Coupling methods in stochastic deposition models
(PhD thesis)

Márton Balázs

(Institute of Mathematics, Technical University Budapest)

Supervisor: Prof. Bálint Tóth

Budapest,
26.08.2003

A szerző köszönetét fejezi ki Dr. Tóth Bálint témavezetőnek, továbbá Dr. Fritz Józsefnek valamint a Matematika Intézet Sztochasztika Tanszék munkatársainak és doktoranduszhallgatóinak a doktori tevékenység során felmerült szakmai problémákkal kapcsolatban tanúsított önzetlen segítségükért.

Ezen értekezés bírálatai és a védésről készült jegyzőkönyv a későbbiekben, a Budapesti Műszaki és Gazdaságtudományi Egyetem Természettudományi Karának Dékáni Hivatalában elérhető.

Alulírott Balázs Márton kijelentem, hogy ezt a doktori értekezést magam készítettem és abban csak a megadott forrásokat használtam fel. Minden olyan részt, amelyet szó szerint, vagy azonos tartalomban, de átfogalmazva más forrásból átvettem, egyértelműen, a forrás megadásával megjelöltem.

....

Balázs Márton

Contents

Magyar nyelvű összefoglaló

Az értekezésben az élet számos területén előforduló, véletlen folyamatok által befolyásolt növekedési, lerakódási, vagy áramlástani jelenségek modellezésére konstruált modelleket vizsgálunk. E modellek fő jellemzője, hogy sok véletlenül viselkedő, de egymással kölcsönható objektum együttes fejlődését írják le. Mi e modellek egy családját egységes keretbe foglalva tárgyaljuk. Ez a család több, a *kölcsönható részecske-rendszerek* témaköréből jól ismert modellt tartalmaz, így például az egyszerű kizárásos folyamatot, illetve a zero range folyamat bizonyos fajtáit. Ez utóbbi kismértékű általánosításaként a dolgozatban egy új modell, a *kőműves modell* is bemutatásra kerül.

Az értekezésben a bevezetőben tárgyalt kitekintés és a modellek definiálása után tárgyaljuk azok csatolását, azaz több modell közös véletlenek által vezérelt egyidejű fejlődését. Ennek során definiáljuk a modell által teremtett véletlen közegben mozgó *másodosztályú részecskéket*, melyek kulcsfontosságú szerepet játszanak a tézisben tárgyalt módszerekben. A tézis első részében utalunk modelljeink és bizonyos elsőrendű nemlineáris parciális differenciálegyenletek (ún. *megmaradási törvények*) kapcsolatára. Egy egyszerű érveléssel az új kőműves modellben is alátámasztjuk a differenciálegyenletek ún. *lökéshullám megoldásainak* és a másodosztályú részecskék kapcsolatát, mely a témához kapcsolódó egyéb munkákból már jól ismert jelenség. Erre alapozva egy bizonyos kőműves modellben olyan eloszlás-családot konstruálunk, amely pontosan megfelel a modellhez tartozó differenciálegyenlet egy lökéshullám-családjának. Mivel ezek az eloszlások kivételesen egyszerű alakúak, eddig - tudomásunk szerint - nem találtak lökéshullámnak megfelelő ennyire egyszerű eloszlást.

Az értekezés második részében a modellek fejlődésének fluktuációit vizsgáljuk. Martingálok használatával hamar eljutunk ahhoz a ponthoz, ahol a másodosztályú részecskét természetes módon tudjuk felhasználni, ehhez azonban szükségünk van azok mozgásának bizonyos tulajdonságaira. Mivel az eleve véletlen közegben mozgó, saját véletlenjétől is függő másodosztályú részecske viselkedése igen bonyolult, hosszú levezetés szól a nekünk fontos tulajdonságok bizonyításáról. Eközben a modellek csatolásának technikáit egy lépéssel tovább finomítjuk, hogy képessé váljunk a különböző modellek másodosztályú részecskéinek egymással való csatolására is.

Az első két részben olyan modellekkel foglalkozunk, melyek matematikai konstrukcióját eddig - tudomásunk szerint - senki sem végezte el. Ezek a modellek bizonyos helyzetekben lényegesen gyorsabb növekedést mutatnak már megkonstruált társaiknál, ezért a harmadik részben - alkalmas kezdeti feltételek megléte esetén - elégséges korlátok létezését bizonyítjuk, melyek biztosítják, hogy ezek a modellek is kezelhetőek maradnak fejlődésük során. Habár a matematikai konstrukció még folyamatban van, így nem lehet része ezen értekezésnek, eredményeink mégis valószínűsíthetően a konstrukció nehezebb részét jelentik.

Summary in English

In the thesis we investigate different behavior of processes constructed for modeling domain growth, deposition processes or current of particles, many phenomena occurring in every day's life. The main feature in these models is that they include many randomly behaving objects, interacting with each other. We consider a family of such processes here in a common framework, which contains some of the well-known models from the area of *interacting particle systems*, e.g. the simple exclusion and the zero range processes. As a slight generalization of the latter, we also introduce the *bricklayers' process* in the present thesis.

After describing many phenomena connected to these types of models, we give a precise definition of our systems, and show how to couple them. The evolution of a coupled pair of models is partially driven by joint randomness, hence their evolution is as close to each other as possible. By coupling we introduce the *second class particle*, an object playing an essential role in our methods. In the first part of the thesis, we refer to the connection between our processes and some first-order non linear partial differential equations, namely, the *conservation laws.* By a simple argument, we also indicate for our new bricklayers' model the connection of the second class particle to the so-called *shock solutions* of the corresponding partial differential equation. This is a well-known phenomenon from other works in this field. Based on this relation, we construct a class of distributions in a type of bricklayers' models, which exactly corresponds to a class of shock solutions in the model's partial differential equation. Since these distributions have extremely simple structure, as far as we know, no such simple distributions were found showing the properties of shocks in a microscopic level.

In the second part, we examine the fluctuations of our processes' growth. By using martingale techniques, we get to the point where the use of second class particles becomes very natural. However, the arguments require knowing some special properties of the second class particle, which are quite difficult to establish since the second class particle performs a random motion in the random environment created by the model. Hence long arguments are set in this part to establish the required properties of the second class particle. These arguments refine the coupling methods providing the ability of coupling second class particles between different models.

In the first two parts we dealt with systems of which the dynamics has, as far as we know, not yet been rigorously constructed. In some situations, these processes perform faster growth than the ones already constructed, hence in the third part we establish stochastic bounds which are sufficient to show that these models stay under control while they evolve. Of course, these bounds only apply in case of appropriate initial conditions. Although the construction of dynamics is in progress, and so it can not be part of the present thesis, our results in this direction probably mean the harder part of the problem.

Part I
Introduction

In the world surrounding us, there are several interesting phenomena, which are only dealt with in the last few decades. Some of these areas are behavior of e.g. infection of plants in a plantation, spreading mildew on the wall, residue in a chemical reaction, electrons in solid matter, cars in a traffic jam, people in queues. Surprisingly, these different areas of life can be modeled by very similar processes.

As people realized that *randomness* is essential in these phenomena, they came up with stochastic models to describe them. The other essential property is that the system has many participants interacting with each other. Hence the randomness of one's behavior is in some sense driven by the other's motion. This is the way how interaction is introduced in the system.

On one hand, these processes are constructed to be simple enough that we can handle them in a mathematically rigorous way. But, on the other hand, the randomness influenced by the interaction of the particles makes the models complicated enough to show new and interesting types of behavior. They show properties never seen before, new exponents in long-time behavior, different phenomena in different rescaling limits. In these limits they also have strong connections to partial differential equations, taking us closer to the understanding of the large-scale deterministic evolution of the systems modeled.

Let us mention some of the interesting problems which can be answered by such stochastic processes. The infection of a disease in a plantation (or the expansion of the area covered by mildew) can be imagined as individual points on a lattice which get infected if their neighborhood is already ill. Of course, this process is not deterministic; we have a probability of getting the next point infected in, let us say, the next hour. This probability may very well depend on the state of the neighboring points. If many of the neighbors are already ill, then we expect this probability to be larger. We already see the two main components of stochastic interacting systems: randomness and interaction. Now, the first question is the speed with which the illness (or mildew) advances. Is it true that it has a well defined speed at all? If yes, what is its value? Can one expect that if the edge of the infected area of trees has moved ten meters today then it will again move about ten meters tomorrow, or is randomness too strong to make things so predictable?

We shall see later, that this is actually not a difficult question to answer in our models; in fact the answer is yes, there is usually a well defined speed value with which the edge moves. The next question is much harder: if one knows the speed of this edge, how much will it fluctuate? Does one need to cut all trees in a far distance, or will it be enough to clear up the forest in a few metered-lane around the expected position of the edge of the area?

By the way, how do we know that there is a well defined edge of the infected domain at all? Isn't it the case that the edge of the domain becomes coarser and coarser anyhow? Which are the initial configurations from which we obtain a somewhat smooth boundary of the infected area? And how can this smoothness be violated if starting with other initial states?

1 Introduction to stochastic deposition models

Let us give a brief description of the models considered. A more detailed and precise introduction is available in later sections. Imagine columns which consist of bricks. These columns are put next to each other, forming a wall this way. The wall itself will represent the infected area of plants, residue in a chemical reaction, or the mildew on a surface. New bricks are deposited on the top of the columns by some stochastic method, modeling the growth of the domains indicated above.

One may think that the speed with which new plants get infected is only influenced by the local circumstances. Hence we assume that the local growth rules do not essentially depend on the size and global shape of the occupied area, they only depend on the local configuration. In our models, this means that the "speed" with which a new brick is added to a column will only depend on the *relative* height of the neighboring columns, not on the proper height of that column.

In fact, we will have a stochastic method to deposit new bricks rather than having a real speed; this is the point where we introduce randomness to the system. We assume that the system has no hidden memory. In other words, knowing the actual state of the process determines exactly the stochastic rules which drive the further evolution of the system. Once we know the current state, the previous history of the process does not influence what happens next. We call this principle conditional independence. Of course, this is just an(other) approximation of reality. Almost all phenomena mentioned before may violate this principle, e.g. a tree may infect the other one more intensively if it has already been infected for a long time. But this would lead to the need of other variables to describe the process entirely. Conditional independence makes the models handleable, as we have arrived to the notion of *Markov-processes* at this point. Between the two possibilities, namely, the *discrete time* and the *continuous time* Markov-processes, we choose the latter. It allows us to give a more subtle description while allowing a not too difficult treatment of the process.

To talk about continuous time Markov-processes, we use exponential waiting times, i.e. random times having exponential distribution. By the well-known renewal property, a process controlled by such waiting times obeys the principle of conditional independence. To be more precise, given a configuration of the wall, a brick is added to the top of each column after an exponential waiting time. In order to introduce interaction in the models, we make the parameters (rates) of these exponential waiting times depend on the *relative* heights of the neighboring columns. We emphasize that they do *not* depend on the absolute height of the columns, only on the difference between neighboring columns. By this rule, the edge of the domain in question advances regardless of the global size of the domain; we have time-homogeneity in this sense. The speed with which the infection moves on depends on how many of the neighboring trees are already ill.

We introduce a bit of notation here. We say that the columns of bricks, put next to each other, are located between *sites*, indexed by integer numbers i. Consider two neighboring columns of bricks and the difference between them: subtract the height of the column to the right-hand side from the height of the one to the left-hand side. The result is an integer number ω_i, this is the quantity

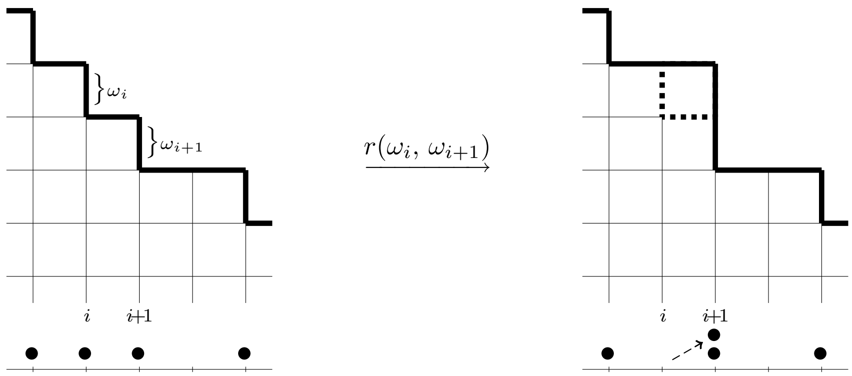

Figure 1: A possible move

we are basically interested in. We calculate it between each neighboring columns i.e. for each site i. When adding a brick to a column between site i and site $i+1$, the difference ω_i to the left-neighboring column decreases by one and, by the same move, the difference ω_{i+1} to the right-neighboring column increases by one. See figure 1, where $\omega_i = 1$, $\omega_{i-1} = 1$ becomes $\omega_i = 0$, $\omega_{i+1} = 2$. This move happens with rate $r(\omega_i,\, \omega_{i+1})$, i.e. after an exponential waiting time having parameter $r(\omega_i,\, \omega_{i+1})$ depending on the neighboring columns relative heights. We shall consider this rate function in more precise forms for different models later on. For an overview of some possibilities, see section 14.2 on page 29.

A natural assumption is some monotonicity condition: the more ill neighbors a tree has the faster it gets infected. In the models the higher neighbors a column has, the bigger rate we have to add the next brick on. The rate function r is non-decreasing in its first, and is non-increasing in its second variable. We call this property of the models *attractivity.* One could very well imagine non-attractive processes of this type. However, if we require some kind of equilibrium, i.e. a notion of the edge of the domain, then we need slower growth for a column which is much higher than its neighbors. Otherwise nothing could stop a column's expansion if it becomes a little larger than the others, and there would not be a front of the domain. Throughout this thesis, we assume attractivity, and we make essential use of it.

2 Particle systems

Before going more into the details, we show how to connect the models to particle systems modeling gases, electrons in matter, cars in traffic jams, people in queues. As we know, ω_i decreases by one and ω_{i+1} increases by one by the same move. This gives the idea to identify the difference ω_i with *number of particles at site i*, see the bottom part of figure 1. A particle disappears from site i and appears at site $i+1$ when a brick is added to the column between sites i and $i+1$. Hence adding a brick corresponds to a jump to the right of a particle over that edge. Of course, we have some difficulty in case ω_i is negative, then we have to introduce antiparticles as well. Observe that in this case, adding a brick to a column means *left* jump of an antiparticle.

As the rate with which a particle jumps from i to $i+1$ (i.e. a brick is added to the corresponding column) depends on ω_i and ω_{i+1}, the behavior of the particles is influenced by the local configuration of the other particles. Our models belong to the field of *interacting particle systems*, an intensively investigated area of mathematics. It is possible to start the models with $\omega_i \geq 0$ for all i, and to set zero-valued rates for any step which would violate this condition. Then we obtain a process with particles only, antiparticles will never be created. This is the way interacting particle systems were introduced and are usually treated. There are two main types of them.

- Zero valued rates can also be set for moves which would imply too many ($> K$) particles at a site. Starting the process from a state with $0 \leq \omega_i \leq K$ for each i, these rates assure this condition for all later times. These models are commonly called totally asymmetric *K-exclusion processes*. Among them, the most famous one is the totally asymmetric *simple exclusion process*, which is the case of $K = 1$. In this model we either have a particle at a site i or not. When having one, it jumps to the right with rate one only if the site to its right is not occupied by another particle. Primarily, these can serve as models for lattice gas, e.g. electrons in solid matter. But the area of applications is much larger than this: one can model many phenomena in which objects excluding each other move in a random way. An interesting example is describing the behavior of cars' motion in a traffic jam, or people walking in crowd. There is again a number of interesting questions to answer: what is the speed of a tagged particle, what is the fluctuation of its position, what happens if something blocks the flow of particles, and what happens when the blocking object is removed from the system?

- For the other kind of processes we have an unbounded number of particles at a site i. Of course, their jump rates depend on the number of them at site i and $i + 1$. A famous example is the *zero range process*, where the jump rate $r(\omega_i, \omega_{i+1})$ doesn't depend on its second variable ω_{i+1} i.e. the number of particles at the arrival site.

Another nice application comes from queuing problems. Consider the distance between two neighboring particles in a simple exclusion process, and represent this as the length of a queue. Then we have an infinite number of queues and servers. When a customer is served (a particle jumps) then he goes to the next queue. We can model the same thing in the zero range process by identifying the number of particles at a site with length of the queue at that site.

3 Coupling the processes

Throughout this thesis, *couplings* are essentially used. The basic idea is quite simple: one can consider the difference between two realizations of the same model. This difference may be realized e.g. in such a way, that for a site i, ω_i is larger in one of the models than in the other. Based on attractivity, it is possible to couple the exponential waiting times and hence the evolution of the two processes. The way this coupling is realized is simply that we consider the waiting times corresponding to similar moves in the two models, and let the faster one (the one with larger rate) happen *always earlier*. Clever use of this principle, together with attractivity, allows us to conserve the number of differences between the two models. Meanwhile, each model of the coupled pair evolves according to its own rules. We call the differences between the models *second class particles*. In some cases, differences of other sign are also introduced, they are the *second class antiparticles*. These objects have their own life with jumps to neighboring sites, annihilation of a second class particle with a second class antiparticle and, of course, interaction with the underlying process. They have one very adorable property: their signed number is conserved by the evolution.

Second class particles are as useful as simple: by realizing the difference between models, they can be atoms in computing correlations, comparing the evolution of two processes, proving domination of one model by the other, showing convergence to some distributions, and characterizing many phenomena shown later.

4 Steady states of the models

In models with stochastic evolution, one can only make probabilistic statements, even if the initial configuration of the process was deterministic. After any time passed, the state realized by the model will be random, having some *distribution*. This distribution may change in time. However, there usually exist distributions which do not change in time, these are called *stationary distributions*. Of course, they may not be unique. Once started from a random state having the stationary distribution, the probabilities of finding the model in given states will not change in time. In this case, we say that the model is in *steady state*.

There is one peculiar property of all our processes: the sum of ω_i-s is conserved. Whenever a move happens, ω_i decreases by one, while ω_{i+1} increases by one, the sum does not change. This phenomenon is clear in the deposition approach: local changes can not effect the sum of the gradient of the wall. It is also obvious if thinking of jumping particles: their total number is conserved by the jumps. We call this phenomenon *conservation of particles*.

As the models evolve on an infinite line, instead of total number of "steps on the wall" or total number of particles, one has to talk about *average slope* of the wall or *average density* of particles. Of course, these quantities do not necessarily exist for all states. However, if looking for the steady state behavior, one expects states in which such space-averages can be computed.

By conservation of particles, the slope of the wall or the particle density is not affected by the evolution of the system. Starting from a state with a given density, it is conserved by the process. Hence the steady state distributions can be characterized by their densities. For any given density, it can be shown that there is a unique stationary distribution; this property is also mentioned as *ergodicity*. See proposition 14.1 on page 31 concerning this question.

As we are primarily interested in the steady state behavior of the processes, we need to handle at least the steady state distributions. Distributions on the state space of our models can be very complicated. For any site i, ω_i can be some integer number, hence our state space is (a subset of) $\mathbb{Z}^{\mathbb{Z}}$. The main difficulty with measures on this space is the presence of correlations: ω_i and ω_j may not be independent for different sites i, j. In view of the interaction between the particles, independence for different sites would be surprising.

The fact is, as we shall see later on, that in some cases the stationary measure has the property that the quantities ω_i are completely independent for different j's. This allows us to handle the stationary measures and to make computations with them.

5 Conservation laws, microscopic shape of shocks

For the large-scale behavior of the models, which is one of the primer interests in view of applications, one can consider the so-called *hydrodynamic limit* of the processes. The idea is that one examines the "local steady state" behavior of the process. Locally, the distribution of the process agrees with the steady state distribution, but one allows its density parameter to change on a large space scale. Of course, this leads out from the steady state distribution of the process, the situation will not be stable anymore. However, things happen slowly in such large scale: one has to rescale time as well. Rescaling space and time this way leads to a deterministic partial differential equation for the density depending on (the rescaled) space and time. Once started from a large-scale density profile, the equation describes its long-time evolution. This equation is usually a non-linear first order hyperbolic equation, called the *hyperbolic conservation law*. Derived by conservation of particles, these equations describe the evolution of a conserved quantity, whence their name comes from.

There are many ways to rescale space and time, see e.g. Tóth and Valkó [29]. The method we are interested in is when time and space are rescaled by the same constant, this is called Euler scaling. In context of Eulerian hydrodynamic limit of our type of models, see Rezakhanlou [22], Seppäläinen [25] and Tóth and Werner [30], we do not deal with this very nice and deep area here.

Solutions of the hyperbolic conservation laws show new and interesting behavior. For a large class of smooth initial data, the solution becomes discontinuous after a finite amount of time passed. The discontinuities are called shocks, and they move with a speed determined by the so-called *Rankine-Hugoniot formula*. For other initial configurations, even a discontinuous solution becomes smooth in an arbitrarily small time interval. These kinds of evolution are called rarefaction waves. Concerning physical applications, shocks and rarefaction waves describe the behavior of the edge between phases (of cars' traffic, electrons, particles) with different densities. There are many extremely interesting questions concerning uniqueness, time-reversal and physical relevance of solutions, see e.g. Smoller [27] in this direction.

