Probability 1, Autumn 2015, Problem sheet 9

To be discussed on the week 30 Nov. . . 4 Dec.
Problems marked with "PrCl" are discussed in the large problem class on Thursday 26 Nov.
Mandatory HW's are marked with "**HW**", they are due on the week 7 Dec. . . 11 Dec, the latest.[1]
Solutions will be available on Blackboard on the 12th Dec.

HW 9.1 The time, in years, a radio of a certain type functions is exponentially distributed with mean 6 years. Being a popular piece of engineering, this radio set has been manufactured for a long period. My friend and I both buy a used (and of course operational) radio of this type. It turns out that mine was made 9 years ago while my friend's set is only 2 years old.

(a) What is the probability that both my and my friend's radio will work 6 years from now?

(b) What is the chance that my radio stops working before my friend's does?

HW 9.2 Bob plays roulette in the casino. Every round he bets 10 tokens on 'red'. After 100 rounds he has lost 300 tokens. Is he reasonable when he thinks that the croupier is cheating? (On the game roulette one has 37 fields, numbered from 0 to 36. Out of these, '0' has color green, and 18 fields are red, and 18 are black. Betting on 'red' pays 10 extra tokens when the roll is red, and looses the 10 tokens bet when it's not red.)

HW 9.3 (a) We randomly select a point on the $[0, 1]$ interval of the x axis. Let D denote the distance between this point and the point at coordinate $(0, 1)$ of the plane. Determine the density function of the distribution of the random variable D.

(b) We randomly select a point on the $[-1, 1]$ interval of the x axis. Let D denote the distance between this point and the point at coordinate $(0, 1)$ of the plane. Determine the density function of the distribution of the random variable D.

9.4 Alice and Bob roll their own fair die in every second. What is the chance that they see their first ⚅'s at the same time?

9.5 Suppose that in the post office we have a Poisson(40) number of customers in the first hour. Each customer independently of everything is a female with probability 55% and male with probability 45%. If 25 female customers visited the post office in the first hour, find the probability that meanwhile there were 20 male customers. *Recall the example from the problem class.*

PrCl **9.6** Let $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$ be independent. Identify the distribution of $(X \,|\, X+Y)$. That is, find and recognise the conditional mass function $\mathfrak{p}_{X|X+Y}(i \,|\, n) = \mathbf{P}\{X = i \,|\, X+Y = n\}$. Is the answer surprising?

9.7 Let X and Y be i.i.d. Geom(p) random variables.

(a) Can you guess the value of $\mathbf{P}\{X = i \,|\, X+Y = n\}$? If the second Head of a sequence of independent (biased) coinflips comes at the n^{th} time, what are the probabilities that the first Head comes at the i^{th} flip, $i = 1, 2, \dots, n-1$?

(b) Verify your intuition by actually calculating $\mathbf{P}\{X = i \,|\, X+Y = n\}$. *Recall that on the problem class we have seen the distribution of* $X+Y$.

9.8 Rolling two dice, let X be the smaller and Y the larger number shown (they can of course be equal). Determine the conditional mass function $\mathfrak{p}_{X|Y}(i \,|\, j)$ for all relevant i, j values.

9.9 It is of course clear from the probabilistic definition, but now prove by computation that the sum of n i.i.d. Bernoulli(p) variables is of Binomial(n, p) distribution. *Use induction on* n.

9.10 Find the probability mass function of the sum of $X \sim \text{Bernoulli}(p_1)$ and an independent $Y \sim \text{Bernoulli}(p_2)$ variable.

9.11 Let $p_1 \neq p_2$, $X \sim \text{Geom}(p_1)$ and $Y \sim \text{Geom}(p_2)$ be independent. Calculate the probability mass function of $X+Y$.

PrCl **9.12** Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Here is a way to do it: substitute $y = \sqrt{2z}$ into the definition $\Gamma(\frac{1}{2}) = \int_0^\infty z^{-1/2} \mathrm{e}^{-z}\,\mathrm{d}z$, and compare with the standard normal density.

9.13 Compute the k^{th} moment of the Gamma(α, λ) distribution for any $k \geq 1$ integer and $\alpha, \lambda > 0$ reals. In particular, verify $\mathbf{E}X = \frac{\alpha}{\lambda}$, $\mathbf{Var}X = \frac{\alpha}{\lambda^2}$.

[1] Details of how to hand in are to be discussed with your tutor.

HW 9.14 Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Gamma}(\alpha, \lambda)$ be independent. Prove $X + Y \sim \text{Gamma}(\alpha + 1, \lambda)$ (for any α, λ positive reals).

9.15 N people arrive to a business dinner one after the other. Upon arrival each of them looks for friends. If they find a friend then they sit to this friend's table, otherwise they open a new table. Suppose that any two people are friends, independently, with probability p and determine the expected number of tables opened by the N people. *HINT: let X_i be the indicator that the i^{th} person to arrive opens a new table upon arrival.*

9.16 A biased coin, that comes up Heads with probability p, is flipped ten times. Let X be the number of runs. (That is, the number of sequences of H's only or of T's only. Example: TTTHHTHHHH has four runs.) Find $\mathbf{E}X$. *HINT: use a clever indicator.*

9.17 We roll a fair die until each of the six numbers occur at least once. What is the expected number of rolls we make?

9.18 In a lecture room of 300 students (that's almost the figure for Probability 1), find the expected number of distinct birthdays that is, the expected number of days with at least one birthday on them. Forget about leap years, make and state the natural assumptions. Use indicators.

9.19 Let X and Y be i.i.d. positive random variables. Find the numerical value of $\mathbf{E}\frac{X}{X+Y}$. *HINT: Symmetry!*

9.20 Let $A_1 \dots A_n$ be events. Their indicator variables will be denoted by $\mathbf{1}\{A_i\}$ $(i = 1 \dots n)$ for this problem. Which event does

$$\prod_{i=1}^{n} \big(1 - \mathbf{1}\{A_i\}\big)$$

indicate? On one hand, taking expectation will give the probability of this event. On the other hand, expand the above product according to a 1, singletons of $\mathbf{1}\{A_i\}$'s, products of pairs $\mathbf{1}\{A_i\}\mathbf{1}\{A_j\}$, products of triplets $\mathbf{1}\{A_i\}\mathbf{1}\{A_j\}\mathbf{1}\{A_k\}$, etc. Apply an expectation on each of these terms and conclude the inclusion-exclusion formula.