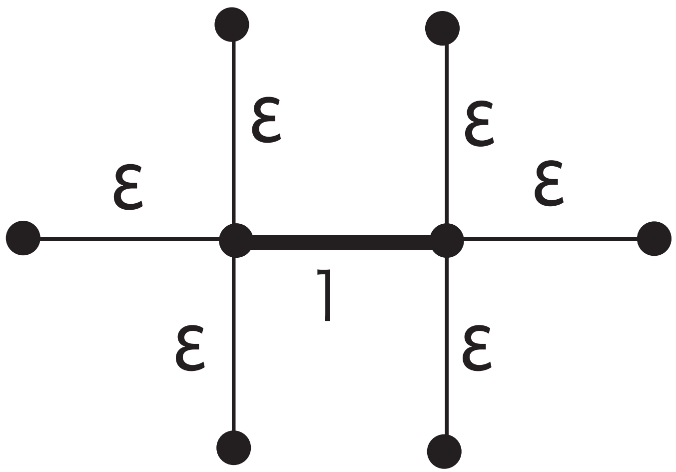
- RW on supercrit. perco.: No anomalous behavior for long time.

On (Q1): Gaussian HK estimates (Barlow '04), Local CLT (Barlow-Hambly '09)

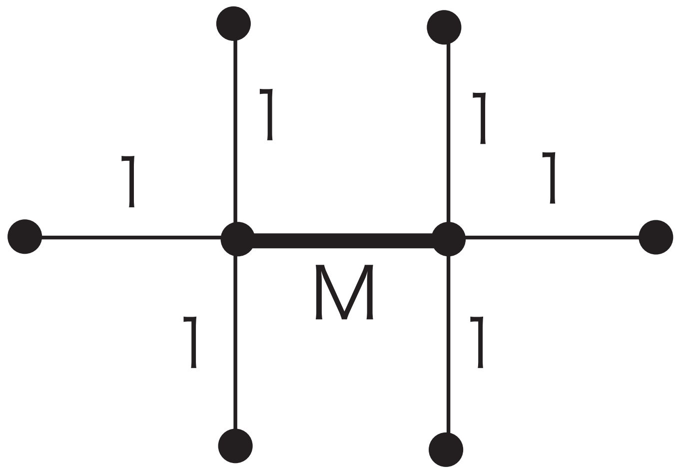
On (Q2): Sidoravicius-Sznitman '04, Berger-Biskup '07, Mathieu-Piatnitski. '07

All the results are 'quenched', i.e. almost surely w.r.t. $\mathbb{P}$.

Issue – Traps –



Trap for both CSRW and VSRW



Trap for CSRW (only)

On (Q1): Heat kernel estimates:

- (Barlow-Deuschel '10) If $\mathbb{P}(1 \leq \mu_e < \infty) = 1$, then for VSRW,

$$\frac{c_1}{t^{d/2}} \exp(-c_2 \frac{d(x,y)^2}{t}) \leq p_t^{\omega}(x,y) \leq \frac{c_3}{t^{d/2}} \exp(-c_4 \frac{d(x,y)^2}{t}),$$

$\mathbb{P}$-a.s. ω for $t \geq d(x,y) \vee \exists U_x$, $x, y \in \mathcal{G}$.

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- Anomalous heat kernel behavior for $\mathbb{P}(\mu_e \leq 1) = 1$

(Fontes-Mathieu '06) Annealed result: VSRW on $\mathbb{Z}^d$ with $\mu_{xy} = \omega(x) \wedge \omega(y)$ where $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. with $\omega(x) \leq 1$ for all x and $\exists \gamma > 0$ s.t.

$$\mathbb{P}(\omega(0) \leq s) \asymp s^\gamma \quad \text{as} \quad s \downarrow 0 \Rightarrow \quad \lim_{t\to\infty} \frac{\log \mathbb{E}[P_\omega^0(Y_t = 0)]}{\log t} = -(\frac{d}{2} \wedge \gamma).$$

(Berger-Biskup-Hoffman-Kozma '08) Quenched HK estimates for discrete time MC:

Theorem 1.1 *Assume* $\mathbb{P}(\mu_e \le 1) = 1$. *(i) For* $\mathbb{P}$*-a.e.* ω, $\exists C_1(\omega) > 0$ *s.t.*

$$P_\omega^n(0,0) \le C_1(\omega) \begin{cases} n^{-d/2}, & d = 2,3, \\ n^{-2}\log n, & d = 4, \\ n^{-2}, & d \ge 5. \end{cases} \qquad \forall n \ge 1 \tag{2}$$

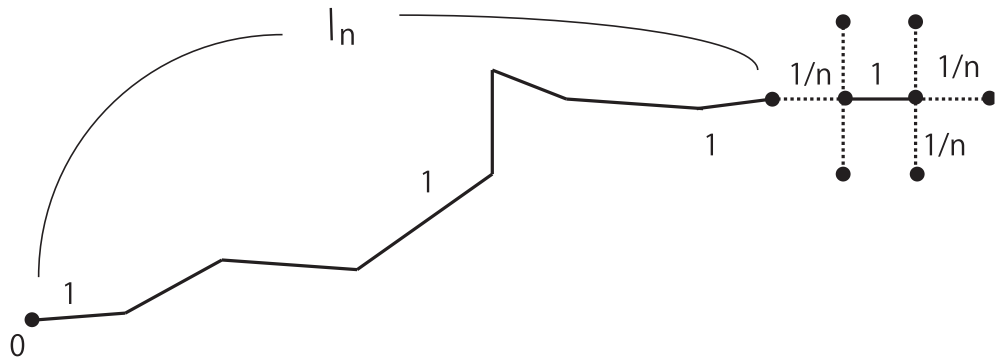
(ii) For any incr. seq. $\{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n \to \infty$, $\exists$ *i.i.d. law* $\mathbb{P}$ *with* $\mathbb{P}(0 < \mu_e \le 1) = 1$ *and* $C_2(\omega), C_3(\omega) > 0$ *s.t. for a.e.* $\omega \in \{|\mathcal{C}(0)| = \infty\}$,

$$P_\omega^{2n}(0,0) \ \ge \ C_3(\omega) n^{-2}\lambda_n^{-1} \quad \text{for} \ \ d \ge 5$$

$$P_\omega^{2n}(0,0) \ \ge \ C_3(\omega) n^{-2}\log n\,\lambda_n^{-1} \quad \text{for} \ \ d = 4. \text{ (Biskup-Boukhadra '11)}$$

along a subsequence that does not depend on ω.

Why n^{-2}?



Suppose $\forall$ large n, the above config. occur w.h.p.

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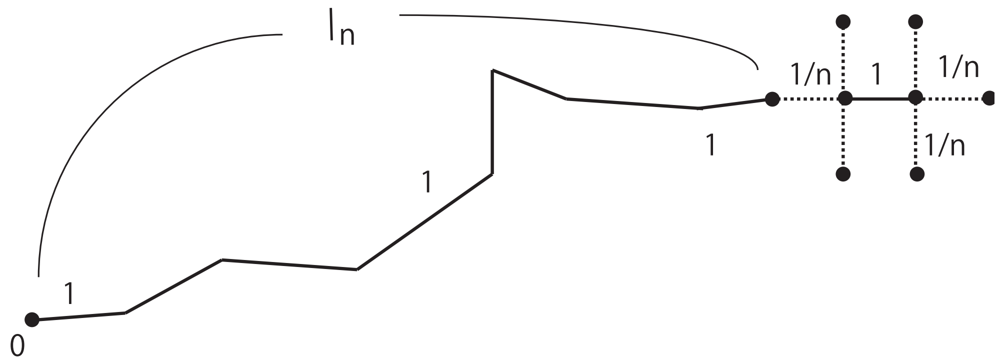

Suppose $\forall$ large n, the above config. occur w.h.p.

<u>Strategy for RW to come back to origin in $2n$ steps (w.p. $\geq n^{-2}$)</u>

(i) RW goes directly towards the trap (costs $e^{O(\ell_n)}$),

(ii) it crosses the weak bond (costs $1/n$), spends time $n - 2\ell_n$ on the strong bond

(costs $(\frac{1}{1+c/n})^{n-2\ell_n} = O(1)$), and crosses a weak bond again (costs $1/n$),

(iii) it goes back to the origin on time (cost $e^{O(\ell_n)}$ term).