While the hydrodynamic limit shows us this rich large-scale *macroscopic* behavior of the models, one is interested in how these structures look like in the *microscopic level*, i.e. in the stochastic model itself. A shock solution describes a discontinuity in a rescaled space variable, but is there really something discontinuous in the microscopic model itself? The surprising fact is that the microscopic objects corresponding to shocks are in close connection to the second class particles. More precisely, these objects are usually constructed by putting one single second class particle to the system, and building up a special distribution on the state space *relative to the position of the second class particle*. Many works consider this problem, e.g. De Masi, Kipnis, Presutti, Saada [5], Derrida, Lebowitz, Speer [6], Ferrari [7], Ferrari, Fontes, Kohayakawa [9], Gärtner and Presutti [10], Rezakhanlou [23]. Part II of the present thesis contains a more detailed description of this problem and a new, simple result in this direction.

The natural question of putting the second class particle into a rarefaction wave raises here. See Kipnis and Ferrari [18] concerning this situation.

6 Fluctuations of the growth

Once started from stationary distribution, it is easy to calculate the average speed of growth of the wall or the average current of the particles. The difficult thing is to determine the fluctuation of this quantity. Ferrari and Fontes [8] calculated it for the case of the simple exclusion process. As a nice application of second class particles, the result is obtained by combinatorial considerations and by knowing that a single second class particle put in a steady state model has a well defined speed. The latter fact is shown in Ferrari [7].

Generalizing this result to our wider class of models seemed to be a work worth to do. It is contained in part III. As in Ferrari and Fontes' work, the arguments are separated into two parts. In the first part, the result is achieved by using martingale considerations and algebraic computations based on the well defined speed of the second class particle put in a steady state model. In the second part, this speed value is determined and its precise probabilistic meaning is shown under some extra conditions on the rate function r. This part includes various new coupling methods and a diffusion-type random process in the random environment provided by the models. Generalizing the results led to better understanding of the formula which describes the asymptotic behavior of the growth fluctuations in large time scale.

However, in some special cases the asymptotic behavior of the fluctuations is uninteresting in the time scale we worked on. In these cases much more complicated and detailed analysis is needed. Spohn and Prähofer [20] indicate results in this field.

7 Bounds on the growth

When talking about a model, the first question to raise is the existence of dynamics. One need to check if there really exists a Markov-process with the desired properties. This is easily done as far as we have a *countable* state space. But many of our arguments (on long-time behavior, for example) use essentially the model on an infinite line, where the values of i are not bounded. Even in the simplest case of the simple exclusion process, this implies an uncountably infinite state space. The construction of similar Markov-processes on such spaces is not trivial at all. There are methods for doing this in Liggett's book [15] in cases when the number of particles at a site, i.e. the values of ω_i are bounded.

The case of zero range and bricklayers' process is different. We do not have a bounded number of particles (or wall-gradients) at a site for these processes. The problem may be here that for some initial configurations an avalanche of particles (or deposited bricks, respectively) comes in from infinity, causing an infinite number of jumps in a finite time-interval over a site i. In this case the process becomes unmanageable, one can not talk about Poisson processes anymore, we say that the dynamics does not exist.

Nevertheless, there are construction methods for this case as well. Liggett [13] gives a construction for the zero range process, when the jump rates r satisfy a regularity assumption. Andjel [1] weakens these assumptions to a Lifshitz condition $|r(z+1) - r(z)| \le K$ for each $z \in \mathbb{Z}$. In this latter case, the growth of the process can be compared to *branching processes*, a well-known area of probability theory, and the desired bounds on the models' growth can be given. Having these bounds, one can show that the process in question is really Markovian, and really has the jump rates (more precisely, Markovian generator) which we wanted. Similar arguments work for the bricklayers' process as well, see Booth [4] or Quant [21].

For the shock-like distribution shown in part II of the thesis, we need exponentially growing rate functions r. Also, when establishing the speed of the second class particle in part III, our coupling methods requires convex rate functions r, which, in some cases, may imply faster than linear growth of them. As far as we know, no construction method is available in this situation.

In the case of attractive bricklayers' and zero range processes, with the help of an auxiliary finite-volumed process with nice stationary distributions, coupling is again applicable to prove stochastic bounds on the growth of a column, hence to show a.s. finiteness of them. The arguments work *regardless the rate of growth of r depending on ω_i's*, and are shown in part IV. As stochastic bounds are usually the harder part of constructing the process, one has the hope that existence of dynamics follows from the present stage of statements; this is a work in progress.

8 The structure of the thesis

The material of this PhD thesis is contained in "two and a half" papers. The first of them on a shock-like distribution is published [2], and is contained in part II. The second one containing the formula for the asymptotic variation of the particle current (or the column's growth, respectively) and the speed of a single second class particle put in a steady state model has been accepted for publication [3], and is contained in part III. Finally, the results so far achieved concerning the existence of dynamics for these models are prepared in a form for a future paper, and are placed in part IV. We decided not to change the original form of the papers, hence each part represents an individual paper. For this reason, introductions of them may at some points overlap, for which the author asks for the Reader's excuse.

Part II

Microscopic shape of shocks in a domain growth model

Abstract

Considering the hydrodynamical limit of some interacting particle systems leads to hyperbolic differential equation for the conserved quantities, e.g. the inviscid Burgers equation for the simple exclusion process. The physical solutions of these partial differential equations develop discontinuities, called shocks. The microscopic structure of these shocks is of much interest and far from being well understood. We introduce a domain growth model in which we find a stationary (in time) product measure for the model, as seen from a defect tracer or second class particle, traveling with the shock. We also show that under some natural assumptions valid for a wider class of domain growth models, no other model has stationary product measure as seen from the moving defect tracer.

Key-words: second class particle; shock solution.

9 Introduction

The hydrodynamical limit of the nearest neighbor asymmetric simple exclusion model leads to the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

which is a special case of the one-component hyperbolic conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial J(u)}{\partial x} = 0 \tag{1}$$

where $u \mapsto J(u)$ is a smooth, typically convex function. (By changing x to $-x$, concave J-s can be transformed to convex ones.) This equation has a shock (weak) solution starting with initial data

$$u(0,\, x) = \begin{cases} u_{\text{left}} \ , & x < 0 \\ u_{\text{right}} \ , & x \geq 0 \end{cases}$$

with $u_{\text{left}} > u_{\text{right}}$. The stable weak solution is of the form

$$u(t,\, x) = \begin{cases} u_{\text{left}} \ , & x < st \\ u_{\text{right}} \ , & x \geq st \end{cases}$$

where the speed s of the traveling shock is determined by the Rankine-Hugoniot formula

$$s = \frac{J(u_{\text{right}}) - J(u_{\text{left}})}{u_{\text{right}} - u_{\text{left}}} \quad , \tag{2}$$

see e.g. [27]. This is what we see on a macroscopic scale. The microscopic structure (i.e. on the level of particles) of the shock is of great interest. It has been considered in the context of the asymmetric simple exclusion process, and rather complicated microscopic structures have been found [5] [6] [7] [9] [10]. In the more general context of attractive particle systems the microscopic structure of the shock was investigated by [23].

In the present note we consider a class of one-dimensional domain growth models, parametrised by a jump rate function, $r : \mathbb{Z} \to \mathbb{R}$. In a special case of the rate function we show that the shock, as seen from a defect tracer (second class particle) has stationary (in time) distribution of product structure which we identify. We also show that this is a peculiarity of the case considered, no other model in the wide class of these models has this property. The structure of the paper is the following:
In section 10 we define the class of models considered and determine the stationary distributions for them.
We describe the hydrodynamic limit of these models and calculate the speed of the shocks using Rankine-Hugoniot formula (2) in section 11.
In section 12 we introduce the defect tracer in our models. Via Rankine-Hugoniot formula, we also give an indication on the fact that, in general, shock solutions are closely related to measures stationary as seen from the defect tracer.
The last section contains our main result on the product structure of such a stationary distribution as seen from the defect tracer. This gives an explicit description of the microscopic shape of some types of shock solutions.

10 The bricklayers' model

10.1 Infinitesimal generator

We consider the phase space

$$\Omega = \{\underline{\omega} = (\omega_i)_{i\in\mathbb{Z}} \ : \ \omega_i \in \mathbb{Z}\} = \mathbb{Z}^{\mathbb{Z}} \quad .$$

For each pair of neighboring sites i and $i+1$ of $\mathbb{Z}$, we can imagine a column built of bricks, above the edge $(i,\, i+1)$. The height of this column is denoted by h_i. If $\underline{\omega}(t) \in \Omega$ for a fixed time $t \in \mathbb{R}$ then $\omega_i(t) = h_{i-1}(t) - h_i(t) \in \mathbb{Z}$ is the negative discrete gradient of the height of the "wall". The growth of a column is described by Poisson processes. A brick can be added to a column:

$$(\omega_i,\, \omega_{i+1}) \longrightarrow (\omega_i - 1,\, \omega_{i+1} + 1) \quad \text{with rate} \quad r(\omega_i) + r(-\omega_{i+1}) \quad .$$

See fig. 2 for some possible instantaneous changes. The process can be represented by bricklayers standing at each site i, laying a brick on the column on their right with rate $r(\omega_i)$ and laying a brick to their left with rate $r(-\omega_i)$. This interpretation gives reason to call these model bricklayers' model. For small ε the conditional expectation of the growth of the column between i and $i+1$ in the time interval $[t,\, t+\varepsilon]$ is $\{r(\omega_i(t)) + r(-\omega_{i+1}(t))\} \cdot \varepsilon + \mathfrak{o}(\varepsilon)$. Note that the process has a left-right mirror symmetry, i.e. the rate of a column's growth is the same as if looking at the reflected configuration. We want the dynamics to smoothen our interface, that is why we assume monotonicity of the rate function

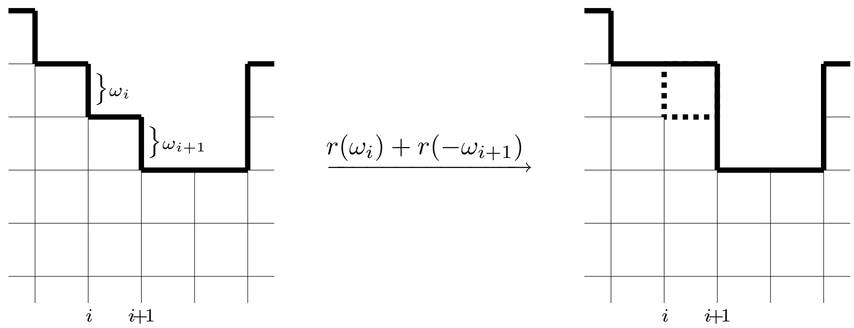

Figure 2: A possible move

r, which means that a column grows more rapidly if it has a higher neighbor on the right or on the left. In later sections we shall impose another restrictive condition on r, see (5).

At time t, the interface mentioned before is described by $\underline{\omega}(t)$. Let $\varphi : \Omega \to \mathbb{R}$ be a bounded cylinder function i.e. φ depends on a finite number of values of ω_i. The growth of this interface is a Markov process, with the formal infinitesimal generator L:

$$(L\varphi)(\underline{\omega}) = \sum_{i\in\mathbb{Z}} \Big\{ [r(\omega_i) + r(-\omega_{i+1})] \cdot [\varphi(\dots, \omega_i - 1, \omega_{i+1} + 1, \dots) - \varphi(\underline{\omega})] \Big\} \quad .$$

Note that for each index i, ω_i can also be negative hence direct particle interpretation fails, see the remark after formula (4).

When constructing the process rigorously, problems may arise due to the unbounded growth rates. The system being one-component and attractive, we assume that existence of dynamics on a set of tempered configurations $\widetilde{\Omega}$ (i.e. configurations obeying some restrictive growth conditions) can be established by applying methods initiated by Liggett and Andjel [1] [12]. Technically we assume that $\widetilde{\Omega}$ is of full measure w.r.t. the canonical Gibbs measures defined in 10.2. We do not deal with this question in the present paper.

The exponential bricklayers' model

A special case of the models is the exponential bricklayers' model (EBL), where for $z \in \mathbb{Z}$

$$r(z) = \mathrm{e}^{-\frac{\beta}{2}}\,\mathrm{e}^{\beta z} \tag{3}$$

with a positive real parameter β.

10.2 Translation invariant stationary product measures

In this subsection we show a natural way to construct a stationary translation invariant product measure for our models. By chapter one of [15], a measure μ

is stationary, iff for any bounded cylinder function φ,

$$\mathbf{E}(L\,\varphi)(\underline{\omega}) = 0$$

is satisfied for a process distributed according to μ. We assume μ to be a product measure with marginals

$$\mu(z) = \mu\{\underline{\omega} \,:\, \omega_i = z\}$$

for $z \in \mathbb{Z}$. By changing variables and using product structure of μ,

$$\begin{aligned}
&\mathbf{E}(L\,\varphi)(\underline{\omega}) = \\
&\quad = \mathbf{E}\sum_{i\in\mathbb{Z}}\Big\{[r(\omega_i) + r(-\omega_{i+1})]\cdot[\varphi(\dots,\,\omega_i - 1,\,\omega_{i+1}+1,\,\dots) - \varphi(\underline{\omega})]\Big\} = \\
&= \mathbf{E}\sum_{i\in\mathbb{Z}}\Big[r(\omega_i{+}1)\cdot\frac{\mu(\omega_i+1)}{\mu(\omega_i)}\cdot\frac{\mu(\omega_{i+1}-1)}{\mu(\omega_{i+1})}{+}r(-\omega_{i+1}{+}1)\cdot\frac{\mu(\omega_i+1)}{\mu(\omega_i)}\cdot\frac{\mu(\omega_{i+1}-1)}{\mu(\omega_{i+1})} \\
&\qquad\qquad - r(\omega_i) - r(-\omega_{i+1})\Big]\cdot\varphi(\underline{\omega}) \quad .
\end{aligned}$$

This expression becomes zero if we make the sum telescopic on the cylinder set supporting φ. Hence stationarity of μ is assured by assuming

$$r(z)\cdot\frac{\mu(z)}{\mu(z-1)} \quad \text{and} \quad r(-z)\cdot\frac{\mu(z)}{\mu(z+1)}$$

to be constants. As a consequence, we obtain the condition

$$r(z)\cdot r(-z+1) = \text{constant} \quad . \tag{4}$$

There are two essentially different choices.

$$r(z)\cdot r(-z+1) = 0$$

defines models of zero range types, we do not consider this possibility here. The other choice is choosing the right-hand side of (4) to be a positive constant. In this case, by rescaling time, we can turn this constant to be one without loss of generality:

$$r(z)\cdot r(-z+1) = 1 \quad . \tag{5}$$

Rates (3) of the EBL model satisfy this condition.

For $n \in \mathbb{N}$, we define

$$r(n)! := \prod_{y=1}^{n} r(y)$$

with the convention that the empty product has value one. Let

$$\bar{\theta} := \log\Big(\liminf_{n\to\infty}\,(r(n)!)^{1/n}\Big) = \lim_{n\to\infty}\log(r(n)) \quad ,$$

which is strictly positive by (5) and by monotonicity of r, and can even be infinite. With a generic real parameter $\theta \in \big(-\bar{\theta},\,\bar{\theta}\big)$, we define

$$Z(\theta) := \sum_{z=-\infty}^{\infty}\frac{\mathrm{e}^{\theta z}}{r(|z|)!}$$

and the product measure $\mu^{(\theta)}$ with marginals

$$\mu^{(\theta)}(z) := \frac{1}{Z(\theta)} \cdot \frac{e^{\theta z}}{r(|z|)!} \quad , \tag{6}$$

which has the property

$$\begin{aligned} r(z) \cdot \frac{\mu^{(\theta)}(z)}{\mu^{(\theta)}(z-1)} &= e^{\theta} \\ r(-z) \cdot \frac{\mu^{(\theta)}(z)}{\mu^{(\theta)}(z+1)} &= e^{-\theta} \quad , \end{aligned} \tag{7}$$

thus it is stationary. We call these measures canonical Gibbs-measures.

For the special case of the EBL model, for $\theta \in (-\infty, \infty)$, we obtain the discrete normal distribution

$$\mu^{(\theta)}(z) = \frac{e^{-\frac{\beta}{2}\left(z-\frac{\theta}{\beta}\right)^2}}{e^{-\frac{\theta^2}{2\beta}} Z(\theta)} = \frac{e^{-\frac{\beta}{2}(z-m)^2}}{\widetilde{Z}(\beta, m)} \tag{8}$$

with the notation $m := \theta/\beta$.

11 Hydrodynamical limit

Being attractive due to monotonicity of r, we can take the hydrodynamical limit of a bricklayers' model by

$$u(t, x) := \mathbf{E}\, \omega_{x/\varepsilon}(t/\varepsilon) \quad . \tag{9}$$

Then via formal computations we obtain differential equation (1)

$$\frac{\partial u}{\partial t} + \frac{\partial J(u)}{\partial x} = 0$$

as $\varepsilon \to 0$, where $J(u)$ is defined as follows. The function $u(\theta) = \mathbf{E}^{(\theta)}(\omega)$ of θ is strictly increasing since the derivative

$$\frac{\mathrm{d}u(\theta)}{\mathrm{d}\theta} = \left(\mathbf{E}^{(\theta)}(\omega^2) - \left(\mathbf{E}^{(\theta)}(\omega)\right)^2\right)$$

is positive $(-\bar{\theta} < \theta < \bar{\theta})$. Let $\theta(u)$ be the inverse function. The quantity $\mathbf{E}^{(\theta)}\left(r(\omega) + r(-\omega)\right)$ depends on θ, and J is defined by

$$J(u) := \mathbf{E}^{(\theta(u))}\left(r(\omega) + r(-\omega)\right) = 2\cosh(\theta(u)) \quad . \tag{10}$$

Proposition 11.1. *There exist $\theta_1 < 0 < \theta_2$ numbers such that $J(u)$ defined above is convex on the interval $(u(\theta_1), u(\theta_2))$.*

Proof. With the notations

$$u'(\theta) := \frac{\mathrm{d}u(\theta)}{\mathrm{d}\theta} \quad \text{and} \quad u''(\theta) := \frac{\mathrm{d}^2 u(\theta)}{\mathrm{d}\theta^2}$$

and by computing derivatives of inverse functions, we obtain from (10)

$$\frac{\mathrm{d}^2 J}{\mathrm{d}u^2} \circ (u(\theta)) = \frac{\cosh(\theta)}{2} \frac{1}{(u'(\theta))^2} - \frac{\sinh(\theta)}{2} \frac{u''(\theta)}{(u'(\theta))^3} \quad .$$

The positivity of the left-hand side is assured in case $\theta = 0$, and is equivalent to the condition

$$\begin{aligned} &\frac{u''(\theta)}{u'(\theta)} < \operatorname{ctanh}(\theta) \ , \text{if } \theta > 0 \quad , \\ &\frac{u''(\theta)}{u'(\theta)} > \operatorname{ctanh}(\theta) \ , \text{if } \theta < 0 \end{aligned} \tag{11}$$

by positivity of $u'(\theta)$. The function $\theta \mapsto u(\theta)$ is analytic in $(-\bar{\theta}, \bar{\theta})$, $u'(\theta)$ is strictly positive, hence the left-hand side of (11) is bounded on the interval $(-\theta^*, \theta^*)$ for each $0 < \theta^* < \bar{\theta}$. Due to the unbounded behavior of $\operatorname{ctanh}(\theta)$ on any interval containing zero, there exist $\theta_1 < 0 < \theta_2$ for which (11) and hence convexity of $J(u)$ is satisfied. □

Using definitions (10) and (9) in Rankine-Hugoniot formula (2), the speed of the shock can now be written as

$$s = \frac{\mathbf{E}^{(\theta(u_{\mathrm{right}}))}\left(r(\omega) + r(-\omega)\right) - \mathbf{E}^{(\theta(u_{\mathrm{left}}))}\left(r(\omega) + r(-\omega)\right)}{\mathbf{E}^{(\theta(u_{\mathrm{right}}))}(\omega) - \mathbf{E}^{(\theta(u_{\mathrm{left}}))}(\omega)} \quad . \tag{12}$$

12 The defect tracer

12.1 Coupling the models

Let $\underline{\omega}^+(0)$ and $\underline{\omega}^-(0)$ be two elements of $\widetilde{\Omega}$. At time $t = 0$ we start with a configuration where these two realizations differ at only one site:

$$\omega_i^+(0) = \omega_i^-(0) \ \text{ if } \ i \neq 0 \quad \text{and} \quad \omega_0^+(0) = \omega_0^-(0) + 1 \quad .$$

One possible representation can be imagined by two walls. At time 0, the walls are the same on the right side of position 0, and every column of the wall $^+$ is higher by one brick than column of wall $^-$ on the left side of zero. We want the two processes to grow together in such a way, that the difference between them remains "one step" at any time $t > 0$:

$$(\forall\, t > 0)\ (\exists_1\, Q(t) \in \mathbb{Z}) : \omega_i^+(t) = \omega_i^-(t) \ \text{ if } \ i \neq Q(t) \quad \text{and} \quad \omega_{Q(t)}^+(t) = \omega_{Q(t)}^-(t) + 1 \quad .$$

We shall call this difference between the two models *defect tracer*, and $Q(t)$ is its position at time t. We show the coupling which preserves the only one defect tracer while both $\underline{\omega}^-$ and $\underline{\omega}^+$ evolves as usual. This coupling for the simple exclusion model is described (with particle notations) in [14] and [15]. Let our defect tracer be posed at point Q (i.e. $\omega_Q^+ = \omega_Q^- + 1$; $\omega_i^+ = \omega_i^-$ if $i \neq Q$), and let $h_i^+ \uparrow$ (or $h_i^- \uparrow$) mean that the column of $\underline{\omega}^+$ (or the column of $\underline{\omega}^-$, respectively) between the points i and $i+1$ has grown by one brick. Then the growing rule for the columns $h_{Q-1}^{\pm}$ and $h_Q^{\pm}$ is shown in table 1. Every line

with rate	$h^-_{Q-1}\uparrow$	$h^+_{Q-1}\uparrow$	$h^-_Q\uparrow$	$h^+_Q\uparrow$	Q has...
$r(-\omega^-_Q)-r(-\omega^+_Q)$	•				decreased
$r(\omega^-_{Q-1})+r(-\omega^+_Q)$	•	•			–
$r(\omega^+_Q)-r(\omega^-_Q)$				•	increased
$r(\omega^-_Q)+r(-\omega^-_{Q+1})$			•	•	–

Table 1: The coupling rules

of that table represents a possible step with rate written on the first column. These rates are non-negative due to monotonicity of r. For each column of this table, summing the rates corresponding to the possible steps assures us that columns of each $\underline{\omega}^+$ and $\underline{\omega}^-$ evolve as usual in the neighborhood of Q. For $i\neq Q-1$ or Q, h^+_i and h^-_i increases at the same time with the original rate $r(\omega^-_i)+r(-\omega^-_{i+1})=r(\omega^+_i)+r(-\omega^+_{i+1})$.

