# Shy Couplings, CAT(0) Spaces, and the Lion and Man

Bristol Probability Seminar

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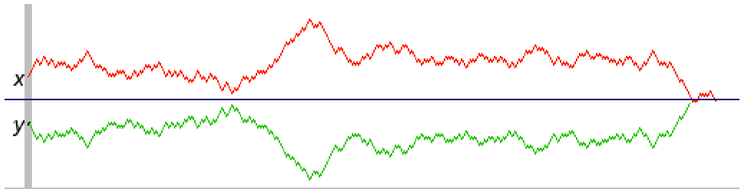
## Wikipedia: Coupling (Probability)

Wikipedia (2013) describes “coupling” thus (after dis-ambiguation):

- A proof technique that allows one to compare two unrelated variables by “forcing” them to be related in some way;
- Examples include the *thematic* cases of synchronous and reflection couplings for random walk.

  Reflection Coupling:
  Make one process meet other by doing mirror-opposite!




Lindvall (1982) “On coupling of Brownian motions.”





## Theoretical framework

Given a random process $X$,

- Intuition: construct two co-dependent copies $X$, $\widetilde{X}$ with different starting points, so as to maximize chance they hit and stick together before a specified time $T$.
- Maximality: upper bound is total variation distance between $\mathcal{L}(X_T)$ and $\mathcal{L}\left(\widetilde{X}_T\right)$.

  Amazingly, this can be achieved for all $T$!

  Price: $\widetilde{X}_t$ generally depends on entire path of $X(s) : s \leqslant T$ (“non-co-adapted”: Griffeath 1975; Pitman 1976; Goldstein 1978).
- Co-adapted: require coupled processes $X$, $\widetilde{X}$ to be defined using same filtration. (More accurately, “immersed”; technically more special: “Markovian”.)





## Applications

Many applications, including:

- Variance reduction for simulations;
- Gradient estimates;
- Perfect simulation;
- Comparison arguments,

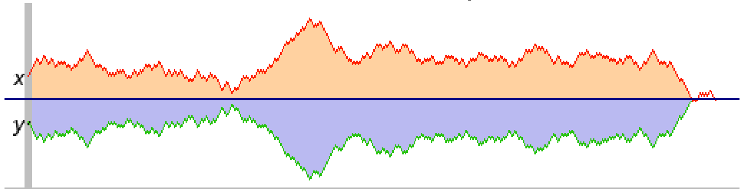
  *particularly*
- Heat-flow monotonicity;
- Notions of efficiency for Markov Chain Monte Carlo.

# Two General Coupling Questions

Coupling theory has been developed both by the force of applications and by considering general questions.

Q1: How *much* can one couple?

(Path functionals as well as particles?)




Q2: When can one *avoid* coupling?

(Thematic: can one couple reflected BM in compact domains so as to stay substantially far apart?)

▸ ANIMATION

Call this Shy coupling.





# Q1: Coupling Path functionals

Brief survey of known results for the first question:

| Path and functionals | Couplings | |
|---|---|---|
| Brownian motion[1] $B$ | refl | Lindvall (1982) |
| $B$, Local time at zero $L$ | refl + sync | WSK (2014) |
| $B, \int B \, \mathrm{d}t$ | refl + sync | (Ben Arous et al. 1995) |
| $B, \int B \, \mathrm{d}t, \int\int B \, \mathrm{d}s \, \mathrm{d}t$ | refl + sync | WSK and Price (2004) |
| $B, \int B \, \mathrm{d}t, \ldots, \int \ldots \int B \, \mathrm{d}s \ldots \mathrm{d}t$ | Morse-Thue | WSK and Price (2004) |
| $BM(\mathbb{R}^2)$, stochastic area[2] | refl + sync | (Ben Arous et al. 1995), WSK (2007) |
| $BM(\mathbb{R}^n)$, $\binom{n}{2}$ stochastic areas | refl + rotate | WSK (2007) |

Coupling single stochastic area: ▸ HEISENBERG ANIMATION

- WSK (2010) results extend to bounds on speed of coupling for (multiple) stochastic areas.
- Couple all invariant diffusions on nilpotent Lie groups?
  all hypoelliptic diffusions?