The cost is $O(1)e^{O(\ell_n)}n^{-2}$ so if $\ell_n = o(\log n)$ then we get $n^{-2}(>> n^{-d/2}$ for $d \geq 5)$.

On (Q2): Quenched invariance principle Let $\{Y_t\}_{t\geq 0}$ be either CSRW or VSRW and

$$Y_t^{(\varepsilon)} := \varepsilon Y_{t/\varepsilon^2}. \tag{3}$$

Theorem 1.2 ($\mu_e \leq 1$ case: Biskup-Prescott '07, Mathieu '08, $\mu_e \geq 1$ case:

Barlow-Deuschel '10, unified: Andres-Barlow-Deuschel-Hambly '13)

(i) Let $\{Y_t\}_{t\geq 0}$ *be the VSRW. Then* $\mathbb{P}$*-a.s.* $Y_t^{(\varepsilon)} \to B_{\sigma_V^2 t}$ *where* $\sigma_V > 0$.

(ii) Let $\{Y_t\}_{t\geq 0}$ *be the CSRW. Then* $\mathbb{P}$*-a.s.* $Y^{(\varepsilon)} \to B_{\sigma_C^2 t}$ *where*

$\sigma_C^2 = \sigma_V^2/(2d\mathbb{E}\mu_e)$ *if* $\mathbb{E}\mu_e < \infty$ *and* $\sigma_C^2 = 0$ *if* $\mathbb{E}\mu_e = \infty$.

Note: When $\mathbb{E}\mu_e < \infty$, "annealed CLT" was already obtained in 80's

(Kipnis-Varadhan '86, De Masi-Ferrari-Goldstein-Wick '89 ($\sigma > 0$))

- Parabolic Harnack ineq. (PHI), Local CLT

(Barlow-Hambly '09) General sufficient condition for PHI and Local CLT

Applied for supercritical percolation case.

(Barlow-Deuschel '10) PHI and Local CLT for $\mathbb{P}(1 \le \mu_e < \infty) = 1$.

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- More general domains (Chen-Croydon-K '13) QIP and HK estimates for RW on supercri. perco. on half/quarter planes (Use D-form theory for the bd. issue)

- General stationary ergodic media

(Andres-Deuschel-Slowik '13, '13+) $\{\mu_e\}_e$: positive and stationary ergodic

$$E[(\sum_y \mu_{xy})^p] < \infty, \quad E[(\sum_y \mu_{xy}^{-1})^q] < \infty$$

1) $1/p + 1/q < 2/d \Rightarrow$ Quenched invariance principle (both CSRW and VSRW)

2) $1/p + 1/q < 2/d$ for CSRW and $1/(p-1) + 1/q < 2/d$ for VSRW

$\Rightarrow$ PHI and Local CLT.

2 New Results

Let $\{\mu_e\}$ be i.i.d. with $\mu_e \leq 1$ for all e and $\exists \gamma > 0$ s.t.

$$\mathbb{P}(\mu_e \leq s) = s^{\gamma}(1 + o(1)) \quad \text{as} \quad s \downarrow 0.$$

Theorem 2.1

i) CSRW case: for any $\gamma > \frac{1}{8}\frac{d}{d-1/2}$, *there exists* $\delta, c_1 > 0$ *and* $T = T(\omega) < \infty$ *s.t.*

$$p_t(x, y) \leq c_1 t^{-d/2} \qquad \forall x, y \in B(0, t^{(1+\delta)/2}), \quad t \geq T.$$

ii) VSRW case: Similar bound holds for $\gamma > 1/4$.