How does an observer following the defect tracer see the surface? We introduce the drifted form $\boldsymbol{\tau}_k\,\underline{\omega}$ of an $\underline{\omega}\in\Omega$ as follows. Let $k\in\mathbb{Z}$, then $\boldsymbol{\tau}_k\,\underline{\omega}\in\Omega$ and

$$(\boldsymbol{\tau}_k\,\underline{\omega})_i:=\omega_{i-k}\quad.$$

From now on, we denote by $\underline{\omega}(t)$ the $\underline{\omega}^-(t)$ process as seen from the position $Q(t)$ of the defect tracer, i.e. $\omega_i:=\omega^-_{Q+i}$. According to the coupling rules, we can write the infinitesimal generator for $\underline{\omega}$:

$$\begin{aligned}(13)\quad &(L^{(\mathrm{d.t.})}\varphi)(\underline{\omega})=\\ &=\sum_{i\neq-1,\,0}\Big\{[r(\omega_i)+r(-\omega_{i+1})]\cdot[\varphi(\dots,\,\omega_i-1,\,\omega_{i+1}+1,\,\dots)-\varphi(\underline{\omega})]\Big\}+\\ &\quad+[r(\omega_{-1})+r(-\omega_0-1)]\cdot[\varphi(\dots,\,\omega_{-1}-1,\,\omega_0+1,\,\dots)-\varphi(\underline{\omega})]+\\ &\quad+[r(\omega_0)+r(-\omega_1)]\cdot[\varphi(\dots,\,\omega_0-1,\,\omega_1+1,\,\dots)-\varphi(\underline{\omega})]+\\ &\quad+[r(-\omega_0)-r(-\omega_0-1)]\cdot[\varphi(\boldsymbol{\tau}_1(\dots,\,\omega_{-1}-1,\,\omega_0+1,\,\dots))-\varphi(\underline{\omega})]+\\ &\quad+[r(\omega_0+1)-r(\omega_0)]\cdot[\varphi(\boldsymbol{\tau}_{-1}\,\underline{\omega})-\varphi(\underline{\omega})]\quad.\end{aligned}$$

12.2 The speed of the defect tracer

The main problem of this note is to find a stationary measure for the process as seen from the defect tracer, i.e. to find a measure $\mu^{(\mathrm{d.t.})}$, for which

$$\mathbf{E}(L^{(\mathrm{d.t.})}\varphi)(\underline{\omega})=0$$

is satisfied. Before giving a partial answer to this question, we give an early indication on the correspondence to shocks of such a measure $\mu^{(\mathrm{d.t.})}$.

Let $a<-1$ (and $b>1$) be sites far on the left side (and far on the right side, respectively) of the defect tracer. We choose the function

$$\varphi(\underline{\omega}):=\sum_{k=a}^{b}\omega_k$$

in (13) to obtain

$$(L^{(\mathrm{d.t.})}\varphi)(\underline{\omega}) = [r(\omega_{a-1}) + r(-\omega_a)] - [r(\omega_b) + r(-\omega_{b+1})] +$$
$$+ [r(-\omega_0) - r(-\omega_0 - 1)] \cdot (\omega_{a-1} - \omega_b) + [r(\omega_0 + 1) - r(\omega_0)] \cdot (\omega_{b+1} - \omega_a) \quad .$$

Let us assume that a measure $\mu^{(d.t.)}$ is stationary for $L^{(d.t.)}$. Let us also assume that as $l \to \pm\infty$, the random variable ω_l becomes asymptotically independent of ω_{-1}, ω_0, ω_1, and the distribution of ω_l converges weakly. Then we have

$$(14) \quad 0 = \mathbf{E}(L^{(\mathrm{d.t.})}\varphi)(\underline{\omega}) = \mathbf{E}\,[r(\omega_a) + r(-\omega_a)] - \mathbf{E}\,[r(\omega_b) + r(-\omega_b)] +$$
$$+ \mathbf{E}\,[r(-\omega_0) - r(-\omega_0 - 1)] \cdot \mathbf{E}(\omega_a - \omega_b) + \mathbf{E}\,[r(\omega_0 + 1) - r(\omega_0)] \cdot \mathbf{E}(\omega_b - \omega_a) +$$
$$+ H(a,\, b) \quad ,$$

where the error function $H(a,\, b)$ tends to zero if $a \to -\infty$ and $b \to \infty$. (For the product measure $\mu^{(\mathrm{d.t.})}$ we find in the next section, $H(a,\, b) = 0$ for any $a < -1,\; b > 1$.) According to the rules of the coupling, and assuming also ergodicity of the process as seen from the view of the defect tracer, we have the law of large numbers

$$v := \lim_{t\to\infty} \frac{Q(t)}{t} = \mathbf{E}\,\{[r(\omega_0 + 1) - r(\omega_0)] - [r(-\omega_0) - r(-\omega_0 - 1)]\} \quad \text{a.s.}$$

for the speed of the defect tracer. Hence we conclude from (14) that

$$v = \lim_{a\to-\infty,\, b\to\infty} \frac{\mathbf{E}\,[r(\omega_b) + r(-\omega_b)] - \mathbf{E}\,[r(\omega_a) + r(-\omega_a)]}{\mathbf{E}(\omega_b) - \mathbf{E}(\omega_a)}$$

in case $\mathbf{E}(\omega_a) \neq \mathbf{E}(\omega_b)$ and their limits are not equal i.e. the slope of the surface is different far on the two sides. This formula is the same as (12), which we obtained for the speed of the shock using the Rankine-Hugoniot formula. This shows that a measure $\mu^{(\mathrm{d.t.})}$ with different asymptotics on the left and on the right can be identified as the microscopic structure of a shock solution of (1).

13 Stationary measures as seen from the defect tracer

In this section we find a stationary product measure satisfying (7) for the defect tracer of the EBL model. We also show that this kind of measure only exists for the EBL model.

Intuitively one expects that far from the defect tracer a stationary measure behaves like the canonical Gibbs-measure $\mu^{(\theta)}$, since the defect tracer is only a local “error” for the evolution of the process. The canonical measure has one parameter θ, but it is not necessary that in this case parameter θ_{left} far on the left side would be equal to the parameter θ_{right} far on the right side. Let

$$\underline{\theta} := \{\theta_i \,:\, i \in \mathbb{Z}\}$$

be a sequence of parameters. Then it seems to be reasonable to assume that the product measure $\mu^{(\underline{\theta})}$ with marginals

$$\mu_i(z) = \mu^{(\underline{\theta})}\,\{\underline{\omega} \,:\, \omega_i = z\} := \mu^{(\theta_i)}(z) = \frac{1}{Z(\theta_i)} \cdot \frac{\mathrm{e}^{\theta_i z}}{r(|z|)!}$$

is stationary for $L^{\mathrm{d.t.}}$ (13) ($i \in \mathbb{Z}$). This measure only differs from the canonical $\mu^{(\theta)}$ (6) in that the parameter of its one-dimensional marginals depends on the position. The question is whether there are any choices of $\underline{\theta}$ for $\mu^{(\underline{\theta})}$ to be stationary.

Theorem 13.1. *For a bricklayers' model, if r is not the constant function, then the measure $\mu^{(\underline{\theta})}$ described above is stationary for $L^{(d.t.)}$ if and only if r is the rate of an EBL model with any parameter $\beta > 0$, and for the $\underline{\theta}$ parameters of $\mu^{(\underline{\theta})}$*

$$\theta_i = \begin{cases} \theta_{left} & \text{if } \; i \le -1 \;, \\ \theta_{right} := \theta_{left} - \beta & \text{if } \; i \ge 0 \end{cases}$$

is satisfied with an arbitrary real number θ_{left}.

Proof. Stationarity means

$$\mathbf{E}^{(\underline{\theta})}(L^{\mathrm{d.t.}}\varphi)(\underline{\omega}) = 0 \quad ,$$

after some changes of variables, by straightforward computations we obtain from (13)

$$0 = \mathbf{E}^{(\underline{\theta})}\Big\{\Big\{A + B + C + D\Big\}\varphi(\underline{\omega})\Big\} \quad , \tag{15}$$

where

$$\begin{aligned}
A = \sum_{i \neq -1} \Big[& [r(\omega_i + 1) + r(-\omega_{i+1} + 1)] \cdot \frac{\mu_i(\omega_i + 1)\, \mu_{i+1}(\omega_{i+1} - 1)}{\mu_i(\omega_i)\, \mu_{i+1}(\omega_{i+1})} - \\
& - [r(\omega_i) + r(-\omega_{i+1})]\Big] \quad , \\
B = {} & [r(\omega_{-1} + 1) + r(-\omega_0)] \cdot \frac{\mu_{-1}(\omega_{-1} + 1)\, \mu_0(\omega_0 - 1)}{\mu_{-1}(\omega_{-1})\, \mu_0(\omega_0)} - \\
& - r(\omega_{-1}) - r(-\omega_0) - r(\omega_0 + 1) + r(\omega_0) \quad , \\
C = {} & [r(-\omega_1 + 1) - r(-\omega_1)] \cdot \frac{\mu_{-1}(\omega_0 + 1)\, \mu_0(\omega_1 - 1)}{\mu_{-1}(\omega_0)\, \mu_0(\omega_1)} \cdot \prod_{j \in \mathbb{Z}} \frac{\mu_{j-1}(\omega_j)}{\mu_j(\omega_j)} \quad , \\
D = {} & [r(\omega_{-1} + 1) - r(\omega_{-1})] \cdot \prod_{j \in \mathbb{Z}} \frac{\mu_{j+1}(\omega_j)}{\mu_j(\omega_j)} \quad .
\end{aligned}$$

We eliminate the expressions μ_k for all $k \in \mathbb{Z}$ with the use of (5) and (7)

$$r(z) \cdot \frac{\mu_k(z)}{\mu_k(z-1)} = \mathrm{e}^{\theta_k} \quad \text{and}$$

$$r(-z) \cdot \frac{\mu_k(z)}{\mu_k(z+1)} = \mathrm{e}^{-\theta_k}$$

to obtain

$$A = \sum_{i \neq -1} \left[e^{\theta_i - \theta_{i+1}} \, r(\omega_{i+1}) + e^{\theta_i - \theta_{i+1}} \, r(-\omega_i) - r(\omega_i) - r(-\omega_{i+1}) \right] \quad ,$$

$$B = e^{\theta_{-1} - \theta_0} \, r(\omega_0) + e^{\theta_{-1} - \theta_0} \, r(-\omega_{-1}) \, \frac{r(\omega_0)}{r(\omega_0 + 1)} - $$
$$- r(\omega_{-1}) - r(-\omega_0) - r(\omega_0 + 1) + r(\omega_0) \quad ,$$

$$C = r(-\omega_0) \left(1 - \frac{r(\omega_1)}{r(\omega_1 + 1)} \right) e^{\theta_{-1} - \theta_0} \prod_{j \in \mathbb{Z}} e^{(\theta_{j-1} - \theta_j) \, \omega_j} \, \frac{Z(\theta_j)}{Z(\theta_{j-1})} \quad ,$$

$$D = [r(\omega_{-1} + 1) - r(\omega_{-1})] \prod_{j \in \mathbb{Z}} e^{(\theta_{j+1} - \theta_j) \, \omega_j} \, \frac{Z(\theta_j)}{Z(\theta_{j+1})} \quad .$$

For $a < -1, \; b > 1$ fixed, let us consider bounded cylinder functions φ, which depend on the variables $\omega_a, \, \omega_{a+1}, \, \dots, \, \omega_b$. By stationarity of $\mu^{(\underline{\theta})}$, (15) is satisfied for all of them. Hence it is necessary that $A + B + C + D$ does not depend on the variables $\omega_a, \, \omega_{a+1}, \, \dots, \, \omega_b$ and its mean is zero according to $\mu^{(\underline{\theta})}$. Only $C, \; D$, and the second term in B are the terms which can contain product of functions of different variables ω_k. Each of them is positive by monotonicity of r. Thus it follows that each of these three terms must not depend on more than one variable. This implies

$$\frac{r(z)}{r(z+1)} = \text{constant} = r(0)^2$$

due to the form of the second term in B. The value $r(0)^2$ of this constant is a consequence of (5). Thus we conclude that r is necessarily exponential, the rates are that of the EBL model (3). C and D can also contain at most one variable, hence we obtain $\theta_k = \theta_{-1}$ for $k \leq -1$ and $\theta_k = \theta_0$ for $k \geq 0$. This means that we have at most two kinds of marginals of $\mu^{(\underline{\theta})}$, one on the left-hand side of the defect tracer and an other one on its right-hand side. In (15), computing the expectation value of φ times the terms of A, summed up for $i \leq a - 2$ and for $i \geq b + 1$, gives zero. The reason for this is that the variables in these terms are independent of the variables which φ depends on. For the rest of the indices, note that we have a telescopic sum for A. Due to this and using the rates of the EBL model, we can simplify our expressions to

$$A = r(-\omega_{a-1}) - r(\omega_{a-1}) + r(\omega_{b+1}) - r(-\omega_{b+1}) +$$
$$+ r(\omega_{-1}) - r(-\omega_{-1}) + r(-\omega_0) - r(\omega_0) \quad ,$$

$$B = e^{\theta_{-1} - \theta_0} \, r(\omega_0) + e^{\theta_{-1} - \theta_0 - \beta} r(-\omega_{-1}) -$$
$$- r(\omega_{-1}) - r(-\omega_0) - e^{\beta} r(\omega_0) + r(\omega_0) \quad ,$$

$$C = \left(1 - e^{-\beta} \right) r(-\omega_0) \, e^{(\theta_{-1} - \theta_0) \, (\omega_0 + 1)} \, \frac{Z(\theta_0)}{Z(\theta_{-1})} \quad ,$$

$$D = \left(e^{\beta} - 1 \right) r(\omega_{-1}) \, e^{(\theta_0 - \theta_{-1}) \, \omega_{-1}} \, \frac{Z(\theta_{-1})}{Z(\theta_0)} \quad .$$

It is easy to check that simply choosing $\theta_{-1} = \theta_0$ does not eliminate the variables $\omega_{-1}, \, \omega_0$ from $A + B + C + D$. Hence this can not be a solution for $\underline{\theta}$ to make equation (15) be satisfied for all φ. This means that the marginals on the left-hand side of the defect tracer are different from those on the right-hand side.

When taking expectation value in (15), this leads to having constant times φ from the terms containing ω_{a-1} and ω_{b+1} in the first part of the expression of A. In order to make (15) be satisfied for all φ, it is necessary that we obtain other constants to have zero together with. They can only come from C and D. Thus we conclude

$$\mathrm{e}^{\theta_{-1}-\theta_0} = \mathrm{e}^{\beta} \tag{16}$$

with the use of the form (3) of r. In view of (8), we have the measures

$$\mu_i(z) = \frac{\mathrm{e}^{-\frac{\beta}{2}\left(z-\frac{\theta_i}{\beta}\right)^2}}{\mathrm{e}^{-\frac{\theta_i^2}{2\beta}} Z(\theta_i)} = \frac{\mathrm{e}^{-\frac{\beta}{2}(z-m_i)^2}}{\widetilde{Z}(\beta,\, m_i)} \tag{17}$$

with $m_i := \theta_i/\beta$. We know that the normalization $\widetilde{Z}(\beta,\, m)$ in the right-hand side of (17) is periodic in the parameter m with period one, which tells us

$$\frac{Z(\theta_{-1})}{Z(\theta_0)} = \frac{Z(\theta_0+\beta)}{Z(\theta_0)} = \frac{\mathrm{e}^{\frac{(\theta_0+\beta)^2}{2\beta}}\widetilde{Z}(\beta,\, \frac{\theta_0}{\beta}+1)}{\mathrm{e}^{\frac{(\theta_0)^2}{2\beta}}\widetilde{Z}(\beta,\, \frac{\theta_0}{\beta})} = \mathrm{e}^{\frac{\beta}{2}+\theta_0} \quad .$$

Using this result together with (16) and with the property $\mathbf{E}^{(\underline{\theta})} r(\pm\omega_i) = \mathrm{e}^{\pm\theta_i}$, we see that (15) is satisfied, which completes the proof. □

The form of the measure described in this theorem shows that the discrete normal distribution of $\omega_i,\ i \le -1$ is shifted by $+1$ compared to the distribution of $\omega_j,\ j \ge 0$. This gives us the picture of a (random) valley with the (randomly) moving defect tracer in its center. Since the position of the defect tracer is not deterministic, we do not see the sharp change between the distribution of the two sides of this valley, if looking the model from outside.

Acknowledgement

The author is grateful to Bálint Tóth for introducing bricklayers' models together with the problem, and for helping him in many questions. Correcting mistakes is also acknowledged to him, as well as to Benedek Valkó.

Part III

Growth fluctuations in a class of deposition models

Abstract

We compute the growth fluctuations in equilibrium of a wide class of deposition models. These models also serve as general frame to several nearest-neighbor particle jump processes, e.g. the simple exclusion or the zero range process, where our result turns to current fluctuations of the particles. We use martingale technique and coupling methods to show that, rescaled by time, the variance of the growth as seen by a deterministic moving observer has the form $|V - C| \cdot D$, where V and C is the speed of the observer and the second class particle, respectively, and D is a constant connected to the equilibrium distribution of the model. Our main result is a generalization of Ferrari and Fontes' result for simple exclusion process. Law of large numbers and central limit theorem are also proven. We need some properties of the motion of the second class particle, which are known for simple exclusion and are partly known for zero range processes, and which are proven here for a type of deposition models and also for a type of zero range processes.

Résumé

On compute les fluctuations du grandissement dans l'état d'équilibre d'une classe vaste des processus de décharge. Ces processus forment aussi bien un cadre pour quelques modèles des bonds voisins des particules, p. e. le modèle simple exclusion ou zero range, où notre résultats deviennent des résultats sur les fluctuations du flux des particules. On utilise de méthode martingale et des techniques des couplages pour présenter que le variance du grandissement, regradué par le temps et vu par un observateur qui avance déterminement à une vitesse V, a la forme $|V - C| \cdot D$, où C est la vélocité de la particule de deuxième classe, et D est une constante connectée à l'état d'équilibre du modèle. Notre résultat principal est une généralisation du résultat de Ferrari et Fontes pour le modèle simple exclusion. La loi des grandes nombres et le théorème de la limite centrale sont aussi démontrés. Nous avons besoin de quelques propriétés du mouvement du particule de deuxième classe, qui sont connues pour simple exclusion et partiellement pour le modèle zero range, et qui sont démontrées ici pour un type des processus de décharge et pour un type des modèles zero range aussi.

Keywords: Current fluctuations; second class particle; coupling methods.
MSC: 60K35, 82C41.

14 Introduction

Stochastic deposition models can be used to obtain microscopic description of domain growths, e.g. a colony of cells or an infected area of plants. The fluctuation of the growth is itself of great interest. Moreover, these models are in close connection to interacting particle systems, where the particle diffusion corresponds to rescaled surface fluctuation. As it is shown below, an additional feature of deposition models is the possibility of handling antiparticles as well as particles in the particle representation of the process. It has been known [8] for the simple exclusion process, that the current fluctuation is in close connection to the motion of the so-called second class particle, and, divided by time, its variance vanishes for an observer moving with the speed of this particle. In this latter case, Prähofer and Spohn [20] suggest this quantity to be in the order of $t^{2/3}$.

In the present note we consider a wide class of one-dimensional deposition models, parameterized by rate functions describing a column's growth depending on the neighboring columns' relative heights. By monotonicity properties of the rate functions, our models are attractive. For a treatment of these models in a hydrodynamical context, without using attractivity, see Tóth and Valkó [29]; Tóth and Werner [30]. Following Rezakhanlou [23], we first show some conditions on the model in order to have product measures as stationary ones for the process. (By stationarity, we mean time-invariance in this paper.) Our description is general enough to include the asymmetric simple exclusion process, some types of the zero range process, and a family of deposition models, which we call bricklayers' models. In this general frame, we compute the growth fluctuations in order $\mathcal{O}(t)$, hence generalize the result of Ferrari and Fontes [8]. In the computations we couple two processes, which only differ at one site. This is the position of the so-called *defect tracer*, or also called second class particle. We need law of large numbers and a second moment condition for the position of this extra particle. These have been established for simple exclusion [7], but, as far as we know, only L^1-convergence is known for most kinds of zero range processes [23]. We prove L^n-convergence with any n for the defect tracer of the totally asymmetric zero range process and for our new bricklayers' models via various coupling techniques.