[1] One-dimensional Brownian motion
[2] Stochastic area: $\int B_i \, \mathrm{d} B_j - B_j \, \mathrm{d} B_i$





# Q2: Avoiding coupling

We say that an (immersion) coupling of two instances $X$ and $\widetilde{X}$ of a Markov process, started at two distinct points, is shy if there is $\varepsilon > 0$ such that

$$\mathbb{P}\left[\inf_{t>0} |X_t - \widetilde{X}_t| > \varepsilon\right] \quad = \quad 1\,.$$

Problem: Describe situations in which shy coupling is impossible.

* We could also envisage non-immersion shy coupling: but very little is known about this.





# Q2: Tools for shy-ness (I):

## Reflecting Brownian motion

Main focus of this talk: shyly immersion-coupling reflecting Brownian motions in suitably regular Euclidean domain $D$.

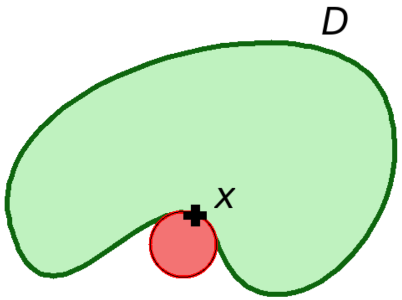
**Theorem**

Reflecting Brownian motion $X$ is given by

$$\mathrm{d}X \quad = \quad \mathrm{d}B - \nu_X \, \mathrm{d}L^X$$

- Here $\nu$ is outward-pointing unit normal vector normal to $\partial D$, $L^X$ is local time spent by $X$ on the boundary $\partial D$.
- Regularity: Saisho (1987) (also Lions and Sznitman 1984) shows that we can make sense of SDE, $\nu$, $L^X$ *via* Skorokhod construction if $D$ has:
  - Uniform Exterior Sphere condition (UESC);
  - Uniform Interior Cone Condition (UICC).

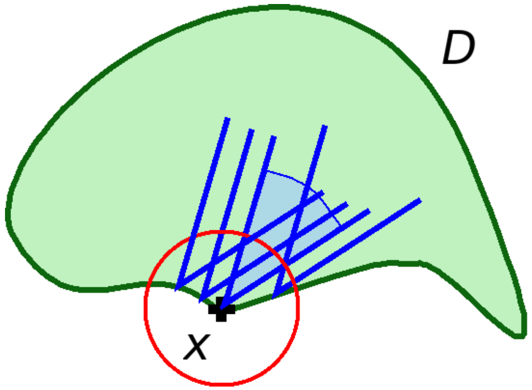
## Uniform Exterior Sphere condition




Uniform Exterior Sphere condition (UESC),
equivalent to “weak convexity” of the domain.



## Uniform Interior Cone Condition




Uniform Interior Cone Condition (UICC),
cone axis can be chosen to be locally constant;
equivalent to the domain being Lipschitz
(local chart using Lipschitz function).



## Tools for shy-ness (II):

### Representation of immersion-coupled reflecting Brownian motions

The following is in the folklore of stochastic calculus.

**Theorem**

Representation: suppose $A$, $B$ are two immersion-coupled Euclidean Brownian motions. Suppose the filtration also supports an independent Euclidean Brownian motion. Then one can construct adapted matrix-valued processes $\mathbb{J}$, $\mathbb{K}$, and a Brownian motion $C$ independent of $B$, with

$$\mathrm{d}\,A \quad = \quad \mathbb{J}^\top \,\mathrm{d}\,B + \mathbb{K}^\top \,\mathrm{d}\,C\,.$$

In fact it also follows that $\mathbb{J}^\top\mathbb{J} + \mathbb{K}^\top\mathbb{K} = \mathbb{I}$ the identity matrix.