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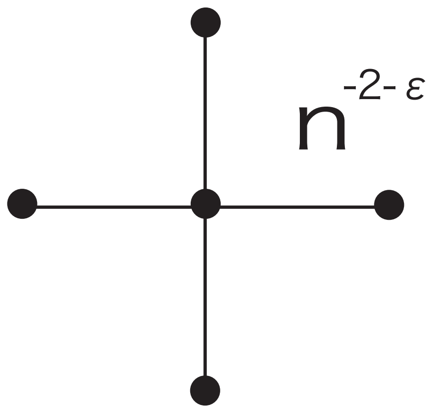
Corollary 2.2 *Under the same cond.,* $E^x_\omega[\tau_{B(x,n)}] \le c_2 n^2$, $\forall x \in B(x_0, n), \forall n \ge \exists R_1(\omega)$.

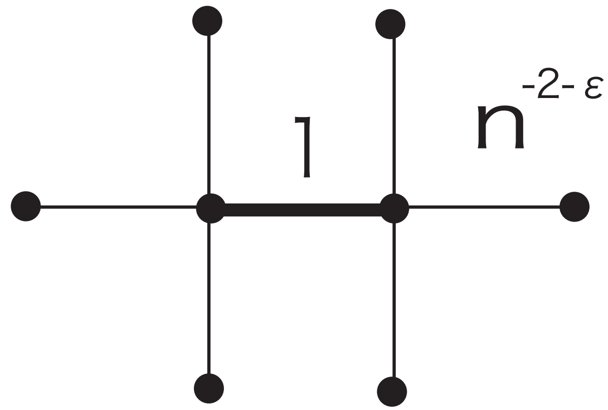
$\gamma = \frac{1}{8}\frac{d}{d-1/2}$ for CSRW, $\gamma = 1/4$ for VSRW are the optimal consts. for Cor 2.2.

I.e. if γ is below that, for a.e. ω, exists $x \in B(0, n)$ s.t. $\tau_{B(0,n)} >> n^2$ when $Y_0 = x$.

Why $\frac{1}{8}\frac{d}{d-1/2}$, $1/4$?



Trap for VSRW (only)



Trap for both CSRW and VSRW

If $\gamma < \frac{1}{8}\frac{d}{d-1/2}$, then $H := (2+\varepsilon)\gamma(4d-2) < d$ for small ε. So

$$\mathbb{P}(\{\forall e \subset B(0,n), \exists b \cap e \neq \emptyset, \mu_b \geq n^{-(2+\varepsilon)}\}) \leq \Big(1 - \mathbb{P}(\mu_b < n^{-(2+\varepsilon)})^{4d-2}\Big)^{c_* n^d}$$

$$= \Big(1 - (cn^{-(2+\varepsilon)\gamma})^{4d-2}\Big)^{c_* n^d} = (1 - c_1 n^{-H})^{c_* n^d} \leq e^{-c_2 n^{d-H}}.$$

By B-C, $\exists e \subset B(0,n)$, cond. $\asymp 1$ s.t. all adj. edges have cond. $<< n^{-2} \Rightarrow \tau_B >> n^2$.

Idea of the proof of Thm 2.1. (CSRW case)

Take $\xi > 0$ s.t. $\mathbb{P}(\mu_e \geq \xi) > p_c(d)$. Let $\mathcal{C}^\xi$ be the unique ∞-cluster.

$$A(t) := \int_0^t 1_{\{Y_s \in \mathcal{C}^\xi\}} ds, \quad X_t^\xi := Y_{A_s^{-1}}, \text{ where } A_t^{-1} := \inf\{s : A_s > t\}.$$

(Key)

$P_\omega^x(A(t) \leq \varepsilon t) \leq c \exp(-c' t^\sigma)$ for all $x \in B(0, t^{(1+\delta)/2})$, t large, $\gamma > \frac{1}{8}\frac{d}{d-1/2}$.