14.1 The model

The class of models described here is a generalization of the so-called misanthrope process. For $-\infty \le \omega^{\min} \le 0$ and $1 \le \omega^{\max} \le \infty$ (possibly infinite valued) integers, we define

$$I := \left\{z \in \mathbb{Z} : \omega^{\min} - 1 < z < \omega^{\max} + 1\right\}$$

and the phase space

$$\Omega = \{\underline{\omega} = (\omega_i)_{i\in\mathbb{Z}} \ : \ \omega_i \in I\} = I^{\mathbb{Z}}.$$

For each pair of neighboring sites i and $i+1$ of $\mathbb{Z}$, we can imagine a column built of bricks, above the edge $(i,\, i+1)$. The height of this column is denoted by h_i. If $\underline{\omega}(t) \in \Omega$ for a fixed time $t \in \mathbb{R}$ then $\omega_i(t) = h_{i-1}(t) - h_i(t) \in I$ is the negative discrete gradient of the height of the "wall". The growth of a column is described by jump processes. A brick can be added:

$$(\omega_i,\, \omega_{i+1}) \longrightarrow (\omega_i - 1,\, \omega_{i+1} + 1) \quad \text{with rate } r(\omega_i,\, \omega_{i+1}).$$

Conditionally on $\underline{\omega}(t)$, these moves are independent. See fig. 3 for some possible instantaneous changes. For small ε, the conditional expectation of the growth of the column between i and $i+1$ in the time interval $[t,\, t+\varepsilon]$ is $r(\omega_i(t),\, \omega_{i+1}(t)) \cdot \varepsilon + \mathfrak{o}(\varepsilon)$.

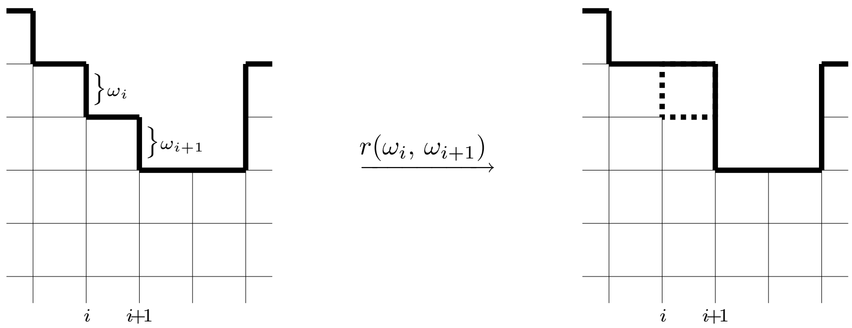

Figure 3: A possible move

The rates must satisfy

$$r(\omega^{\min},\, \cdot\,) \equiv r(\,\cdot\,,\, \omega^{\max}) \equiv 0$$

whenever either $\omega^{\min}$ or $\omega^{\max}$ is finite. We assume r to be non-zero in all other cases. We want the dynamics to smoothen our interface, that is why we assume monotonicity in the following way:

$$r(z+1,\, y) \geq r(z,\, y), \qquad r(y,\, z+1) \leq r(y,\, z) \tag{18}$$

for $y,\, z,\, z+1 \in I$. This means that the higher neighbors a column has, the faster it grows. Our model is hence *attractive*.

We are going to use product property of the model's stationary measure. For this reason, similarly to Rezakhanlou [23], we assume that for any $x,\, y,\, z \in I$

$$r(x,\, y) + r(y,\, z) + r(z,\, x) = r(x,\, z) + r(z,\, y) + r(y,\, x), \tag{19}$$

and for $\omega^{\min} < x,\, y,\, z < \omega^{\max} + 1$

$$r(x, y-1) \cdot r(y, z-1) \cdot r(z, x-1) = r(x, z-1) \cdot r(z, y-1) \cdot r(y, x-1). \tag{20}$$

These two conditions imply product structure of the stationary measure, see section 14.3. Equation (20) is equivalent to the condition $r(y,\, z) = s(y,\, z+1) \cdot f(y)$ for some function f and a symmetric function s.

At time t, the interface mentioned above is described by $\underline{\omega}(t)$. Let $\varphi\, :\, \Omega \to \mathbb{R}$ be a finite cylinder function i.e. φ depends on a finite number of values of ω_i. The growth of this interface is a Markov process, with the formal infinitesimal generator L:

$$(L\varphi)(\underline{\omega}) = \sum_{i\in\mathbb{Z}} r(\omega_i,\, \omega_{i+1}) \cdot \left[\varphi(\dots,\, \omega_i - 1,\, \omega_{i+1} + 1,\, \dots) - \varphi(\underline{\omega})\right]. \tag{21}$$

When constructing the process rigorously, problems may arise due to the unbounded growth rates. The system being one-component and attractive, we assume that, with appropriate growth conditions on the rates, existence of dynamics on a set of tempered configurations $\widetilde{\Omega}$ (i.e. configurations obeying some restrictive growth conditions) can be established by applying methods initiated by Liggett and Andjel [12] [1]. Technically we assume that $\widetilde{\Omega}$ is of full measure w.r.t. the canonical Gibbs measures defined in section 14.3. In fact this has been proved for some kinds of these models, see below. We do not deal with questions of existence of dynamics in the present paper.

14.2 Examples

There are three essentially different cases of these models, all of them are of nearest neighbor type.

1. **Generalized exclusion processes** are described by our models in case both $\omega^{\min}$ and $\omega^{\max}$ are finite.

   - **The totally asymmetric simple exclusion process (SE)** introduced by F. Spitzer [28] is described this way by $\omega^{\min} = 0,\ \omega^{\max} = 1$,

     $$r(\omega_i,\, \omega_{i+1}) = \omega_i \cdot (1 - \omega_{i+1}).$$

     Here ω_i is the occupation number for the site i, and $r(\omega_i,\, \omega_{i+1})$ is the rate for a particle to jump from site i to $i + 1$. Conditions (18), (19) and (20) for these rates are satisfied.

   - **A particle-antiparticle exclusion process** is also shown to demonstrate the generality of the frame described above. Let $\omega^{\min} = -1,\ \omega^{\max} = 1$. Fix c (*creation*), a (*annihilation*) positive rates with $c \le a/2$. Put

     $$r(0,\, 0) = c, \quad r(0,\, -1) = \frac{a}{2}, \quad r(1,\, 0) = \frac{a}{2}, \quad r(1,\, -1) = a,$$

     and all other rates are zero. If ω_i is the number of particles at site i, with $\omega_i = -1$ meaning the presence of an antiparticle, then this model describes a totally asymmetric exclusion process of particles and antiparticles with annihilation and particle-antiparticle pair creation. These rates also satisfy our conditions.

   Other generalizations are possible allowing a bounded number of particles (or antiparticles) to jump to the same site. By the bounded jump rates and by nearest-neighbor type of interaction, the construction of dynamics of these processes is well understood, see e.g. Liggett [15].

2. **Generalized misanthrope processes** are obtained by choosing $\omega^{\min} > -\infty,\ \omega^{\max} = \infty$.

   - **The zero range process (ZR)** is included by $\omega^{\min} = 0,\ \omega^{\max} = \infty$,

     $$r(z,\, y) = f(z)$$

     with an arbitrary $f : \mathbb{Z}^+ \to \mathbb{R}^+$ nondecreasing function and $f(0) = 0$. Here ω_i represents the number of particles at site i. These rates trivially satisfy conditions (18), (19), (20). The dynamics of this process is constructed by Andjel [1] under the condition that the rate function f obeys the growth condition $|f(z+1) - f(z)| \le K$ for some $K > 0$ and all $z \ge 0$.

3. **General deposition processes** are the type of these models where $\omega^{\min} = -\infty$ and $\omega^{\max} = \infty$. In this case, the height difference between columns next to each other can be arbitrary in $\mathbb{Z}$. Hence the presence of antiparticles can not be avoided when trying to give a particle representation of the process.

   - **Bricklayers' models (BL).** Let

     $$r(z, y) := f(z) + f(-y)$$

     with the property

     $$f(z) \cdot f(-z+1) = 1$$

     for the nondecreasing function f and for any $z \in \mathbb{Z}$. This process can be represented by bricklayers standing at each site i, laying a brick on the column on their left with rate $f(-\omega_i)$ and laying a brick to their right with rate $f(\omega_i)$. This interpretation gives reason to call these models bricklayers' model. Conditions (18), (19) and (20) hold for r. Similarly to the ZR process, this model is constructed by Booth and Quant [21] only in case $|f(z+1) - f(z)|$ is bounded in $\mathbb{Z}$.

14.3 Translation invariant stationary product measures

We are interested in translation invariant stationary measures for these processes, i.e. canonical Gibbs-measures. We construct such measures similarly to Rezakhanlou [23] of the following form. Fix $f(1) > 0$ and define

$$f(z) := \frac{r(z, 0)}{r(1, z-1)} \cdot f(1) \tag{22}$$

for $\omega^{\min} < z < \omega^{\max} + 1$. Then f is a nondecreasing strictly positive function. For $I \ni z > 0$ we define

$$f(z)! := \prod_{y=1}^{z} f(y),$$

while for $I \ni z < 0$ let

$$f(z)! := \frac{1}{\prod\limits_{y=z+1}^{0} f(y)},$$

finally $f(0)! := 1$. Then we have

$$f(z)! \cdot f(z+1) = f(z+1)!$$

for all $z \in I$. Let

$$\bar{\theta} := \begin{cases} \log\left(\liminf\limits_{z\to\infty} \left(f(z)!\right)^{1/z}\right) = \lim\limits_{z\to\infty} \log(f(z)) & , \text{ if } \omega^{\max} = \infty \\ \infty & , \text{ else} \end{cases}$$

and

$$\underline{\theta} := \begin{cases} \log\left(\limsup\limits_{z\to\infty} \left(f(-z)!\right)^{1/z}\right) = \lim\limits_{z\to\infty} \log(f(-z)) & , \text{ if } \omega^{\min} = -\infty \\ -\infty & , \text{ else.} \end{cases}$$

By monotonicity of f, we have $\bar{\theta} \geq \underline{\theta}$. We assume $\bar{\theta} > \underline{\theta}$. With a generic real parameter $\theta \in (\underline{\theta}, \bar{\theta})$, we define

$$Z(\theta) := \sum_{z \in I} \frac{\mathrm{e}^{\theta z}}{f(z)!}.$$

Let the product-measure $\underline{\mu}_\theta$ have marginals

$$\mu_\theta(z) = \underline{\mu}_\theta \{\underline{\omega} : \omega_i = z\} := \frac{1}{Z(\theta)} \cdot \frac{\mathrm{e}^{\theta z}}{f(z)!}. \tag{23}$$

By definition it has the property

$$\frac{\mu_\theta(z+1)}{\mu_\theta(z)} = \frac{\mathrm{e}^\theta}{f(z+1)}$$

which implies

$$r(z+1,\, y-1) \cdot \frac{\mu_\theta(z+1)\, \mu_\theta(y-1)}{\mu_\theta(z)\, \mu_\theta(y)} = r(y,\, z) \tag{24}$$

due to (22) and (20). Hence stationarity of $\underline{\mu}_\theta$ follows via (19).

As can be verified, the expectation value $\varrho(\theta) := \mathbf{E}_\theta(\omega_i)$ is a strictly increasing function of θ. We introduce its inverse $\theta(\varrho)$ and the function

$$\mathcal{H}(\varrho) := \mathbf{E}_{\theta(\varrho)} \{r(\omega_i,\, \omega_{i+1})\}, \tag{25}$$

playing an important role in hydrodynamical considerations. For the SE model, the construction leads to the well-known Bernoulli product-measure with marginals

$$\mu(1) = \underline{\mu}\{\underline{\omega} : \omega_i = 1\} := \varrho,$$
$$\mu(0) = \underline{\mu}\{\underline{\omega} : \omega_i = 0\} := 1 - \varrho$$

with a real number ϱ between zero and one (the density of the particles). In our notations, $-\varrho$ describes the average slope of the interface.

For the particle-antiparticle exclusion process, the relative probability of having a particle or an antiparticle as a function of the rates goes as $\sqrt{c/a}$, independently for the sites. The density of particles relative to antiparticles can be set by an arbitrary parameter.

Both for the ZR process and for BL models, it turns out that f defined in (22) and f in the definition of the rates agree.

It is not hard to show ergodicity of these models, which also implies extremality of the invariant measures $\underline{\mu}_\theta$:

Proposition 14.1. *The processes given in subsection 14.1, distributed according to their stationary measures* $\underline{\mu}_\theta$ (23)*, are ergodic.*

Proof. We need to show that any (time-) stationary bounded measurable function defined on the trajectories of the process is constant a.s. By proposition V.2.4 of Neveu [17], this follows once we see that any bounded function

φ on $\widetilde{\Omega}$ satisfying $P\varphi = \varphi$ is constant for $\underline{\mu}$-almost all $\underline{\omega}$ with the Markov-transition operator P. Hence ergodicity of the process follows if $L\varphi = 0$ implies $\varphi(\underline{\omega}) =$ constant for almost all $\underline{\omega} \in \widetilde{\Omega}$. We compute the Dirichlet-form

$$- \mathbf{E}_\theta(\varphi \cdot L\varphi) = \\ = \frac{1}{2}\mathbf{E}_\theta \left\{ \sum_{i\in\mathbb{Z}} r(\omega_i,\, \omega_{i+1}) \cdot [\varphi(\dots,\, \omega_i - 1,\, \omega_{i+1} + 1,\, \dots) - \varphi(\underline{\omega})]^2 \right\}.$$

By positivity of the rates, this shows that assuming $L\,\varphi \equiv 0$ results in

$$\varphi(\dots,\, \omega_i - 1,\, \omega_{i+1} + 1,\, \dots) = \varphi(\underline{\omega})$$

for almost all $\underline{\omega} \in \widetilde{\Omega}$. Consecutive use of this equation shows that any function obeying $L\varphi = 0$ does a.s. not depend on any finite cylinder set in $\widetilde{\Omega}$. Especially, for $\varepsilon > 0$ and a constant $K \in \operatorname{Ran}(\varphi)$, the event

$$\{\varphi(\underline{\omega}) \in (K,\, K + \varepsilon]\}$$

does not depend on any finite cylinder set. Hence by Kolmogorov's 0-1 law, the probability of these events is zero or one w.r.t. the product measure $\underline{\mu}$. Partitioning the bounded image of φ, this shows that this function is constant for almost all $\underline{\omega}$. □

14.4 Results

We start our model in a canonical Gibbs-distribution, with parameter θ. For a fixed speed value $V > 0$ we define

$$J^{(V)}(t) := h_{\lfloor Vt \rfloor}(t) - h_0(0),$$

the height of column at site $\lfloor Vt \rfloor$ at time t, relative to the initial height of the column at the origin. For $V < 0$, we introduce

$$J^{(V)}(t) := h_{\lceil Vt \rceil}(t) - h_0(0),$$

which is the mirror-symmetric form of $J^{(V)}$ defined above for positive V's. For $V = 0$ we write

$$J(t) = J^{(0)}(t) := h_0(t) - h_0(0).$$

In particle notations of the models, $J^{(V)}(t)$ is the current, i.e. the algebraic number of particles jumping through the moving window positioned at Vt, in the time interval $[0,\, t]$. We prove law of large numbers for this quantity:

$$\lim_{t\to\infty} \frac{J^{(V)}}{t} = \mathbf{E}(r) - V\,\mathbf{E}(\omega) \quad \text{a.s.} \tag{26}$$

We need law of large numbers and a second-moment condition for the position $Q(t)$ of the defect tracer (also called second class particle, see section 16 for its definition) if one of the coupled models is started from its canonical Gibbs-measure:

Condition 14.2. *With initial distribution $\underline{\mu}_\theta$ of $\underline{\omega}$, weak law of large numbers*

$$\lim_{t\to\infty} \mathbf{P}_\theta \left(\left| \frac{Q(t)}{t} - C(\theta) \right| > \delta \right) = 0 \tag{27}$$

for a speed value $C(\theta)$ and for any $\delta > 0$ holds, and the bound

$$\mathbf{E}_\theta \left(\frac{Q(t)^2}{t^2} \right) < K < \infty \tag{28}$$

is satisfied for all large t for the position $Q(t)$ of the defect tracer.

Inequality (28) is obvious in case of bounded rates, since in this situation, the process $|Q(t)|$ is bounded by some Poisson-process.

Theorem 14.3 (Main). *Assume condition 14.2. Then*

$$\lim_{t\to\infty} \frac{\mathbf{Var}_\theta(J^{(V)}(t))}{t} = |V - C(\theta)| \cdot \mathbf{Var}_\theta(\omega_0) =: D_J(\theta) \tag{29}$$

for any $V \in \mathbb{R}$, where $\mathbf{Var}_\theta$ stands for the variance w.r.t. μ_θ.

Theorem 14.4 (Central limit theorem). *Assuming condition 14.2,*

$$\lim_{t\to\infty} \mathbf{P}_\theta \left(\frac{\widetilde{J}^{(V)}(t)}{\sqrt{D_J(\theta)} \cdot \sqrt{t}} \leq x \right) = \Phi(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}} \, \mathrm{d}y,$$

i.e. $\widetilde{J}^{(V)}(t)/\sqrt{t}$ converges in distribution to $N(0, D_J(\theta))$, a centered normal random variable with variance $D_J(\theta)$ of (29). Tilde means here that the mean value of $J^{(V)}(t)$ is subtracted.

For the SE model, (27) is proven in [7]. It is shown there that

$$\lim_{t\to\infty} \frac{Q(t)}{t} = 1 - 2\varrho \quad \text{a.s.}$$

Condition 14.2 is satisfied by this law, hence theorem 14.3 gives

$$\lim_{t\to\infty} \frac{\mathbf{Var}(J^{(V)}(t))}{t} = \varrho\,(1-\varrho)\,|(1-2\,\varrho) - V|,$$

and the central limit theorem 14.4 also holds. These results have been known for SE by Ferrari and Fontes [8].

For the ZR and BL models, we need a condition on the growth rates:

Condition 14.5. *For ZR and BL processes defined above, the rate function f is convex.*

For the ZR process, under this condition and assuming either strict convexity or concavity of $\mathcal{H}(\varrho)$ defined in (25), more than (27), namely, L^1-convergence is established by Rezakhanlou [23] with speed

$$C(\theta) = \frac{\mathrm{e}^\theta}{\mathbf{Var}_\theta(\omega)}. \tag{30}$$

As far as we know, the second-moment condition (28) has not yet been proven for this model.

Theorem 14.6. *For ZR and BL models satisfying condition 14.5 with initial distribution* $\underline{\mu}_\theta$ *of* $\underline{\omega}$*, and for any* $n \in \mathbb{Z}^+$*,*

$$\frac{Q(t)}{t} \to C(\theta) \qquad \text{in } L^n,$$

where $C(\theta)$ *is defined in* (30) *for the ZR process, and*

$$C(\theta) := \frac{2\sinh(\theta)}{\mathbf{Var}_\theta(\omega)} \tag{31}$$

for the BL model.

Hence under condition 14.5, condition 14.2 and thus theorem 14.3 and 14.4 hold for both ZR and BL models with $C(\theta)$ defined in (30) and (31), respectively. As we expect by mirror symmetric properties of the BL model, the speed $C(\theta)$ of the defect tracer is zero in case $\theta = 0$ in this model.

Our methods do not rely on hydrodynamic limits. $C(\theta)$ is a nondecreasing function for the totally asymmetric ZR process and BL model under condition 14.5, see remark 18.10. This shows (non strict) convexity of the function $\mathcal{H}(\varrho)$ of (25) for these models, since

$$C(\theta(\varrho)) = \frac{\mathrm{d}\mathcal{H}(\varrho)}{\mathrm{d}\varrho}$$

after some computations, and $\theta(\varrho)$ is also a monotone function.

Proposition 14.7. *Under condition 14.5, the function* $\mathcal{H}(\varrho)$ *is strictly convex for the BL model. For the ZR process satisfying 14.5, linearity of* $\mathcal{H}(\varrho)$ *is equivalent to linearity of the rate function* f *on* $\mathbb{Z}$*, which is the case of independent random walk of the particles. If this is not the case, then* $\mathcal{H}(\varrho)$ *is strictly convex.*

This is an important observation for [2], since this property is only proved for small θ values there. It is also remarkable for [23], where strict convexity is just assumed.

We remark that rates for removal of the bricks can also be introduced to obtain a model with both growth and decrease of columns. In particle notations this represents possible left jumps of particles (or right jump of antiparticles, respectively). Therefore, not only the totally asymmetric case, but the general asymmetric case of particle processes (SE or ZR, for example) can also be included in the description. The extension of the proof of theorems 14.3 and 14.4 to this case is straightforward. However, the coupling arguments used to establish condition 14.2 for ZR and BL models in later sections are not applicable in case of brick-removal.