(So we can parametrize immersion couplings using $\mathbb{J}$.)



## Shy-ness (I)

Brief survey of previous results for Question 2.

Shy-ness clearly relates to convexity . . .

- Evidently shy coupling *can* occur in an annulus.

  ▸ SHY ANIMATION (II)

- However it is reasonable to suppose that domain-convexity precludes shy coupling.

  | | |
  |---|---|
  | Convex $C^2$ planar domain, regularity[1] | (Benjamini et al. 2007) |
  | Convex planar domain | WSK (2009) |
  | Convex domain in $\mathbb{R}^n$, regularity[3] | WSK (2009) |

- WSK (2009) method of proof: potential theory; view coupling as a degenerate problem in stochastic control; find an appropriate function which is a supermartingale under all couplings.

[1] supporting lines touch boundary only at isolated points

# Shy-ness (II)

Can one go further than convexity-related conditions? YES!

- Bramson suggested (2008, personal communication): no shy-ness in any planar simply-connected domain!
- Bramson, Burdzy, and WSK (2013) prove this, so long as domain is Lipschitz (UICC) and satisfies UESC. (*Nearly* required for strong reflecting BM: Saisho 1987.)
- This is a special case of a much stronger result: no shy-ness in CAT(0) (regular) domains!

▸ SHY ANIMATION (III)

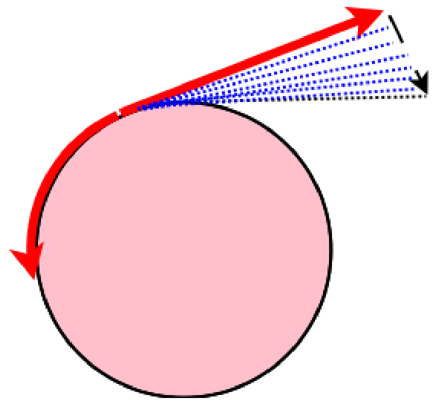
Bramson, Burdzy, and WSK (2013): Shy Couplings, CAT(0) Spaces, and the Lion and Man. We now introduce further tools required for this investigation.





# Health warning

Itô analysis of intrinsic distance produces singularities in drift away from zero!



So it appears that generalizations need an approach working with $1^{st}$- rather than $2^{nd}$-order quantities.

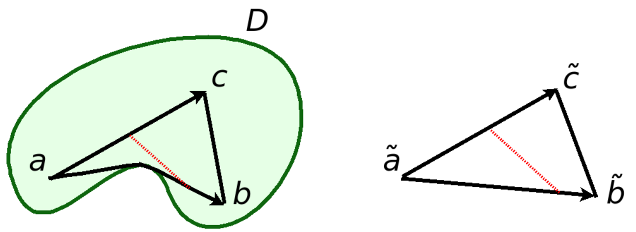




# CAT(0) spaces

CAT(0) is an integrated form of a curvature constraint, so may be such a first-order quantity. Consider a connected (open) subset $D$ of Euclidean space.

- Furnish it with the intrinsic metric; the distance between two points is the least length of a connecting path lying completely in $D$.
- Say $D$ is a CAT(0) domain if intrinsic geodesic triangles are skinnier than comparable Euclidean triangles.








# Reshetnyak majorization

A powerful result capturing the intuition that CAT(0) geometry can be guessed from Euclidean analogues:

**Theorem**

(Rešetnjak 1968) Given a regular closed unit-speed curve $\zeta$ in a CAT(0) domain $D$, one can construct

- a convex subset $C$ of the plane bounded by a closed unit-speed curve $\overline{\zeta}$,
- and a distance-non-increasing map $\phi : C \to \mathbb{R}^2$, such that $\phi \circ \overline{\zeta} = \zeta$, with $\phi$ preserving arc-length distance between $\overline{\zeta}$ and $\zeta$.