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$$\begin{aligned} P_\omega^x(Y_t = x) &\leq \frac{2}{t} \int_{t/2}^t P_\omega^x(Y_v = x) dv = \frac{2}{t} E_\omega^x \left[\int_{A(t/2)}^{A(t)} 1_{\{X_u^\xi = x\}} du \right] \\ &\leq \frac{2}{t} \int_{\varepsilon t/2}^t P_\omega^x(X_u^\xi = x) du + \frac{2}{t} \int_0^t P_\omega^x(A(t/2) \leq \varepsilon\, t/2) du \end{aligned}$$

for $x \in \mathcal{C}^\xi$. Using $P_\omega^x(X_u^\xi = x) \leq c_1 t^{-d/2}$ (due to (1)), (Key) and $\mu_x \geq \xi$, we have

$$p_t^\mu(x,x) \leq c_2 t^{-d/2} + c_3 \exp(-c_4 t^\sigma) \leq c_3 t^{-d/2} \qquad \square$$

– Proof of (Key) involves percolation est. and spectral gap est.

(Further HK estimates)

Proposition 2.3 *Let* $\gamma > \frac{1}{8}\frac{d}{d-1/2}$ *for CSRW and* $\gamma > 1/4$ *for VSRW.*

(i) For each $x_1, x_2 \in \mathbb{Z}^d$*, there exists* $T_1 = T_1(x_0) > 0$ *and* ε *small, such that if*

$$c(d(x_1,x_2) \vee t^{1/(2-\varepsilon)}) \geq T_1(x_0)^2 \quad \text{and} \quad d(x_0,x_1) \leq c'(d(x_1,x_2) \vee t^{1/(2-\varepsilon)}), \quad \textit{then}$$

$$p_t(x_1,x_2) \leq c_1 t^{-d/2} \exp(-c_2 d(x_1,x_2)^2/t), \qquad \forall t \geq d(x_1,x_2),$$

$$p_t(x_1,x_2) \leq c_3 \exp(-c_4 d(x_1,x_2)(1 \vee \log(d(x_1,x_2)/t))), \qquad \forall t < d(x_1,x_2).$$

(ii) There exist $c, \delta_0, \delta_1 > 0$ *and* $T_2 = T_2(x_0) < \infty$ *such that*

$$p_t(x,y) \geq c t^{-d/2}, \quad \forall x,y \in B(x_0, \delta_0 t^{(1+\delta)/2}) \textit{ with } |x-y| \leq \delta_1 t^{1/2}$$

for all $t \geq T_2(x_0)$.

Remark on Proposition 2.3 (i)

Various ways to deduce the off-diagonal HK upper bound from that of on-diagonal.

Note we (only) have $p_t(x, y) \le c_1 t^{-d/2}$, $\forall x, y \in B(0, t^{(1+\delta)/2})$, $t \ge T(\omega)$.

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I) Davies' method: (Perturbation method) cf. Carlen-Kusuoka-Stroock

$\times$ Since it requires the Nash ineq. (full time, full space)

II) Grigor'yan's method:

Deduce $p_t(x_1, x_2) \le ..$ for $t \ge T^2 \wedge d(x_1, x_2)$ from $p_t(x_i, x_i) \le ...$ for $t \ge T$.

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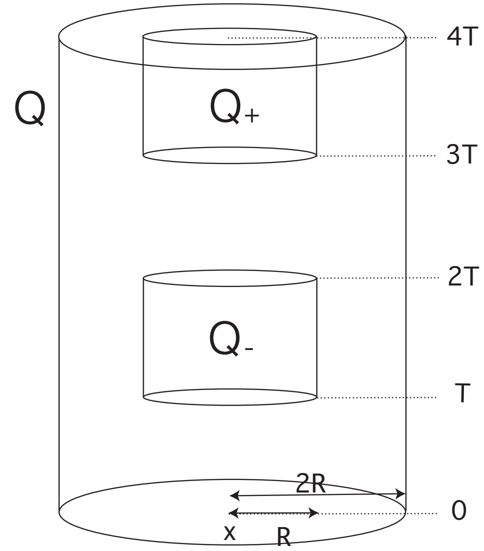
III) $\bigcirc$ Method used on diffusions on fractals + Carne-Varopolous

$c_1 r^2 \le E^x[\tau_{B(x,r)}] \le c_2 r^2$ for $x \in B(x_0,n) \Rightarrow P^x(\tau_{B(x,r)} \le t) \le c_3 \exp(-c_4 r^2/t)$,

$p_t(x,y) \le \frac{c_5}{\sqrt{\theta_x \theta_y}} \exp(-c_4 d(x,y)^2/t)$ for $t \ge d(x,y)$.