We see that $\lim\limits_{t\to\infty} \frac{\mathbf{Var}(J^{(V)}(t))}{t}$ vanishes if we observe this quantity from the moving position $Vt = C(\theta)t$, having the characteristic speed of the hydrodynamical equation. This has been known for the SE model with strongly restricted values of ω_i, and now it is proven for the class of more general models with possibly $\omega_i \in \text{all}\,\mathbb{Z}$ also. The interesting question, of which the answer is strongly suggested for some models [20], is the correct exponent of t leading to nontrivial limit of $\mathbf{Var}(J^{(C)}(t))/t^{2\alpha}$ as $t \to \infty$. α is believed to be $1/3$, in close connection to $t^{2/3}$ order fluctuations of the position $Q(t)$ of the defect tracer.

The structure of the paper is the following: after some definitions on the reversed chain, we begin with separating martingales from $\mathbf{Var}(J(t))$ in section 15.2. Then we proceed in section 15.3 by computing the generator's inverse on the rates and then by transforming $\mathbf{Var}(J(t))$ into nontrivial correlations. These correlations can be computed using monotonicity thus coupling possibilities of the model, this is done in section 16. This section also includes a technical lemma showing an interesting relation of space-time correlations to the motion of the defect tracer. After $J(t)$, we deal with $J^{(V)}(t)$, the growth in non-vertical directions in section 17. Our results are proven in this section, except for theorem 14.6, which is proven in the last section for the totally asymmetric ZR process and for BL models. This last section includes the introduction of a new random walk depending on our processes, and new coupling techniques based on convexity of the rate function f. As another consequence of these methods, this part is followed by a proof of strict convexity of the function $\mathcal{H}(\varrho)$.

15 The growth and correlations

In this section we obtain a formula for $\mathbf{Var}(J(t))$, which contains only space-time correlations of $\omega_i(t)$'s as non-trivial expressions.

15.1 The reversed chain

The formal infinitesimal generator L^* for the reversed chain is of the form

$$(L^*\varphi)(\underline{\omega}) = \sum_{i\in\mathbb{Z}} r^*(\omega_i,\, \omega_{i+1}) \cdot [\varphi(\dots,\, \omega_i + 1,\, \omega_{i+1} - 1,\, \dots) - \varphi(\underline{\omega})]$$

on the finite cylinder functions. The rates r^* of the reversed process w.r.t. $\underline{\mu}_\theta$ can be determined by the equation

$$\mathbf{E}_\theta\left(\psi(\underline{\omega}) \cdot L\varphi(\underline{\omega})\right) = \mathbf{E}_\theta\left(\varphi(\underline{\omega}) \cdot L^*\psi(\underline{\omega})\right).$$

Proposition 15.1. *For $\omega^{min} \le z,\, y \le \omega^{max}$,*

$$r^*(z,\, y) = r(y,\, z). \tag{32}$$

Note that the rates of the reversed process do not depend on the parameter θ of the original process' distribution.

Proof. Let $\psi,\ \varphi$ be finite cylinder functions, and let $\mathcal{I} \subset \mathbb{Z}$ be a finite discrete interval of which the size can be divided by three, and which contains the set

$$\{i \in \mathbb{Z} :\, \psi, \text{ or } \varphi \text{ depends on } \omega_i \text{ or on } \omega_{i-1}\}.$$

Then the summation index i in the definition (21) of the generator can be run on the set $\mathcal{I}$. We begin by changing variables ω_i, ω_{i+1}:

$$\mathbf{E}_\theta\left(\psi(\underline{\omega})\cdot L\varphi(\underline{\omega})\right) =$$
$$= \mathbf{E}_\theta \sum_{i\in\mathcal{I}} \left\{ r(\omega_i,\,\omega_{i+1})\cdot[\psi(\underline{\omega})\varphi(\dots,\,\omega_i-1,\,\omega_{i+1}+1,\,\dots)-\psi(\underline{\omega})\cdot\varphi(\underline{\omega})]\right\} =$$
$$= \mathbf{E}_\theta \sum_{i\in\mathcal{I}} \Bigg\{ r(\omega_i+1,\,\omega_{i+1}-1)\cdot\frac{\mu_\theta(\omega_i+1)\,\mu_\theta(\omega_{i+1}-1)}{\mu_\theta(\omega_i)\,\mu\theta(\omega_{i+1})}\times$$
$$\times\,\psi(\dots,\,\omega_i+1,\,\omega_{i+1}-1,\,\dots)\varphi(\underline{\omega})\Bigg\} - \mathbf{E}_\theta\left\{\left(\sum_{i\in\mathcal{I}} r(\omega_i,\,\omega_{i+1})\right)\cdot\psi(\underline{\omega})\varphi(\underline{\omega})\right\}.$$

Since $|\mathcal{I}|$ can be divided by three, we can apply (19) in order to show that

$$\sum_{i\in\mathcal{I}} r(\omega_i,\,\omega_{i+1}) = \sum_{i\in\mathcal{I}} r(\omega_{i+1},\,\omega_i)$$

in the second term. By using (24) for the first term we finally obtain

$$\mathbf{E}_\theta\left(\psi(\underline{\omega})\cdot L\varphi(\underline{\omega})\right) =$$
$$= \mathbf{E}_\theta \sum_{i\in\mathcal{I}} \left\{ r(\omega_{i+1},\,\omega_i)\cdot[\psi(\dots,\,\omega_i+1,\,\omega_{i+1}-1,\,\dots)\varphi(\underline{\omega})-\psi(\underline{\omega})\cdot\varphi(\underline{\omega})]\right\},$$

which equals to $\mathbf{E}_\theta\left(\varphi(\underline{\omega})\cdot L^*\psi(\underline{\omega})\right)$ by choosing r^* according to (32). □

Combining (24) with (32) leads to

$$r^*(z,\,y) = \frac{\mu_\theta(z+1)\,\mu_\theta(y-1)}{\mu_\theta(z)\,\mu_\theta(y)}\cdot r(z+1,\,y-1)\,, \tag{33}$$

which is the natural formula suggested by considering conditional expectation values.

In order to simplify notations, let

$$r(t) := r(\omega_0(t),\,\omega_1(t)), \qquad\qquad r^*(t) := r^*(\omega_0(t),\,\omega_1(t)).$$

15.2 Preparatory computations

For a quantity $A(\underline{\omega})$ with $\mathbf{E}|A| < \infty$, let

$$\widetilde{A} = \widetilde{A}(\underline{\omega}) := A - \mathbf{E}A.$$

Lemma 15.2. $\mathbf{Var}(J(t)) = t\,\mathbf{E}(r) + 2\int_0^t (t-v)\,\mathbf{E}\left(\widetilde{r}(v)\,r^*(0)\right)\,\mathrm{d}v.$

Proof. By definition, $\mathbf{E}(J(t)\,|\,\underline{\omega}(0)) = t\,r(0) + \mathfrak{o}(t)$, hence

$$M(t) := J(t) - \int_0^t r(s)\,\mathrm{d}s$$

is a martingale with $M(0)=0$. Using this,

$$\mathbf{Var}(J(t)) = \mathbf{E}M(t)^2 + 2\,\mathbf{E}\left(M(t)\int_0^t \widetilde{r}(s)\,\mathrm{d}s\right) + \mathbf{E}\left(\left(\int_0^t \widetilde{r}(s)\,\mathrm{d}s\right)^2\right). \tag{34}$$

Due to $\mathbf{E}\left(M(t)^2 \,|\, \underline{\omega}(0)\right) = t\,r(0) + \mathfrak{o}(t)$, the process

$$N(t) := M(t)^2 - \int_0^t r(s)\,\mathrm{d}s$$

is also a martingale with $N(0)=0$. Hence

$$\mathbf{E}M(t)^2 = t\,\mathbf{E}(r).$$

Using the martingale property of M, the second term of (34) can be written as

$$2\int_0^t \mathbf{E}\left(M(t)\,\widetilde{r}(s)\right)\,\mathrm{d}s = 2\int_0^t \mathbf{E}\left(M(s)\,\widetilde{r}(s)\right)\,\mathrm{d}s.$$

Simply changing the limits of integration in the third term of (34), we have

$$\mathbf{E}\left(\left(\int_0^t \widetilde{r}(s)\,\mathrm{d}s\right)^2\right) = 2\int_0^t \mathbf{E}\left(\widetilde{r}(s)\int_0^s \widetilde{r}(u)\,\mathrm{d}u\right)\,\mathrm{d}s.$$

These calculations lead to

$$\begin{aligned}\mathbf{Var}(J(t)) &= t\,\mathbf{E}(r) + 2\int_0^t \mathbf{E}\left(\widetilde{r}(s)\left(M(s) + \int_0^s \widetilde{r}(u)\,\mathrm{d}u\right)\right)\,\mathrm{d}s = \\ &= t\,\mathbf{E}(r) + 2\int_0^t \mathbf{E}\left(\widetilde{r}(s)\,J(s)\right)\,\mathrm{d}s.\end{aligned} \tag{35}$$

In order to handle $\mathbf{E}\left(\widetilde{r}(s)\,J(s)\right)$, we introduce $J^{(s)\,*}$, the quantity corresponding to J in the reversed model by

$$J^{(s)\,*}(u) := J(s) - J(s-u) \qquad (s \geq u \geq 0).$$

This is the number of bricks removed from the column in the reversed model started from time s. As in case of $J(t)$, a reversed martingale can be separated by

$$M^{(s)\,*}(u) := J^{(s)\,*}(u) - \int_0^u r^*(s-v)\,\mathrm{d}v.$$

For this reversed object, $M^{(s)\,*}(0) = 0$ and $\mathbf{E}\left(M^{(s)\,*}(u) \,|\, \mathcal{F}_{[t,\,\infty)}\right) = M^{(s)\,*}(s-t)$ if $0 \leq s-t \leq u$, where $\mathcal{F}$ stands for the natural filtration of the (forward) process. In view of this,

$$\mathbf{E}\left(\widetilde{r}(s)\, J(s)\right) = \mathbf{E}\left[\widetilde{r}(s)\,\mathbf{E}\left(J^{(s)\,*}(s) \mid \mathcal{F}_{[s,\,\infty)}\right)\right] =$$
$$= \mathbf{E}\left(\widetilde{r}(s)\int_0^s r^*(s-v)\,\mathrm{d}v\right) = \int_0^s \mathbf{E}\left(\widetilde{r}(v)\, r^*(0)\right)\,\mathrm{d}v,$$

where in the last step we used time-invariance of the measure. Using this result, we obtain

$$\mathbf{Var}(J(t)) = t\,\mathbf{E}(r) + 2\int_0^t (t-v)\;\mathbf{E}\left(\widetilde{r}(v)\, r^*(0)\right)\,\mathrm{d}v$$

from (35) by changing the order of integration. □

15.3 Occurrence of space-time correlations

In this subsection we denote $r(\omega_i,\,\omega_{i+1})$ and $\widetilde{r}(\omega_i,\,\omega_{i+1})$ by r_i and $\widetilde{r}_i$, respectively. For $k \in \mathbb{Z}$, let

$$d_k \quad : \quad \Omega \to I \quad ; \quad d_k(\underline{\omega}) = \omega_k$$

be the k-th coordinate of Ω. Then

$$(36) \qquad \begin{aligned} \left(Ld_k\right)(\underline{\omega}) &= r_{k-1} - r_k \quad \text{and} \\ \left(L^* d_k\right)(\underline{\omega}) &= -r^*_{k-1} + r^*_k\,, \end{aligned}$$

where L^* is the infinitesimal generator (15.1) for the reversed process.

Lemma 15.3. *For $0 < \alpha < 1$ the expressions*

$$(37) \qquad \varphi_\alpha := \sum_{k=1}^{\infty} \alpha^{k-1} d_k \qquad\qquad \psi_\alpha := \sum_{k=0}^{\infty} \alpha^k d_{-k}$$

exist a.s., and

$$\lim_{\alpha\to 1}(L\varphi_\alpha)(\underline{\omega}) = -\lim_{\alpha\to 1}(L\psi_\alpha)(\underline{\omega}) = \widetilde{r},$$
$$\lim_{\alpha\to 1}(L^*\psi_\alpha)(\underline{\omega}) = -\lim_{\alpha\to 1}(L^*\varphi_\alpha)(\underline{\omega}) = \widetilde{r^*}$$

in L^2.

Proof. The a.s. existence of the sums above can be easily shown by using the Borel-Cantelli lemma for the sets

$$A_n := \{\underline{\omega} \;:\; |\omega_n| \geq n\}\,.$$

We show the first equation for φ_α. By (36)

$$(38) \qquad \left(L\varphi_\alpha\right)(\underline{\omega}) = r_0 + (\alpha-1)\sum_{k=1}^{\infty} r_k \alpha^{k-1} =$$
$$= r_0 - \mathbf{E}\,r + (\alpha-1)\sum_{k=1}^{\infty}(r_k - \mathbf{E}\,r)\alpha^{k-1} = \widetilde{r}_0 + (\alpha-1)\sum_{k=1}^{\infty}\widetilde{r}_k\alpha^{k-1}.$$

By independence of ω_i and ω_j for $i \neq j$, $\mathbf{E}(\widetilde{r}_l \cdot \widetilde{r}_k) = 0$ if $|l-k| > 1$ and $|\mathbf{E}(\widetilde{r}_l \cdot \widetilde{r}_k)| \leq \mathbf{E}(\widetilde{r}_l \cdot \widetilde{r}_l) = ||\widetilde{r}||_2^2$, if $|k-l| = 0$ or 1. Hence the L^2-norm of the second term on the right-hand side of (38) tends to zero as $\alpha \to 1$:

$$\left|\left|(\alpha-1)\sum_{k=1}^{\infty}\widetilde{r}_k\alpha^{k-1}\right|\right|_2^2 \leq (\alpha-1)^2\sum_{k=1}^{\infty}||\widetilde{r}_k||_2^2\alpha^{2k-2}+$$

$$+2(\alpha-1)^2\sum_{k=1}^{\infty}||\widetilde{r}_k||_2^2\alpha^{2k-3} = \frac{(\alpha-1)^2}{1-\alpha^2}||\widetilde{r}||_2^2(1+2\alpha^{-1}) \underset{\alpha\to 1}{\longrightarrow} 0.$$

The proof of the other three equations is similar. □

Now we can compute the integrals in our expression for $\mathbf{Var}(J)$.

Theorem 15.4.

$$\mathbf{Var}(J(t)) = t\,\mathbf{E}(r) - 2\,t\,\mathbf{E}(r^*(0)\cdot\widetilde{\omega}_1(0)) + 2\sum_{n=1}^{\infty} n\mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t)) =$$

$$= t\,\mathbf{E}(r) + 2\,t\,\mathbf{E}(r^*(0)\cdot\widetilde{\omega}_0(0)) + 2\sum_{n=1}^{\infty} n\mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_{-n}(t)). \tag{39}$$

As can be seen in the next session, the sums on the right-hand side are convergent.

Proof. Using L^2 convergence stated in lemma 15.3 and Cauchy's inequality, we rewrite the integral in the result of lemma 15.2. We can write $\widetilde{r}^*(0)$ instead of $r^*(0)$ there, since $\mathbf{E}(\widetilde{A}\,B) = \mathbf{E}(\widetilde{A}\,\widetilde{B})$ if both sides exist.

$$\left|\int_0^t (t-v)\mathbf{E}(\widetilde{r}(v)\,\widetilde{r}^*(0))\,\mathrm{d}v - \lim_{\alpha\to 1}\int_0^t (t-v)\mathbf{E}(L\varphi_\alpha(v)\,\widetilde{r}^*(0))\,\mathrm{d}v\right| \leq$$

$$\leq \lim_{\alpha\to 1}\int_0^t (t-v)\sqrt{\mathbf{E}\left([\widetilde{r}(v)-L\varphi_\alpha(v)]^2\right)\cdot\mathbf{E}(\widetilde{r}^*(0)^2)}\,\mathrm{d}v =$$

$$= \lim_{\alpha\to 1}\sqrt{\mathbf{E}\left([\widetilde{r}(0)-L\varphi_\alpha(0)]^2\right)\cdot\mathbf{E}(\widetilde{r}^*(0)^2)}\int_0^t (t-v)\,\mathrm{d}v = 0,$$

hence we can apply integration by parts:

$$\mathbf{Var}(J(t)) = t\,\mathbf{E}(r) + 2\lim_{\alpha\to 1}\int_0^t (t-v)\,\mathbf{E}\left(L\varphi_\alpha(v)\,\widetilde{r}^*(0)\right)\,\mathrm{d}v =$$

$$= t\,\mathbf{E}(r) + 2\lim_{\alpha\to 1}\int_0^t (t-v)\,\frac{\mathrm{d}}{\mathrm{d}v}\mathbf{E}(\varphi_\alpha(v)\,\widetilde{r}^*(0))\,\mathrm{d}v =$$

$$= t\,\mathbf{E}(r) - 2\,t\lim_{\alpha\to 1}\mathbf{E}(\widetilde{\varphi}_\alpha(0)\,\widetilde{r}^*(0)) + 2\lim_{\alpha\to 1}\int_0^t \mathbf{E}(\widetilde{\varphi}_\alpha(v)\,\widetilde{r}^*(0))\,\mathrm{d}v.$$

The last integral here can be transformed in the same way, using lemma 15.3 again:

$$\int_0^t \mathbf{E}(\widetilde{\varphi}_\alpha(v)\,\widetilde{r}^*(0))\,\mathrm{d}v = \int_0^t \mathbf{E}(\widetilde{\varphi}_\alpha(0)\,\widetilde{r}^*(-v))\,\mathrm{d}v =$$

$$= \lim_{\gamma\to 1}\int_0^t \mathbf{E}(\widetilde{\varphi}_\alpha(0)\,L^*\psi_\gamma(-v))\,\mathrm{d}v = \lim_{\gamma\to 1}\int_0^t \frac{\mathrm{d}}{\mathrm{d}v}\mathbf{E}(\widetilde{\varphi}_\alpha(0)\,\psi_\gamma(-v))\,\mathrm{d}v =$$

$$= \lim_{\gamma\to 1}\mathbf{E}(\widetilde{\varphi}_\alpha(0)\,\widetilde{\psi}_\gamma(-t)) - \lim_{\gamma\to 1}\mathbf{E}(\widetilde{\varphi}_\alpha(0)\,\widetilde{\psi}_\gamma(0)).$$

Hence with definitions (37), the variance of $J(t)$ can now be written as

$$\begin{aligned}\mathbf{Var}(J(t)) &= t\,\mathbf{E}(r) - 2t\lim_{\alpha\to 1}\mathbf{E}(\widetilde{\varphi}_\alpha(0)\,\widetilde{r}^*(0)) + \\ &\quad + 2\lim_{\alpha,\gamma\to 1}\mathbf{E}(\widetilde{\varphi}_\alpha(0)\,\widetilde{\psi}_\gamma(-t)) - 2\lim_{\alpha,\gamma\to 1}\mathbf{E}(\widetilde{\varphi}_\alpha(0)\,\widetilde{\psi}_\gamma(0)) = \\ &= t\,\mathbf{E}(r) - 2t\lim_{\alpha\to 1}\mathbf{E}\left(\sum_{k=1}^{\infty}\alpha^{k-1}\,\widetilde{\omega}_k(0)\,\widetilde{r}^*(0)\right) + \\ &\quad + 2\lim_{\alpha,\gamma\to 1}\mathbf{E}\left(\sum_{k=1}^{\infty}\alpha^{k-1}\,\widetilde{\omega}_k(0)\sum_{l=0}^{\infty}\gamma^l\,\widetilde{\omega}_{-l}(-t)\right) - \\ &\quad - 2\lim_{\alpha,\gamma\to 1}\mathbf{E}\left(\sum_{k=1}^{\infty}\alpha^{k-1}\,\widetilde{\omega}_k(0)\sum_{l=0}^{\infty}\alpha^l\,\widetilde{\omega}_{-l}(0)\right).\end{aligned}$$

Using product property of the measure at time $t = 0$ and the fact that r^* depends only on ω_0 and ω_1, most of our expressions become simple (recall that all quantities with tilde are centered random variables):

$$\begin{aligned}\mathbf{Var}(J(t)) &= t\,\mathbf{E}(r) - 2t\,\mathbf{E}(\widetilde{\omega}_1(0)\,\widetilde{r}^*(0)) + 2\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\mathbf{E}(\widetilde{\omega}_k(0)\,\widetilde{\omega}_{-l}(-t)) - 0 = \\ &= t\,\mathbf{E}(r) - 2t\,\mathbf{E}(\widetilde{\omega}_1(0)\,\widetilde{r}^*(0)) + 2\sum_{n=1}^{\infty} n\,\mathbf{E}(\widetilde{\omega}_n(t)\,\widetilde{\omega}_0(0)).\end{aligned}$$

In the last step, we used translation- and time-invariance of the measure.

We needed $L\varphi_\alpha \to \widetilde{r}$ and $L^*\psi_\alpha \to \widetilde{r}^*$ in L^2 so far. The properties $-L\psi_\alpha \to \widetilde{r}$ and $-L^*\varphi_\alpha \to \widetilde{r}^*$ can be used in a similar way to prove the second equation of the theorem. However, we need both φ_α and ψ_α: using only one of them would have lead to a divergent sum in the last step. □

The first two expressions of formula (39) can be computed easily. The difficulty is in determining the space-time correlations $\mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_k(t))$. In order to do this, we use coupling technique.

16 Coupling and correlations

In this section, we show how to couple a pair of our models, with the help of the so-called second class particles. We can use second particles to compute our expressions containing space-time correlations.