There is a similar result for CAT(1) spaces, referring to the unit sphere, subject to the constraint that the curve $\zeta$ should have length less than $2\pi$.

## The Lion and Man

A problem in recreational mathematics:

- Richard Rado (1925) proposed the Lion and Man problem: Lion $X$ chases Man $Y$ around disk. Both move at unit speed, are arbitrarily agile, and tireless. Can the Lion catch the Man?
- Obviously yes; $X$ to centre of disk, $Y$ moves as far away as possible and keeps running, $X$ can capture $Y$ by moving on circle of half radius.
- Never trust an argument containing the word "obviously". Besicovitch showed that if $Y$ moves slightly away from boundary then $Y$ can avoid $X$ for ever (pretty argument revolving around standard criterion for convergence / divergence of $\sum n^{-\alpha}$).
- The Lion gets arbitrarily close, but never actually catches up with Man. What has this to do with shy coupling?





## Shy-ness ideas of proof (I)

The idea is fairly simple, but careful new CAT(0) geometry arguments are required.

CAT(0) version of Lion-and-Man problem;

**Theorem**

The Lion can draw arbitrarily close to the Man in a bounded CAT(0) domain.

IDEA:

- Lion uses "greedy" strategy of direct pursuit;
- Lion draws close if Man does not run directly away;
- Man runs out of domain if he does not curve enough.





## Shy-ness ideas of proof (II)

- Derive vector-field $\chi(X, Y)$ from "greedy" pursuit strategy using CAT(0) arguments;
- Impose large multiple of $\chi$ on SDE for coupled reflecting BMs (WSK 2009):

$$
\begin{aligned}
\mathrm{d}X &= \mathrm{d}B & &+ n\chi(X,Y)\,\mathrm{d}t & &-\nu_X\,\mathrm{d}L^X\,,\\
\mathrm{d}Y &= \left(\mathbb{J}^\top \mathrm{d}B + \mathbb{K}^\top \mathrm{d}A\right) & &+ n\mathbb{J}^\top\chi(X,Y)\,\mathrm{d}t & &-\nu_Y\,\mathrm{d}L^Y\,;
\end{aligned}
$$

- Weak convergence, time-change $\Rightarrow$ deterministic Lion-and-Man $\Rightarrow$ $X$ gets close to $Y$ for large $n$;
- Use Cameron-Martin-Girsanov theorem to translate vector-field into change-of-measure;
- Deduce positive chance for $X$, $Y$ to break shy-ness *however coupled.*

  Technical part:
  establish regularity of $\chi$, make above quantitative.





## Shyness and CAT(0) domains

**Theorem**

Suppose $D$ is a bounded CAT(0) domain satisfying UESC and UICC. Then there are no shy immersion couplings of reflected Brownian motion in $D$.

**Corollary**

Suppose $D$ is a bounded planar simply-connected domain satisfying UESC and UICC. Then there are no shy immersion couplings of reflected Brownian motion in $D$.

## Geometry of shy-ness?

(Bramson, Burdzy, and WSK 2014, ignoring technicalities)

- What geometric structures might be connected to shy-ness?
- Notion of contractible rubber band: a loop which is length-monotonic homotopic to a point.
- Well-contractible rubber band: the homotopy can be chosen so that relative rate of contraction is bounded away from unity.
- RB-contractible domain: every loop is well-contractible (with uniformly specified bound).
- Stable rubber band: no concatenation power of the loop $\ell$ can be *locally* perturbed to a loop which is quantifiably far from $\ell$ and quantifiably shorter.





## Stable rubber bands and evasion

**Theorem**

Suppose $D$ is a bounded CAT$(1)$ domain and contains a stable rubber band. If the Man starts on the rubber band, and the Lion starts away from the Man, then the Man has a successful evasion strategy.