For $x \in \mathbb{Z}^d$ and $R, T > 0$, let $Q(x, R, T) := (0, 4T] \times B(x, 2R)$. Define

$$Q_-(x, R, T) := [T, 2T] \times B(x, R), \quad Q_+(x, R, T) := [3T, 4T] \times B(x, R).$$

Let $u(n, x) : [0, 4T] \times \bar{B}(x, 2R) \longrightarrow \mathbb{R}$.

We say $u(n, x)$ is caloric on Q if for $0 \leq n \leq 4T - 1$ and $y \in B(x, 2R)$,

$$u(n + 1, y) - u(n, y) = \mathcal{L}_\theta u(n, y).$$

Theorem 2.4 *Under the same condition as in Thm 2.1, the following hold.*

(i) **(Parabolic Harnack inequalities)** $\exists c_1, R_2(x_0) > 0$ *s.t.* $\forall R \geq R_2(x_0)$, *and* $\forall u = u(n,x) \geq 0$ *which is caloric on* $Q(x_0, R, R^2)$, *it holds that*

$$\sup_{(n,x)\in Q_-(x_0,R,R^2)} u(n,x) \leq c_1 \inf_{(n,x)\in Q_+(x_0,R,R^2)} u(n,x).$$

(ii) $\exists c_1, \theta, R_3(x_0) > 0$ *s.t.* $\forall R \geq R_3(x_0), T_* \geq R^2 + 1$, *suppose* $u > 0$ *is caloric on* $Q(x_0, \sqrt{T}_*, T_*)$. *Then* $\forall x_1, x_2 \in B(x_0, R)$ *and* $\forall n_1, n_2 \in [4(T_* - R^2), 4T_*]$, *we have*

$$|u(n_1,x_1) - u(n_2,x_2)| \leq c_1 (R/T_*^{1/2})^\theta \sup_{Q_+(x_0,\sqrt{T}_*,T_*)} u.$$

Theorem 2.5 *Under the same condition as in Thm 2.1, the following hold.*

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(ii) $\exists c_1, \theta, R_3(x_0) > 0$ *s.t.* $\forall R \geq R_3(x_0), T_* \geq R^2 + 1$, *suppose* $u > 0$ *is caloric on* $Q(x_0, \sqrt{T}_*, T_*)$. *Then* $\forall x_1, x_2 \in B(x_0, R)$ *and* $\forall n_1, n_2 \in [4(T_* - R^2), 4T_*]$, *we have*

$$|u(n_1,x_1) - u(n_2,x_2)| \leq c_1 (R/T_*^{1/2})^\theta \sup_{Q_+(x_0,\sqrt{T}_*,T_*)} u.$$

Proposition 2.6 *Let* $k_t(x) = (2\pi t\sigma_*^2)^{-d/2}\exp(-|x|^2/(2\sigma_*^2 t))$ *and* $M, T_1, T_2 > 0$. *Then*

$$\lim_{n\to\infty} \sup_{|x|\leq M} \sup_{t\in[T_1,T_2]} |n^{d/2} p^\omega_{nt}(0, [n^{1/2}x]) - k_t(x)| = 0, \quad \mathbb{P}-a.s.$$

– VSRW ($\gamma > 1/4$) case is already in Andres-Deuschel-Slowik ('13+).

Theorem 2.5 holds under certain general setting. (Idea from Grigor'yan-Telcs '01)

Proposition 2.7

On-diagonal upper bound ($p_t(x,y) \le c_1 t^{-d/2}$ *for* $t \ge T_0(x_0)$)

$+\ c_2 r^2 \le E^x[\tau_{B(x,r)}]$ *for* $r \ge R_0(x_0)$

+Elliptic Harnack ineq.

(+ (CSRW case) $\mu(B(x_0,R)) \asymp R^d$ *for* $R \ge R_2(x_0)$, $\lim_{R\to\infty} R^{-d}\mu(B(x_0,R)) = c_5$*)*

$\Rightarrow$ *Conclusion of Theorem 2.5 holds.*

Thank you!