16.1 The basic coupling

We consider two realizations of a model, namely, $\underline{\zeta}$ and $\underline{\eta}$. We show the basic coupling preserving

$$\zeta_i(t) \geq \eta_i(t), \tag{40}$$

if this property holds initially for $t = 0$. We say that $n = \zeta_i(t) - \eta_i(t) \geq 0$ is the number of *second class particles* present at site i at time t. During the evolution of the processes, the total number of these particles is preserved, and each of them performs a nearest neighbor random walk.

The height of the column of $\underline{\zeta}$ (or $\underline{\eta}$) between sites i and $i+1$ is denoted by g_i (or h_i, respectively). (These quantities are just used for easier understanding, they are not essential for the processes.) Let $g_i \uparrow$ (or $h_i \uparrow$) mean that the column of $\underline{\zeta}$ (or the column of $\underline{\eta}$, respectively) between the sites i and $i+1$ has grown by one brick. Then the coupling rules are shown in table 2. Each line of this table represents a possible move, with rate written in the first column. In the last column, $\curvearrowright$ (or $\curvearrowleft$) means that a second class particle has jumped from i to $i+1$ (or from $i+1$ to i, respectively). This coupling for the SE model is described (with particle notations) in Liggett [14], [15] and [16]. The rates of these steps are non-negative due to (40) and monotonicity (18) of r. These rules clearly preserve property (40), since the rate of any move which could destroy this condition becomes zero. Summing the rates corresponding to either $g_i \uparrow$ or to $h_i \uparrow$ shows that each $\underline{\zeta}$ and $\underline{\eta}$ evolves according to its own rates. It would be possible to couple models possessing rates for removal of bricks as well.

with rate	$g_i \uparrow$	$h_i \uparrow$	a second class particle
$r(\zeta_i,\, \zeta_{i+1}) - r(\eta_i,\, \zeta_{i+1})$	$\bullet$		$\curvearrowright$
$r(\eta_i,\, \eta_{i+1}) - r(\eta_i,\, \zeta_{i+1})$		$\bullet$	$\curvearrowleft$
$r(\eta_i,\, \zeta_{i+1})$	$\bullet$	$\bullet$	

Table 2: Growth coupling rules

16.2 Correlations and the defect tracer

We introduce the notation $\underline{\delta}_i \in \Omega$, a configuration being one at site i and zero at all other sites. Let $\underline{\omega}$ be a model distributed according to $\underline{\mu}_\theta$, and $\underline{\zeta}(0) = \underline{\omega}(0) + \underline{\delta}_0$, i.e. we have a single one second class particle between $\underline{\zeta}$ and $\underline{\omega}$, initially at site 0. In order to avoid confusions, we call this particle the *defect tracer*. According to the basic coupling, this single defect tracer is conserved for any time t:

$$\underline{\zeta}(t) = \underline{\omega}(t) + \underline{\delta}_{Q(t)} \tag{41}$$

The quantity $Q(t)$ is the position of the defect tracer, performing a nearest neighbor random walk on $\mathbb{Z}$.

In this subsection we consider the process $(\underline{\omega}(t),\, Q(t))$, the model distributed according to the Gibbs measure $\underline{\mu}$ and the random walk $Q(t)$ connected to it with $Q(0) = 0$. Using condition 14.2, we prove theorem 14.3 for $V = 0$. We begin with a technical lemma, showing how to make use of the defect tracer.

Lemma 16.1. *For the pair* $(\underline{\omega}(t), Q(t))$ *defined above, and for a function* $F : I \to \mathbb{R}$ *with* $F(\omega^{max}) = 0$ *and with finite expectation value* $\sum F(z)\,\mu(z)$*,*

$$\mathbf{E}\left(\omega_n(t)\,\left[\frac{F(\omega_0(0)-1)\,\mu(\omega_0(0)-1)}{\mu(\omega_0(0))} - F(\omega_0(0))\right]\right) = \\ = \mathbf{E}\left(\mathbf{1}\{Q(t)=n\}\,F(\omega_0(0))\right). \tag{42}$$

Proof. We take conditional expectation value of (41):

$$\mathbf{E}\left(\zeta_n(t)\,|\,\omega_0(0)=z\right) = \mathbf{E}\left(\omega_n(t)\,|\,\omega_0(0)=z\right) + \mathbf{P}\left(Q(t)=n\,|\,\omega_0(0)=z\right). \tag{43}$$

Initially, $\underline{\zeta}(0) = \underline{\omega}(0) + \underline{\delta}_0$. Therefore, $\underline{\zeta}$ itself is also a model with initial distribution $\underline{\mu}$, except for the origin. Hence

$$\mathbf{E}\left(\zeta_n(t)\,|\,\omega_0(0)=z\right) = \mathbf{E}\left(\zeta_n(t)\,|\,\zeta_0(0)=z+1\right) = \mathbf{E}\left(\omega_n(t)\,|\,\omega_0(0)=z+1\right),$$

and (43) can be written as

$$\mathbf{E}\left(\omega_n(t)\,|\,\omega_0(0)=z+1\right) - \mathbf{E}\left(\omega_n(t)\,|\,\omega_0(0)=z\right) = \mathbf{P}\left(Q(t)=n\,|\,\omega_0(0)=z\right).$$

We multiply both sides with $F(z)\,\mu(z)$ and then add up for all $z \in I$ to obtain

$$\sum_{z\in I}\mathbf{E}\left(\omega_n(t)\,|\,\omega_0(0)=z\right)\cdot\left(F(z-1)\,\mu(z-1) - F(z)\,\mu(z)\right) = \\ = \sum_{z\in I}\mathbf{P}\left(Q(t)=n\,|\,\omega_0(0)=z\right)\cdot F(z)\,\mu(z).$$

Here we used that $F(\omega^{\max}) = 0$ and we write $\mu(\omega^{\min}-1) = 0$. We know that $\mathbf{P}(\omega_0(0)=z) = \mu(z)$, hence the proof follows. □

Corollary 16.2. *We use the convention that the empty sum equals zero. Let*

$$g(z) := z - \sum_{y\in I} y\,\mu(y).$$

For $n \in \mathbb{Z}$*,*

$$\mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t)) = \mathbf{E}\left(\mathbf{1}\{Q(t)=n\}\cdot\sum_{z=\omega_0+1}^{\omega^{max}} g(z)\frac{\mu(z)}{\mu(\omega_0)}\right).$$

Proof. By the previous lemma, our goal is now to find the correct function F, for which

$$\frac{F(z-1)\,\mu(z-1)}{\mu(z)} - F(z) = g(z) = z - \sum_{y\in I} y\,\mu(y)$$

is satisfied. By inverting the operation on the left side, we find

$$F(z) := \sum_{y=z+1}^{\omega^{\max}} g(y)\,\frac{\mu(y)}{\mu(z)}.$$

This function satisfies the conditions of the lemma. Using (42),

$$\mathbf{E}(\widetilde{\omega}_n(t)\,\widetilde{\omega}_0(0)) = \mathbf{E}(\omega_n(t)\,\widetilde{\omega}_0(0)) = \mathbf{E}(\omega_n(t)\cdot g(\omega_0(0))) =$$
$$= \mathbf{E}\left(\omega_n(t)\,\left[\frac{F(\omega_0(0)-1)\,\mu(\omega_0(0)-1)}{\mu(\omega_0(0))} - F(\omega_0(0))\right]\right) =$$
$$= \mathbf{E}\left(\mathbf{1}\{Q(t)=n\}\,F(\omega_0(0))\right) =$$
$$= \mathbf{E}\left(\mathbf{1}\{Q(t)=n\}\cdot\sum_{y=\omega_0(0)+1}^{\omega^{\max}} g(y)\frac{\mu(y)}{\mu(\omega_0(0))}\right).$$

□

Now it becomes clear that we need to know something about the motion of the defect tracer. $\underline{\zeta}$ and $\underline{\omega}$ can not be started together from their original stationary distribution due to the initial difference between them, present at the origin. We could follow our defect tracer. Knowing a measure stationary as seen from site $Q(t)$ for all time t would help us to state the law of large numbers for the $Q(t)$ process. In general, we don't know such a stationary measure which has the same asymptotics far on the left and far on the right side. It is shown in [2], that under some weak assumptions for BL models, this measure can not be a product-distribution. (Instead, a shock-like stationary product-measure is described there for certain type of rates, under which the slope of the surface differs on the left side from that on the right side.)

For SE and some types of ZR processes, law of large numbers (27) is known. This law and the second moment condition (28) for BL and ZR models possessing convexity condition 14.5 are proven in section 18. As shown in the next theorem, this allows us to do further computations on the space-time correlations of the models. We need the following properties of the canonical measure:

Lemma 16.3. *(i) The sum*

$$\sum_{z\in I}\sum_{y=z+1}^{\omega^{max}} g(y)\frac{\mu(y)}{\sqrt{\mu(z)}}$$

is convergent, and
(ii) the sum

$$\sum_{z\in I}\sum_{y=z+1}^{\omega^{max}} g(y)\,\mu(y) = \mathbf{Var}(\omega_0)$$

is convergent and the equality holds.

Proof. For $\theta \in (\underline{\theta},\,\bar{\theta})$, the tails of the measure $\mu_\theta(\cdot)$ have exponential decay. Hence the convergence in both expressions holds. The identity in (ii) is straightforward and is left to the reader. □

The next lemma shows the essential connection of the defect tracer to space-time correlations in the model.

Lemma 16.4. *Assume condition 14.2 with speed value C. Let $B(t)$ be a real-valued function with $\lim_{t\to\infty} B(t) = B \in \mathbb{R}$, $n_1,\, n_2 \in \mathbb{Z}$, $A \in \mathbb{R}$, $V_1 < V_2$ in $\mathbb{R}\cup\{-\infty,\,\infty\}$ and the real interval $\mathcal{V} := [V_1,\, V_2]$. If either* *(i)* $C \neq V_1, V_2$, *or*

(ii) $C \in \mathbb{R}$ *and* $A \cdot C = -B$

holds, then

$$\lim_{t\to\infty} \sum_{n=\lceil tV_1\rceil+n_1}^{\lfloor tV_2\rfloor+n_2} \left(\frac{n}{t}A + B(t)\right) \cdot \mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t)) =$$
$$= (A\,C + B) \cdot \mathbf{1}\{C \in \mathcal{V}\} \cdot \mathbf{Var}(\omega_0),$$

where $\mathbf{Var}(\omega_0)$ *is the variance of* ω_0 *w.r.t. the canonical Gibbs-measure.*

Proof. We define $\mathcal{V}^t$ by

$$\mathcal{V}^t := \left[V_1 + \frac{n_1}{t},\ V_2 + \frac{n_2}{t}\right].$$

By corollary 16.2,

$$\lim_{t\to\infty} \sum_{n=\lceil tV_1\rceil+n_1}^{\lfloor tV_2\rfloor+n_2} \left(\frac{n}{t}A + B(t)\right) \cdot \mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t)) = \tag{44}$$
$$= \lim_{t\to\infty} \sum_{n=\lceil tV_1\rceil+n_1}^{\lfloor tV_2\rfloor+n_2} \left(\frac{n}{t}A + B(t)\right) \mathbf{E}\left(\mathbf{1}\{Q(t)=n\} \cdot \sum_{y=\omega_0(0)+1}^{\omega^{\max}} g(y)\frac{\mu(y)}{\mu(\omega_0(0))}\right) =$$
$$= \lim_{t\to\infty} \mathbf{E}\left(\left(A\frac{Q(t)}{t} + B(t)\right) \cdot \mathbf{1}\{Q(t)/t \in \mathcal{V}^t\} \cdot \sum_{y=\omega_0(0)+1}^{\omega^{\max}} g(y)\frac{\mu(y)}{\mu(\omega_0(0))}\right) =$$
$$= \lim_{t\to\infty} \sum_{z\in I} \mathbf{E}\left(\left(A\frac{Q(t)}{t} + B(t)\right) \cdot \mathbf{1}\{Q(t)/t \in \mathcal{V}^t\} \cdot \mathbf{1}\{\omega_0(0) = z\}\right) \times$$
$$\times \sum_{y=z+1}^{\omega^{\max}} g(y)\frac{\mu(y)}{\mu(z)}.$$

We show that the limit and the summation can be interchanged in this expression. We use Cauchy's inequality to obtain

$$\left|\mathbf{E}\left(\left(A\frac{Q(t)}{t} + B(t)\right) \cdot \mathbf{1}\{Q(t)/t \in \mathcal{V}^t\} \cdot \mathbf{1}\{\omega_0(0) = z\}\right)\right| \leq$$
$$\leq \sqrt{\mathbf{E}\left(\left(A\frac{Q(t)}{t} + B(t)\right)^2\right)} \cdot \sqrt{\mathbf{P}\left(\frac{Q(t)}{t} \in \mathcal{V}^t \text{ and } \omega_0(0) = z\right)} \leq$$
$$\leq K' \cdot \sqrt{\mathbf{P}\left(\frac{Q(t)}{t} \in \mathcal{V}^t \,\middle|\, \omega_0(0) = z\right)} \cdot \sqrt{\mu(z)} \leq K' \cdot \sqrt{\mu(z)}$$

for some constant K' by (28). Since $g(y)$ is monotone in y and

$$\sum_{y=\omega^{\min}}^{\omega^{\max}} g(y)\,\mu(y) = 0,$$

the sum

$$\sum_{y=z+1}^{\omega^{\max}} g(y)\,\mu(y)$$

is non-negative for any $z \in I$. Hence we can bound from above the absolute value of the terms in (44) for each $z \in I$ by

$$K' \cdot \sum_{y=z+1}^{\omega^{\max}} g(y) \frac{\mu(y)}{\sqrt{\mu(z)}},$$

and the sum

$$\sum_{z\in I} K' \cdot \sum_{y=z+1}^{\omega^{\max}} g(y) \frac{\mu(y)}{\sqrt{\mu(z)}}$$

is convergent by lemma 16.3. Using dominated convergence, we write

$$\lim_{t\to\infty} \sum_{n=\lceil tV_1\rceil+n_1}^{\lfloor tV_2\rfloor+n_2} \Big(\frac{n}{t}A + B(t)\Big) \cdot \mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t)) =$$
$$= \sum_{z\in I} \lim_{t\to\infty} \mathbf{E}\left(\left(A\frac{Q(t)}{t} + B(t)\right) \cdot \mathbf{1}\{Q(t)/t \in \mathcal{V}^t\} \cdot \mathbf{1}\{\omega_0(0) = z\}\right) \times$$
$$\times \sum_{y=z+1}^{\omega^{\max}} g(y) \frac{\mu(y)}{\mu(z)}.$$

We introduce the set $\mathcal{V}^t_\varepsilon := \mathcal{V}^t \cap \mathcal{B}_\varepsilon(C)$, where for $\varepsilon > 0$, $\mathcal{B}_\varepsilon(C) = (C-\varepsilon,\, C+\varepsilon) \subset \mathbb{R}$. Hence $\mathcal{V}^t = \mathcal{V}^t_\varepsilon \cup (\mathcal{V}^t \setminus \mathcal{B}_\varepsilon(C))$:

$$(45)\quad \lim_{t\to\infty} \sum_{n=\lceil tV_1\rceil+n_1}^{\lfloor tV_2\rfloor+n_2} \Big(\frac{n}{t}A + B(t)\Big) \cdot \mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t)) =$$
$$= \sum_{z\in I} \lim_{t\to\infty} \mathbf{E}\left(\left(A\frac{Q(t)}{t} + B(t)\right) \cdot \mathbf{1}\{Q(t)/t \in \mathcal{V}^t_\varepsilon\} \cdot \mathbf{1}\{\omega_0(0) = z\}\right) \times$$
$$\times \sum_{y=z+1}^{\omega^{\max}} g(y) \frac{\mu(y)}{\mu(z)} +$$
$$+ \sum_{z\in I} \lim_{t\to\infty} \mathbf{E}\left(\left(A\frac{Q(t)}{t} + B(t)\right) \cdot \mathbf{1}\{Q(t)/t \in \mathcal{V}^t \setminus \mathcal{B}_\varepsilon(C)\} \cdot \mathbf{1}\{\omega_0(0) = z\}\right) \times$$
$$\times \sum_{y=z+1}^{\omega^{\max}} g(y) \frac{\mu(y)}{\mu(z)}.$$

(45) contains two terms. We use Cauchy's inequality on the second term as we

have done before:

$$\left|\mathbf{E}\left(\left(A\frac{Q(t)}{t}+B(t)\right)\cdot\mathbf{1}\{Q(t)/t\in\mathcal{V}^t\setminus\mathcal{B}_\varepsilon(C)\}\cdot\mathbf{1}\{\omega_0(0)=z\}\right)\right|\le$$
$$\le K'\cdot\sqrt{\mathbf{P}\left(\frac{Q(t)}{t}\in\mathcal{V}^t\setminus\mathcal{B}_\varepsilon(C)\text{ and }\omega_0(0)=z\right)}\le$$
$$\le K'\cdot\sqrt{\mathbf{P}\left(\frac{Q(t)}{t}\notin\mathcal{B}_\varepsilon(C)\right)}\to 0$$

as $t\to\infty$ by the law of large numbers (27). Only the first term of (45) remained, for which we write

$$\lim_{t\to\infty}\sum_{n=\lceil tV_1\rceil+n_1}^{\lfloor tV_2\rfloor+n_2}\left(\frac{n}{t}A+B(t)\right)\cdot\mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t))=\tag{46}$$
$$=\sum_{z\in I}\lim_{t\to\infty}(A\cdot C+B(t)+\mathcal{O}(\varepsilon))\cdot\mathbf{P}\left(Q(t)/t\in\mathcal{V}^t_\varepsilon\text{ and }\omega_0(0)=z\right)\times$$
$$\times\sum_{y=z+1}^{\omega^{\max}}g(y)\frac{\mu(y)}{\mu(z)}.$$

We have three possibilities.
(i) If $C\in\mathcal{V}$, $C\neq V_1$, V_2, then for small ε and large t, $\mathcal{V}^t_\varepsilon=\mathcal{B}_\varepsilon(C)$, and by (27),

$$\lim_{t\to\infty}\mathbf{P}\left(Q(t)/t\in\mathcal{V}^t_\varepsilon\text{ and }\omega_0(0)=z\right)=$$
$$=\lim_{t\to\infty}\mathbf{P}\left(Q(t)/t\in\mathcal{B}_\varepsilon(C)\text{ and }\omega_0(0)=z\right)=\mathbf{P}(\omega_0(0)=z)=\mu(z).$$

Hence we can continue (46) by

$$\lim_{t\to\infty}\sum_{n=\lceil tV_1\rceil+n_1}^{\lfloor tV_2\rfloor+n_2}\left(\frac{n}{t}A+B(t)\right)\cdot\mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t))=$$
$$=\sum_{z\in I}\lim_{t\to\infty}(A\cdot C+B(t)+\mathcal{O}(\varepsilon))\cdot\sum_{y=z+1}^{\omega^{\max}}g(y)\mu(y)\to$$
$$(A\cdot C+B)\cdot\sum_{z\in I}\sum_{y=z+1}^{\omega^{\max}}g(y)\mu(y)=(A\cdot C+B)\cdot\mathbf{Var}(\omega_0)$$

as $\varepsilon\to 0$. The last equality is a result of lemma 16.3.
(ii) If $A\cdot C=-B$, then the right-hand side of (46) tends to $\mathcal{O}(\varepsilon)$ as $t\to\infty$ for all $\varepsilon>0$, hence is zero in this limit. Here we used that

$$\mathbf{P}\left(Q(t)/t\in\mathcal{V}^t_\varepsilon\text{ and }\omega_0(0)=z\right)\le\mathbf{P}(\omega_0(0)=z)=\mu(z),$$

and that

$$\sum_{z\in I}\sum_{y=z+1}^{\omega^{\max}}g(y)\mu(y)$$

is convergent.
(iii) In case $C \notin \mathcal{V}$, then $\mathcal{V}_\varepsilon^t$ is empty for ε small and t large enough, and hence the right-hand side of (46) is zero.
The result of these three cases completes the proof the lemma. $\square$

Now we are able to compute $\lim_{t\to\infty} \frac{\mathbf{Var}(J^V(t))}{t}$ for $V = 0$. The proof of the general formula (29) requires some more computations in the next subsection.

Theorem 16.5. *Assume condition 14.2 with speed C. Then*

$$\lim_{t\to\infty} \frac{\mathbf{Var}(J(t))}{t} = \mathbf{E}(r) - 2\,\mathbf{E}(r^*(0)\cdot\widetilde{\omega}_1(0)) + 2\,C^+\cdot\mathbf{Var}(\omega_0) =$$
$$= \mathbf{E}(r) + 2\,\mathbf{E}(r^*(0)\cdot\widetilde{\omega}_0(0)) + 2\,C^-\cdot\mathbf{Var}(\omega_0). \tag{47}$$

Here $0 \le C^\pm$ is the positive or the negative part of C, respectively.