▸ Graphic of idea to show evasion for stable rubber band





## RB-contractible domains and capture

**Theorem**

Suppose $D$ is a bounded CAT$(1)$ domain and RB-contractible. If the Lion starts within $\pi$ of the Man, then the Lion can draw arbitrarily close to the Man.

▸ Graphic of idea to show capture in RB-contractible domain





## RB-contractible domains and shy coupling

We can now describe a wide class of domains which support no shy coupling.

**Theorem**

Suppose $D$ is bounded, CAT$(1)$ and RB-contractible, and suppose $D$ is UESC and UICC. Then there can be no shy immersion coupling for reflected Brownian motion in $D$.

**Corollary**

Suppose $D$ is bounded, starlike, and is UESC and UICC. Then there can be no shy immersion coupling for reflected Brownian motion in $D$.

## Further Questions

These results suggest some significant foundational questions for Coupling (Probability)[1].

- Can one say anything about bounded domains in which shy coupling occurs? Conjecture: in such domains one can implement a shy coupling using domain symmetries.
- Bold conjecture:
  It is impossible to be shy in simply-connected bounded domains of any dimension.
- Can one develop a theory for *non-co-adapted* shy coupling?

THE END

[1] Immersion coupling! (unless otherwise stated . . . )





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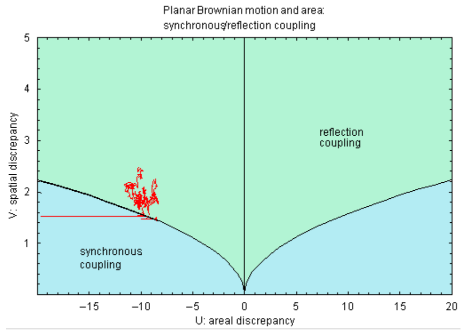
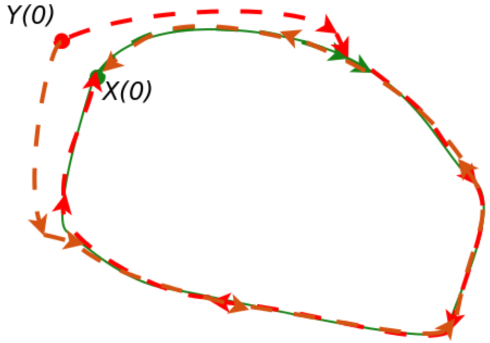
## Coupling BM and stochastic area




WSK (2010) distributional asymptotics for coupling time: $(U_0^2/V_0^2)\times$ reciprocal of Gamma; uses Lamperti (1972). BACK

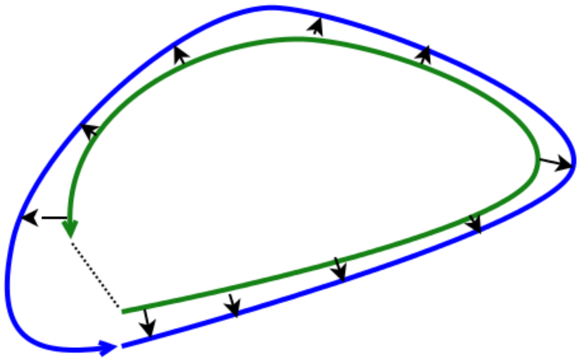
## Stable rubber bands and evasion




- Man $X$ moves along stable rubber band;
- Lion $Y$ in $\varepsilon$-hot-pursuit;
- When Lion close at $t$, follow geodesic to $X(t)$;
- Lion then chases Man round rubber band.
- Ditto for opposite direction, contradicts stability.



## RB-contractible domains and shy coupling



- Man $X$ escapes: construct doubly-infinite geodesic;
- Dense by boundedness: build closed curve $K$ by appending small segment;
- Consider any curve $\widetilde{K}$ tracing $K$ closely;
- Trap off curvilinear rectangles. Reshetnyak majorization and spherical trigonometry show $\widetilde{K}$ cannot be much shorter than $K$.