Proof. We consider the result of theorem 15.4. Dividing (39) by t and taking the limit $t\to\infty$ allows us to use the result of lemma 16.4. For the first equality of (39), we use this lemma with parameters $V_1 = 0,\ V_2 = \infty,\ n_1 = 1,\ n_2 = 0,\ A = 1,\ B(t) = 0$. Then we obtain

$$\lim_{t\to\infty} \frac{\mathbf{Var}J(t)}{t} = \mathbf{E}(r) - 2\,\mathbf{E}(r^*(0)\cdot\widetilde{\omega}_1(0)) + 2\,C\cdot\mathbf{1}\{C\ge 0\}\cdot\mathbf{Var}(\omega_0).$$

For the second equality of (39), we rewrite the sum as

$$\lim_{t\to\infty} \frac{\mathbf{Var}J(t)}{t} = \mathbf{E}(r) + 2\,\mathbf{E}(r^*(0)\cdot\widetilde{\omega}_0(0)) - 2\sum_{n=-\infty}^{-1} n\,\mathbf{E}(\widetilde{\omega}_0(0)\,\widetilde{\omega}_n(t)),$$

in order to use lemma 16.4 with parameters $V_1 = -\infty,\ V_2 = 0,\ n_1 = 0,\ n_2 = -1,\ A = 1,\ B(t) = 0$. Hence

$$\lim_{t\to\infty} \frac{\mathbf{Var}J(t)}{t} = \mathbf{E}(r) + 2\,\mathbf{E}(r^*(0)\cdot\widetilde{\omega}_0(0)) - 2\,C\cdot\mathbf{1}\{C\le 0\}\cdot\mathbf{Var}(\omega_0),$$

which proves the second equality of the theorem. $\square$

We obtained two formulas for the variance of $J(t)$. If the characteristic speed C exists, then we can compute it by subtracting the two lines of (47).

17 The growth in non-vertical directions

We have examined so far $\mathbf{Var}(J(t))$, the growth fluctuation of a fixed column, i.e. the fluctuation of vertical growth. In this section we deal with the growth fluctuation of the surface in equilibrium, but considered in a slanting direction, namely, $\mathbf{Var}(J^{(V)}(t))$. From now on, we assume without loss of generity $h_0(0) = 0$.

Proof of (26). By definition $\omega_j(t) = h_{j-1}(t) - h_j(t)$, we have

$$h_i(t) = h_0(t) - \sum_{j=1}^{i} \omega_j(t) \tag{48}$$

for any site $i > 0$, hence for $V > 0$,

$$J^{(V)}(t) = h_{\lfloor Vt \rfloor}(t) = h_0(t) - \sum_{j=1}^{\lfloor Vt \rfloor} \omega_j(t) = h_0(t) - \lfloor Vt \rfloor \frac{1}{\lfloor Vt \rfloor} \sum_{j=1}^{\lfloor Vt \rfloor} \omega_j(t).$$

By ergodicity, the first term has the limit $\mathbf{E}r$ a.s. when divided by t. The second term is $\lfloor Vt \rfloor$ times the average of an increasing number of different iid. variables. These variables have finite moments, hence the fourth-moment argument (see e.g. [11, Theorem 7.1]) is applicable with the discretization series $t_n := n/V$ to show that

$$\lim_{n\to\infty} \frac{1}{\lfloor t_n V \rfloor} \sum_{j=1}^{\lfloor t_n V \rfloor} \omega_j(t_n) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \omega_j(t_n) = \mathbf{E}(\omega) \quad \text{a.s.}$$

This shows (26) for the limit taken along the subsequence t_n. For any $t \in \mathbb{R}^+$, there is a unique $n_t \in \mathbb{Z}^+$ for which $t_{n_t} \le t < t_{n_t+1}$, and $J^{(V)}(t) - J^{(V)}(t_{n_t})$ is the number of bricks laid on column n_t in a time interval shorter than $1/V$, hence dividing it by t leads a.s. to zero in the limit. Therefore (26) holds for the limit of $J^{(V)}(t)/t$ as well. Similar computation works for $V < 0$, and finally, the case $V = 0$ is trivial. □

Now we consider the fluctuations (with tilde meaning the mean value subtracted).

$$(49) \quad \mathbf{Var}(J^{(V)}(t)) = \mathbf{E}\left\{ \left(J^{(V)}(t) - \mathbf{E}J^{(V)}(t) \right)^2 \right\} =$$

$$= \mathbf{E}\left\{ \left(\widetilde{h}_{\lfloor Vt \rfloor}(t) \right)^2 \right\} = \mathbf{E}\left\{ \left(\left[\widetilde{h}_{\lfloor Vt \rfloor}(t) - \widetilde{h}_{\lfloor Vt \rfloor}(0) \right] + \widetilde{h}_{\lfloor Vt \rfloor}(0) \right)^2 \right\} =$$

$$= \mathbf{E}\left\{ \left(\widetilde{h}_{\lfloor Vt \rfloor}(t) - \widetilde{h}_{\lfloor Vt \rfloor}(0) \right)^2 \right\} - \mathbf{E}\left\{ \left(\widetilde{h}_{\lfloor Vt \rfloor}(0) \right)^2 \right\} + 2\,\mathbf{E}\left(\widetilde{h}_{\lfloor Vt \rfloor}(t)\, \widetilde{h}_{\lfloor Vt \rfloor}(0) \right).$$

By translation-invariance, the first term is $\mathbf{Var}(J(t))$, computed in the previous sections. By (48) and by product structure of the measure, the second term of the right-hand side of (49) is

$$-\mathbf{E}\left\{ \left(\widetilde{h}_{\lfloor Vt \rfloor}(0) \right)^2 \right\} = -\lfloor Vt \rfloor \cdot \mathbf{E}(\widetilde{\omega}_0(0)^2) = -\lfloor Vt \rfloor \cdot \mathbf{Var}(\omega).$$

The limit of the third term divided by t in (49) is computed in the following two lemmas:

Lemma 17.1. *For $V > 0$,*

$$(50) \quad \mathbf{E}\left(\widetilde{h}_{\lfloor Vt \rfloor}(t)\, \widetilde{h}_{\lfloor Vt \rfloor}(0) \right) =$$

$$= \sum_{n=-\infty}^{\lfloor Vt \rfloor - 1} (\lfloor Vt \rfloor - n) \mathbf{E}(\widetilde{\omega}_n(t)\, \widetilde{\omega}_0(0)) + \sum_{n=-\infty}^{-1} n \mathbf{E}(\widetilde{\omega}_n(t)\, \widetilde{\omega}_0(0)).$$

Proof. Using (48) again,

$$\text{(51)}\quad \mathbf{E}\left(\widetilde{h}_{\lfloor Vt\rfloor}(t)\,\widetilde{h}_{\lfloor Vt\rfloor}(0)\right) =$$

$$= -\mathbf{E}\left(h_0(t)\sum_{j=1}^{\lfloor Vt\rfloor}\widetilde{\omega}_j(0)\right) + \mathbf{E}\left(\sum_{i=1}^{\lfloor Vt\rfloor}\widetilde{\omega}_i(t)\sum_{j=1}^{\lfloor Vt\rfloor}\widetilde{\omega}_j(0)\right) =$$

$$= -\sum_{j=1}^{\lfloor Vt\rfloor}\mathbf{E}(h_0(t)\,\widetilde{\omega}_j(0)) + \sum_{i=1}^{\lfloor Vt\rfloor}\sum_{j=1}^{\lfloor Vt\rfloor}\mathbf{E}(\widetilde{\omega}_i(t)\,\widetilde{\omega}_j(0)).$$

A martingale

$$H(t) := h_0(t) - \int_0^t r_0(s)\,\mathrm{d}s$$

with $H(0) = 0$ can be separated in order to show that

$$\mathbf{E}(h_0(t)\,\widetilde{\omega}_j(0)) = \mathbf{E}(H(t)\,\widetilde{\omega}_j(0)) + \int_0^t \mathbf{E}(r_0(s)\,\widetilde{\omega}_j(0))\,\mathrm{d}s =$$

$$= \int_0^t \mathbf{E}(\widetilde{r}_0(s)\,\widetilde{\omega}_j(0))\,\mathrm{d}s.$$

Now we use an argument very similar to the proof of theorem 15.4. By lemma 15.3, the L^2-convergence

$$-\lim_{\alpha\to 1}(L\psi_\alpha)(\underline{\omega}) = \widetilde{r}_0$$

can be used to replace our integral: for $j \geq 1$ we continue by

$$\mathbf{E}(h_0(t)\,\widetilde{\omega}_j(0)) = \int_0^t \mathbf{E}(\widetilde{r}_0(s)\,\widetilde{\omega}_j(0))\,\mathrm{d}s = -\lim_{\alpha\to 1}\int_0^t \mathbf{E}(L\psi_\alpha(s)\,\widetilde{\omega}_j(0))\,\mathrm{d}s =$$

$$= -\lim_{\alpha\to 1}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{E}(\psi_\alpha(s)\,\widetilde{\omega}_j(0))\,\mathrm{d}s = \mathbf{E}(\psi_\alpha(t)\,\widetilde{\omega}_j(0)) - \mathbf{E}(\psi_\alpha(0)\,\widetilde{\omega}_j(0)).$$

Using definition (37) of ψ_α and product structure of the canonical measure,

$$\mathbf{E}(h_0(t)\,\widetilde{\omega}_j(0)) = \mathbf{E}\left(-\sum_{i=0}^{\infty}\omega_{-i}(t)\,\widetilde{\omega}_j(0)\right) - \mathbf{E}\left(-\sum_{i=0}^{\infty}\omega_{-i}(0)\,\widetilde{\omega}_j(0)\right) =$$

$$= -\sum_{i=0}^{\infty}\mathbf{E}(\widetilde{\omega}_{-i}(t)\,\widetilde{\omega}_j(0)) = -\sum_{i=-\infty}^{0}\mathbf{E}(\widetilde{\omega}_i(t)\,\widetilde{\omega}_j(0)).$$

Combining this expression with (51) leads to

$$\mathbf{E}\left(\widetilde{h}_{\lfloor Vt\rfloor}(t)\,\widetilde{h}_{\lfloor Vt\rfloor}(0)\right) = \sum_{i=-\infty}^{0}\sum_{j=1}^{\lfloor Vt\rfloor}\mathbf{E}(\widetilde{\omega}_i(t)\,\widetilde{\omega}_j(0)) + \sum_{i=1}^{\lfloor Vt\rfloor}\sum_{j=1}^{\lfloor Vt\rfloor}\mathbf{E}(\widetilde{\omega}_i(t)\,\widetilde{\omega}_j(0)) =$$
$$= \sum_{i=-\infty}^{\lfloor Vt\rfloor}\sum_{j=1}^{\lfloor Vt\rfloor}\mathbf{E}(\widetilde{\omega}_i(t)\,\widetilde{\omega}_j(0)) = \sum_{i=-\infty}^{\lfloor Vt\rfloor}\sum_{j=1}^{\lfloor Vt\rfloor}\mathbf{E}(\widetilde{\omega}_{i-j}(t)\,\widetilde{\omega}_0(0))$$

by translation-invariance. Changing the summation indices leads to the proof of the lemma. □

Lemma 17.2. *Assume condition 14.2. Then for* $V > 0$,

$$\lim_{t\to\infty}\frac{1}{t}\,\mathbf{E}\left(\widetilde{h}_{\lfloor Vt\rfloor}(t)\,\widetilde{h}_{\lfloor Vt\rfloor}(0)\right) = (V - C^+)^+\cdot\mathbf{Var}(\omega).$$

Proof. We use lemma 16.4 for the two terms on the right-hand side of (50). For the first one we set $V_1 = -\infty,\ V_2 = V,\ n_1 = 0,\ n_2 = -1,\ A = -1,\ B(t) = \lfloor Vt\rfloor/t$, while for the second term in (50) we put $V_1 = -\infty,\ V_2 = 0,\ n_1 = 0,\ n_2 = -1,\ A = 1,\ B(t) = 0$. One can easily check that for any $C \in \mathbb{R}$ and $V > 0$, one of the cases (i) or (ii) of lemma 16.4 apply. Consequently, we obtain

$$\lim_{t\to\infty}\frac{1}{t}\,\mathbf{E}\left(\widetilde{h}_{\lfloor Vt\rfloor}(t)\,\widetilde{h}_{\lfloor Vt\rfloor}(0)\right) =$$
$$= [(V - C)\cdot\mathbf{1}\{C \le V\} + C\cdot\mathbf{1}\{C \le 0\}]\cdot\mathbf{Var}(\omega) = (V - C^+)^+\cdot\mathbf{Var}(\omega).$$

□

Now we divide equation (49) by t and take the limit $t \to \infty$. We use the result of lemma 17.2 to obtain

$$\lim_{t\to\infty}\frac{\mathbf{Var}(J^{(V)}(t))}{t} = \lim_{t\to\infty}\frac{\mathbf{Var}(J(t))}{t} + [2(V - C^+)^+ - V]\cdot\mathbf{Var}(\omega) \tag{52}$$

for $V > 0$.

For $V < 0$, we proceed as we did above with $J^{(V)}$ for positive V's. The only important difference is using φ_α instead of $-\psi_\alpha$ in the proof of lemma 17.1. The result of a similar lemma for $V < 0$ is

$$\mathbf{E}(\widetilde{h}_{\lceil Vt\rceil}(t)\,\widetilde{h}_{\lceil Vt\rceil}(0)) =$$
$$= -\sum_{n=\lceil Vt\rceil+1}^{\infty}(\lceil Vt\rceil - n)\mathbf{E}(\widetilde{\omega}_n(t)\,\widetilde{\omega}_0(0)) - \sum_{n=1}^{\infty}n\mathbf{E}(\widetilde{\omega}_n(t)\,\widetilde{\omega}_0(0)).$$

Therefore, lemma 16.4 is applicable in a similar way as in lemma 17.2 above. The result of this application is

$$\lim_{t\to\infty}\frac{1}{t}\,\mathbf{E}\left(\widetilde{h}_{\lceil Vt\rceil}(t)\,\widetilde{h}_{\lceil Vt\rceil}(0)\right) = (V + C^-)^-\cdot\mathbf{Var}(\omega).$$

Computing $\mathbf{Var}(J^{(V)})$ for $V < 0$ as we did in (49) leads then to

$$\lim_{t\to\infty}\frac{\mathbf{Var}(J^{(V)}(t))}{t} = \lim_{t\to\infty}\frac{\mathbf{Var}(J(t))}{t} + [2(V + C^-)^- + V]\cdot\mathbf{Var}(\omega). \tag{53}$$

Now, assuming condition 14.2, we can prove (29) by the result of theorem 16.5.

Proof of theorem 14.3. All time arguments of our variables for this proof are thought to be zero without mentioning it. By (33),

$$\begin{aligned}
&\mathbf{E}\left(r^* \cdot (\widetilde{\omega}_0 - \widetilde{\omega}_1)\right) = \mathbf{E}\left(r^* \cdot (\omega_0 - \omega_1)\right) = \\
&\qquad = \mathbf{E}\left(r(\omega_0 + 1,\, \omega_1 - 1) \cdot \frac{\mu(\omega_0 + 1)\,\mu(\omega_1 - 1)}{\mu(\omega_0)\,\mu(\omega_1)} \cdot (\omega_0 - \omega_1)\right) = \\
&\mathbf{E}(r\cdot(\omega_0-\omega_1)) - 2\mathbf{E}(r) = \mathbf{E}(r\cdot(\widetilde{\omega}_0-\widetilde{\omega}_1)) - 2\mathbf{E}(r) = -\mathbf{E}(r^*\cdot(\widetilde{\omega}_0-\widetilde{\omega}_1)) - 2\mathbf{E}(r),
\end{aligned}$$

we used (32) in the last step. Hence we obtain

$$\mathbf{E}\left(r^* \cdot (\widetilde{\omega}_0 - \widetilde{\omega}_1)\right) = -\mathbf{E}(r).$$

We have two formulas for the variance $\mathbf{Var}(J(t))$ by theorem 16.5, which are used together with (52) and (53) to obtain

$$\begin{aligned}
\lim_{t\to\infty} \frac{\mathbf{Var}(J^{(V)}(t))}{t} &= \mathbf{E}(r) - 2\,\mathbf{E}(r^* \cdot \widetilde{\omega}_1) + (|V - C| + C) \cdot \mathbf{Var}(\omega) = \\
&= \mathbf{E}(r) + 2\,\mathbf{E}(r^* \cdot \widetilde{\omega}_0) + (|V - C| - C) \cdot \mathbf{Var}(\omega).
\end{aligned}$$

We take the average of these two formulas:

$$\begin{aligned}
\lim_{t\to\infty} \frac{\mathbf{Var}(J^{(V)}(t))}{t} = \mathbf{E}(r) + \mathbf{E}\left(r^* \cdot (\widetilde{\omega}_0 - \widetilde{\omega}_1)\right) + |V - C| \cdot \mathbf{Var}(\omega) = \\
= |V - C| \cdot \mathbf{Var}(\omega).
\end{aligned}$$

□

Now it is easy to prove central limit theorem for $J^{(V)}$.

Proof of theorem 14.4. We introduce the drifted form of $J^{(C)}$ by $i \in \mathbb{Z}$:

$$J_i^{(C)}(t) := h_{\lfloor Ct \rfloor + i}(t) - h_i(0)$$

for $C \geq 0$, and

$$J_i^{(C)}(t) := h_{\lceil Ct \rceil + i}(t) - h_i(0)$$

for $C < 0$. Due to translation-invariance, the distribution of this quantity is independent of i. Hence by (29), for $C \geq 0$ and $V \geq 0$, the variance of

$$J^{(C)}_{\lfloor Vt \rfloor - \lfloor Ct \rfloor}(t) = h_{\lfloor Vt \rfloor}(t) - h_{\lfloor Vt \rfloor - \lfloor Ct \rfloor}(0) = J^{(V)}(t) - h_{\lfloor Vt \rfloor - \lfloor Ct \rfloor}(0)$$

is $\mathfrak{o}(t)$ as $t \to \infty$. Thus it follows that we only need central limit theorem for $h_{\lfloor Vt \rfloor - \lfloor Ct \rfloor}(0)$, which is, by (48) and by $h_0(0) = 0$, the sum of $|\lfloor Vt \rfloor - \lfloor Ct \rfloor|$ number of iid. $\omega_i(0)$ variables with finite moments. Hence the theorem follows for $V \geq 0,\ C \geq 0$. For $V \geq 0,\ C < 0$,

$$J^{(C)}_{\lfloor Vt \rfloor - \lceil Ct \rceil}(t) = h_{\lfloor Vt \rfloor}(t) - h_{\lfloor Vt \rfloor - \lceil Ct \rceil}(0) = J^{(V)}(t) - h_{\lfloor Vt \rfloor - \lceil Ct \rceil}(0),$$

here we have (and we only need) central limit theorem for the sum of $|\lfloor Vt \rfloor - \lceil Ct \rceil|$ number of iid. $\omega_i(0)$ variables, which proves the theorem. Similar argument works for $V < 0$ also. □

18 The motion of the defect tracer

With the help of another type of coupling, with any $n \in \mathbb{Z}^+$, we prove L^n-convergence for $Q(t)/t$ of BL and totally asymmetric ZR models in this section. This coupling only works under convexity condition 14.5, which we assume for the rest of the paper. The idea of the proof is the following: we fix our $(\underline{\omega}, Q)$ pair and compare it with another model $\underline{\zeta}$. The difference between $\underline{\omega}$ and $\underline{\zeta}$ is realized by second class particles. The current of these particles satisfies law of large numbers by separate ergodicity of $\underline{\omega}$ and $\underline{\zeta}$, and we compare their motion to our defect tracer Q placed on $\underline{\omega}$. The main difficulty is finding the way to couple the defect tracer to the second class particles. As shown later, this coupling can not be made directly; we need to introduce a new process called the *S-particles*, a random process defined in terms of the second class particles.

We set $\theta_1 < \theta_2$, then there exists a two dimensional measure μ on $\mathbb{Z}\times\mathbb{Z}$, which has marginals μ_{θ_1} and μ_{θ_2}, respectively, and for which $\mu(x, y) = 0$ if $x > y$. We fix two configurations $\underline{\eta}$ and $\underline{\zeta}$ of our model, distributed initially according to a product measure with marginals $\mathbf{P}(\eta_i(0) = x, \zeta_i(0) = y) = \mu(x, y)$. Therefore, $\underline{\eta}$ is itself in distribution $\underline{\mu}_{\theta_1}$, $\underline{\zeta}$ is in distribution $\underline{\mu}_{\theta_2}$, and $\eta_i(0) \le \zeta_i(0)$ for each site i is satisfied. According to the basic coupling described in subsection 16.1, $\eta_i(t) \le \zeta_i(t)$ holds for all later times t, and we have a positive density of second class particles between these two models. The number of these particles at site i is $\zeta_i - \eta_i \ge 0$. Hence they are initially distributed according to a product measure but, at later times, only the marginal distributions of $\underline{\eta}$ or of $\underline{\zeta}$ will possess a product structure. Note that the joint distribution of the processes is translation invariant.

18.1 The Palm distribution

For further applications, we want to select "a typical second class particle". We do it as follows. We introduce the drifted form of the models: for $k \in \mathbb{Z}$,

$$(\boldsymbol{\tau}_k\, \underline{\eta})_i := \eta_{i+k}, \qquad (\boldsymbol{\tau}_k\, \underline{\zeta})_i := \zeta_{i+k}.$$

If $N \in \mathbb{Z}^+$ is large enough, we choose uniformly one second class particle among the particles present at sites $-N \le i \le N$. We determine the distribution of the values of a function g depending on $(\underline{\eta}, \underline{\zeta})$, as seen from the position k of the randomly selected second class particle. For N large enough, the total number

$$\sum_{j=-N}^{N} (\zeta_i - \eta_i)$$

of second class particles at sites $-N \le i \le N$ is positive, and then

$$\begin{aligned}
\mathbf{E}^{(N)}\left(g(\boldsymbol{\tau}_k\, \underline{\eta}, \boldsymbol{\tau}_k\, \underline{\zeta})\right) &= \mathbf{E}\left[\mathbf{E}\left(g(\boldsymbol{\tau}_k\, \underline{\eta}, \boldsymbol{\tau}_k\, \underline{\zeta}) \,\middle|\, \underline{\eta}, \underline{\zeta}\right)\right] = \\
&= \mathbf{E}\left(\sum_{i=-N}^{N} g(\boldsymbol{\tau}_i\, \underline{\eta}, \boldsymbol{\tau}_i\, \underline{\zeta}) \cdot \frac{\zeta_i - \eta_i}{\sum_{j=-N}^{N}(\zeta_j - \eta_j)}\right) = \\
&= \mathbf{E}\left(\frac{\frac{1}{2N+1}\sum_{i=-N}^{N} g(\boldsymbol{\tau}_i\, \underline{\eta}, \boldsymbol{\tau}_i\, \underline{\zeta}) \cdot (\zeta_i - \eta_i)}{\frac{1}{2N+1}\sum_{j=-N}^{N}(\zeta_j - \eta_j)}\right).
\end{aligned}$$

For bounded g, the random variable we see in the last line of the display is bounded, and is the quotient of two random variables, both having a.s. limit as $N \to \infty$ by translation invariance and ergodicity of translations. Hence our expression converges due to dominated convergence, and have the limit

$$(54) \qquad \widehat{\mathbf{E}}\left(g(\underline{\eta},\,\underline{\zeta})\right) := \lim_{N\to\infty} \mathbf{E}^{(N)}\left(g(\boldsymbol{\tau}_k\,\underline{\eta},\,\boldsymbol{\tau}_k\,\underline{\zeta})\right) = \frac{\mathbf{E}\left(g(\underline{\eta},\,\underline{\zeta})\cdot(\zeta_0-\eta_0)\right)}{\mathbf{E}\left(\zeta_0-\eta_0\right)}.$$

The distribution $\underline{\widehat{\mu}}$ defined by (54) is called the Palm distribution of the process. The Palm measure can be extended to non-negative functions g, see [19]. Note that $\widehat{\mathbf{P}}(\zeta_0(0)-\eta_0(0)>0)=1$ according to this measure, i.e. we necessarily have at least one second class particle at the origin, if looking the process "as seen from a typical second class particle".

By initial product distribution of $(\underline{\eta},\,\underline{\zeta})$, $\underline{\widehat{\mu}}$ is initially also a product measure, consisting of the original marginals μ for sites $i \neq 0$, and of marginal

$$(55) \qquad \widehat{\mu}(x,\,y) := \frac{\mu(x,\,y)\cdot(y-x)}{\mathbf{E}(\zeta_0-\eta_0)}$$

for site $i=0$. For later use, we introduce the pair $(\underline{\eta}'(t),\,\underline{\zeta}'(t))$ started from this initial product distribution $\underline{\widehat{\mu}}$.

18.2 Random walk on the second class particles

We label the second class particles between $\underline{\eta}$ and $\underline{\zeta}$ in space-order. Let $U^{(m)}(t)$ denote the position of the m-th second class particle at time t. Initially, we look for the first site possessing second class particle on the right side of the origin. We choose one of the particles at this site, giving it label $m=0$:

$$U^{(0)}(0) := \min\{i \geq 0 \,:\, \zeta_i > \eta_i\}.$$

We label the particles at $t=0$ in such a way that $U^{(m)}(0) \leq U^{(m+1)}(0)$ $(\forall m \in \mathbb{Z})$ (the order of particles at the same site is not important). We define $J_i^{(2^{\mathrm{nd}})}(t)$ to be the algebraic number of second class particles passing the column between i and $i+1$ in the time interval $[0,\,t]$. This quantity is determined by the evolution of the processes $\underline{\eta}$ and $\underline{\zeta}$. For $t=0$, we define

$$(56) \qquad m_i(0) := \max\{m \,:\, U^{(m)}(0) \leq i\},$$

while for $t>0$,

$$m_i(t) := m_i(0) - J_i^{(2^{\mathrm{nd}})}(t).$$

We label the particles at later times such that (56) holds at any time t as well. This method assures $U^{(m)}(t) \leq U^{(m+1)}(t)$ for all time t. The particles labeled from $m_{i-1}+1$ up to m_i, exactly $\zeta_i-\eta_i = m_i - m_{i-1}$ of them are at site i. (At sites i for which $m_i = m_{i-1}$, there is no second class particle).

We have defined so far the coupled pair $\underline{\eta}$ and $\underline{\zeta}$ with the $U^{(m)}(t)$ process of the second class particles indexed in space order at any time t. The latter will serve us as a background environment for a new random process, $\left(s^{(n)}(t)\right)_{n\in\mathbb{Z}}$. Initially, we put $s^{(n)}(0) := n$ for each n. Assume that just before a second class particle jumps from a site i at a time t, $s^{(n)}(t) \in \{m_{i-1}(t)+1,\, m_{i-1}(t)+$

$2, \ldots, m_i(t)\}$, which means $U^{(s^{(n)})}(t) = i$ just before the jump. Then by the time $t+0$ of this jump, $s^{(n)}(t+0) := \Pi_i(s^{(n)}(t))$, where Π_i is a random uniform permutation on the integer set $\{m_{i-1}(t)+1,\, m_{i-1}(t)+2, \ldots, m_i(t)\}$.

We can represent this new process as follows. Initially, we put an extra particle, which we call S-particle, on each second class particle. The S-particles are labeled by n, and initially we put the n-th S-particle on the n-th second particle. $s^{(n)}(t)$ stands for the index of the second class particle carrying the n-th S-particle. Whenever a jump of second class particle happens from site i, we permute uniformly and randomly the S-particles present at site i just before the jump. According to the labeling of second class particles, one jumping to the right (or to the left, respectively) from site i has index $m_i(t)$ (or $m_{i-1}(t)+1$, respectively) and is carrying exactly the n-th S-particle, for which $s^{(n)}(t+0) = \Pi_i(s^{(n)}(t)) = m_i(t)$ (or $m_{i-1}(t)+1$, respectively). Hence a uniformly and randomly chosen S-particle is taken from the site i with the jumping second class particle.

For simplicity, we define $s(t) := s^{(0)}(t)$ and $S(t) := U^{(s(t))}$, and by simply saying the S-particle, we mean the zeroth S-particle at site $S(t)$. Then $S(t)$ represents a random walk moving always together with a second class particle, but having always probability $1/(m_i - m_{i-1}) = 1/(\zeta_i - \eta_i)$ of jumping together with a second class particle jumping from the site i. As can be derived from table 2, the rate for a second class particle to jump to the left (or to the right) is $f(-\eta_i) - f(-\zeta_i)$ (or $f(\zeta_i) - f(\eta_i)$, respectively). Hence the rate for the S-particle to jump to the left (or to the right) together with the jumping second class particle from site $i = S(t)$ is

$$\frac{f(-\eta_i) - f(-\zeta_i)}{\zeta_i - \eta_i} \quad \left(\text{or } \frac{f(\zeta_i) - f(\eta_i)}{\zeta_i - \eta_i}, \text{ respectively}\right). \tag{57}$$

Recall that $S(0) = U^{(s(0))}(0) = U^{(0)}(0)$ is the first site on the right-hand side of the origin initially with second class particles. We introduce the notation $(\underline{\eta}''(t),\, \underline{\zeta}''(t)) := (\boldsymbol{\tau}_{S(0)}\underline{\eta}(t),\, \boldsymbol{\tau}_{S(0)}\underline{\zeta}(t))$, which is the $(\underline{\eta}(t),\, \underline{\zeta}(t))$ process shifted to this initial position $S(0)$ of the S-particle. We also introduce its S''-particle: $S''(t) := S(t) - S(0)$. Hence the initial distribution of $(\underline{\eta}''(0),\, \underline{\zeta}''(0))$ is modified according to this random shifting-procedure; we show the details in the proof of the next lemma.

Using the Palm measures, we show that the expected rates for S to jump are bounded in time.

Lemma 18.1. *Let $n \in \mathbb{Z}^+,\ k \in \mathbb{Z}$, and*

$$c_i(t) := f(\zeta_i(t)) - f(\eta_i(t)) + f(-\eta_i(t)) - f(-\zeta_i(t)) \tag{58}$$

the rate for any second class particle to jump from site i. Then

$$\mathbf{E}\left([c_{S(t)}(t)]^n \cdot [\zeta_{S(t)}(t) - \eta_{S(t)}(t)]^k\right) \le K(n,\, k)$$

uniformly in time.

Proof. First we consider the pair $(\underline{\eta}'(0),\, \underline{\zeta}'(0))$ defined following (55). As described there, this is in fact the pair $(\underline{\eta}(0),\, \underline{\zeta}(0))$ at time $t = 0$, as seen from "a typical second class particle", or equivalently, as seen from "a typical S-particle". In this pair, we have at least one second class particle at the origin, which we call S'. We let our process $(\underline{\eta}', \underline{\zeta}')$ evolve, and we follow this "typical" S'-particle. Started from the Palm-distribution, this tagged S'-particle keeps on "being typical" (see [19]), i.e. for a function g of the process as seen by S',

$$\mathbf{E}\left(g(\boldsymbol{\tau}_{S'(t)}\underline{\eta}'(t),\, \boldsymbol{\tau}_{S'(t)}\underline{\zeta}'(t))\right) = \widehat{\mathbf{E}}\left(g(\underline{\eta}(t),\, \underline{\zeta}(t))\right)$$

with definition (54).

Now we first show the desired result for the S'-particle of $(\underline{\eta}', \underline{\zeta}')$ instead of the S-particle of $(\underline{\eta}, \underline{\zeta})$. In the previous display, we put the function

$$g(\underline{\eta}(t),\, \underline{\zeta}(t)) := [c_0(t)]^n \cdot [\zeta_0(t) - \eta_0(t)]^k,$$

and we denote by k^+ the positive part of k. We know that $\zeta_0(t) - \eta_0(t) \geq 1$ holds $\widehat{\mathbf{P}}$-a.s., hence

$$\mathbf{E}\left([c_{S'(t)}(t)]^n \cdot [\zeta'_{S'(t)}(t) - \eta'_{S'(t)}(t)]^k\right) = \widehat{\mathbf{E}}\left([c_0(t)]^n \cdot [\zeta_0(t) - \eta_0(t)]^k\right) \leq$$

$$\leq \widehat{\mathbf{E}}\left([c_0(t)]^n \cdot [\zeta_0(t) - \eta_0(t)]^{k^+}\right) = \frac{\mathbf{E}\left([c_0(t)]^n \cdot [\zeta_0(t) - \eta_0(t)]^{k^+ + 1}\right)}{\mathbf{E}(\zeta_0(t) - \eta_0(t))}$$

by (54). The function $c_0(t)$ consists of sums of $f(\pm\eta_0(t))$ and $f(\pm\zeta_0(t))$, hence the numerator is an $n + k^+ + 1$-order polinom of these functions and of $\zeta_0(t)$, $\eta_0(t)$. These are all random variables with all moments finite. Therefore, using Cauchy's inequality, the numerator can be bounded from above by products of moments of either $f(\eta_0(t))$ or $f(\zeta_0(t))$ or $\eta_0(t)$, or $\zeta_0(t)$. The models $\underline{\eta}$ and $\underline{\zeta}$ are both separately in their stationary distributions, hence these bounds are constants in time. The denominator is a positive number due to $\theta_2 > \theta_1$ and strict monotonicity of $\mathbf{E}_\theta(z)$ in θ. We see that we found a bound, uniform in time for the function g of $(\underline{\eta}', \underline{\zeta}')$ as seen from S'.

We need to find similar bound for a function g of the original pair $(\underline{\eta}, \underline{\zeta})$, as seen from S. This is equivalent to finding a bound for g of $(\underline{\eta}'', \underline{\zeta}'')$ defined above, as seen from S'' of this pair. Let us consider first the initial distribution of $(\underline{\eta}'', \underline{\zeta}'')$, which we shall call $\underline{\mu}''$. By definition, it is clear that this distribution is the product of the original marginals μ for sites $i > 0$. Fix a K positive integer and two vectors $\underline{x}, \underline{y} \in \mathbb{Z}^{\mathbb{Z}}$. For simplicity we introduce the notations

$$\underline{\eta}_{[a,\,b]} := (\eta_a,\, \ldots,\, \eta_b) \qquad \text{and} \qquad \underline{\zeta}_{[a,\,b]} := (\zeta_a,\, \ldots,\, \zeta_b)\,,$$
$$\underline{x}_{[a,\,b]} := (x_a,\, \ldots,\, x_b) \qquad \text{and} \qquad \underline{y}_{[a,\,b]} := (y_a,\, \ldots,\, y_b)$$

and, where not written, we consider our models at time zero. We break the events according to the initial position $S(0)$ of the S-particle in the original pair

$(\underline{\eta}, \underline{\zeta})$:

$$\mathbf{P}\left(\underline{\eta}''_{[-K,\,0]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}''_{[-K,\,0]} = \underline{y}_{[-K,\,0]}\right) =$$

$$= \mathbf{P}\left(\underline{\eta}_{[S(0)-K,\,S(0)]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}_{[S(0)-K,\,S(0)]} = \underline{y}_{[-K,\,0]}\right) =$$

$$= \sum_{n=0}^{K} \mathbf{P}\left(\underline{\eta}_{[n-K,\,n]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}_{[n-K,\,n]} = \underline{y}_{[-K,\,0]},\ S(0) = n\right) +$$

$$+ \sum_{n=K+1}^{\infty} \mathbf{P}\left(\underline{\eta}_{[n-K,\,n]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}_{[n-K,\,n]} = \underline{y}_{[-K,\,0]},\ S(0) = n\right) =$$

$$= \sum_{n=0}^{K} \mathbf{P}\left(\underline{\eta}_{[n-K,\,n]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}_{[n-K,\,n]} = \underline{y}_{[-K,\,0]}\right) \cdot E_n(\underline{x},\,\underline{y}) +$$

$$\sum_{n=K+1}^{\infty} \mathbf{P}\left(\underline{\eta}_{[n-K,\,n]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}_{[n-K,\,n]} = \underline{y}_{[-K,\,0]}\right) \cdot E_K(\underline{x},\,\underline{y}) \cdot \mathbf{P}\{F_{n-K}\},$$

where the function E_n of $\underline{x}$ and $\underline{y}$ is an indicator defined by

$$E_n(\underline{x},\,\underline{y}) := \mathbf{1}\left\{x_{-n} = y_{-n},\, x_{-n+1} = y_{-n+1},\, \ldots,\, x_{-1} = y_{-1},\, x_0 < y_0\right\},$$

and the event F_{n-K} is

$$F_{n-K} := \{\eta_0 = \zeta_0,\, \eta_1 = \zeta_1,\, \ldots,\, \eta_{n-K-1} = \zeta_{n-K-1}\}.$$

The last equality follows from the product structure of $\underline{\mu}$ and from the fact that $S(0)$ is the first site to the right of the origin where $\eta_i \neq \zeta_i$. Continuing the computation results in

$$\mathbf{P}\left(\underline{\eta}''_{[-K,\,0]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}''_{[-K,\,0]} = \underline{y}_{[-K,\,0]}\right) =$$

$$= \prod_{i=-K}^{0} \mu(x_i,\, y_i) \cdot \left[\sum_{n=0}^{K} E_n(\underline{x},\,\underline{y}) + E_K(\underline{x},\,\underline{y}) \cdot \sum_{n=K+1}^{\infty} \mu\left\{\eta_0 = \zeta_0\right\}^{n-K}\right]$$

$$= \prod_{i=-K}^{0} \mu(x_i,\, y_i) \cdot \left[\sum_{n=0}^{K} E_n(\underline{x},\,\underline{y}) + E_K(\underline{x},\,\underline{y}) \cdot \frac{\mu\left\{\eta_0 = \zeta_0\right\}}{\mu\left\{\eta_0 < \zeta_0\right\}}\right]$$

using translation-invariance.

For later purposes, we are interested in the Radon-Nikodym derivative of the distribution $\underline{\mu}''$ of $(\underline{\eta}'',\, \underline{\zeta}'')$ w.r.t. the Palm distribution $\widehat{\underline{\mu}}$ of $(\underline{\eta}',\, \underline{\zeta}')$. Since both have product of marginals μ for sites $i > 0$, we only have to deal with the left part of the origin. Passing to the limit $K \to \infty$, we have

$$\frac{\mathrm{d}\underline{\mu}''}{\mathrm{d}\underline{\widehat{\mu}}}(\underline{x},\, \underline{y}) = \lim_{K\to\infty} \frac{\mathbf{P}\left(\underline{\eta}''_{[-K,\,0]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}''_{[-K,\,0]} = \underline{y}_{[-K,\,0]}\right)}{\mathbf{P}\left(\underline{\eta}'_{[-K,\,0]} = \underline{x}_{[-K,\,0]},\ \underline{\zeta}'_{[-K,\,0]} = \underline{y}_{[-K,\,0]}\right)} =$$

$$= \lim_{K\to\infty} \frac{\prod_{i=-K}^{0} \mu(x_i,\, y_i)}{\prod_{i=-K}^{-1} \mu(x_i,\, y_i)\widehat{\mu}(x_0,\, y_0)} \cdot \left[\sum_{n=0}^{K} E_n(\underline{x},\, \underline{y}) + E_K(\underline{x},\, \underline{y}) \cdot \frac{\mu\,\{\eta_0 = \zeta_0\}}{\mu\,\{\eta_0 < \zeta_0\}}\right] =$$

$$= \frac{\mu(x_0,\, y_0)}{\widehat{\mu}(x_0,\, y_0)} \cdot \left[\sum_{n=0}^{\infty} E_n(\underline{x},\, \underline{y}) + \lim_{K\to\infty} E_K(\underline{x},\, \underline{y}) \cdot \frac{\mu\,\{\eta_0 = \zeta_0\}}{\mu\,\{\eta_0 < \zeta_0\}}\right] =$$

$$= \frac{\mu(x_0,\, y_0)}{\widehat{\mu}(x_0,\, y_0)} \cdot \sum_{n=0}^{\infty} E_n(\underline{x},\, \underline{y})$$

for $\underline{\widehat{\mu}}$-almost all configurations $(\underline{x},\, \underline{y})$. Note that the sum on the right-hand side gives exactly the distance between the origin and the first position i to the left of the origin with $x_i \neq y_i$. Hence this sum is finite for $\underline{\widehat{\mu}}$-almost all configurations $(\underline{x},\, \underline{y})$.

In view of this result, we can now obtain our estimates. The main idea here is that the pairs $(\underline{\eta}',\, \underline{\zeta}')$ and $(\underline{\eta}'',\, \underline{\zeta}'')$ only differ in their initial distribution, hence their behavior conditioned on the same initial configuration agree. This is used for obtaining the third expression, and Cauchy's inequality is used for the fourth one below.

$$\mathbf{E}\left([c_{S''(t)}(t)]^n \cdot [\zeta_{S''(t)}(t) - \eta_{S''(t)}(t)]^k\right) =$$

$$= \int\limits_{\widetilde{\Omega}\cap\{x_0<y_0\}} \mathbf{E}\left([c_{S''(t)}(t)]^n \cdot [\zeta_{S''(t)}(t) - \eta_{S''(t)}(t)]^k \,|\, \underline{\eta}''(0) = \underline{x},\, \underline{\zeta}''(0) = \underline{y}\right) \times$$

$$\times\, \mathrm{d}\underline{\mu}''(\underline{x},\, \underline{y}) =$$

$$= \int\limits_{\widetilde{\Omega}\cap\{x_0<y_0\}} \mathbf{E}\left([c_{S'(t)}(t)]^n \cdot [\zeta_{S'(t)}(t) - \eta_{S'(t)}(t)]^k \,|\, \underline{\eta}'(0) = \underline{x},\, \underline{\zeta}'(0) = \underline{y}\right) \times$$

$$\times\, \frac{\mu(x_0,\, y_0)}{\widehat{\mu}(x_0,\, y_0)} \cdot \sum_{n=0}^{\infty} E_n(\underline{x},\, \underline{y})\, \mathrm{d}\underline{\widehat{\mu}}(\underline{x},\, \underline{y}) \leq$$

$$\leq \Bigg[\int\limits_{\widetilde{\Omega}\cap\{x_0<y_0\}} \left[\mathbf{E}\left([c_{S'(t)}(t)]^n \cdot [\zeta_{S'(t)}(t) - \eta_{S'(t)}(t)]^k \,|\, \underline{\eta}'(0) = \underline{x},\, \underline{\zeta}'(0) = \underline{y}\right)\right]^2 \times$$

$$\times\, \mathrm{d}\underline{\widehat{\mu}}(\underline{x},\, \underline{y})\Bigg]^{\frac{1}{2}} \cdot \Bigg[\int\limits_{\widetilde{\Omega}\cap\{x_0<y_0\}} \left[\frac{\mu(x_0,\, y_0)}{\widehat{\mu}(x_0,\, y_0)} \cdot \sum_{n=0}^{\infty} E_n(\underline{x},\, \underline{y})\right]^2 \cdot \mathrm{d}\underline{\widehat{\mu}}(\underline{x},\, \underline{y})\Bigg]^{\frac{1}{2}} \leq$$

$$\leq \Bigg[\int\limits_{\widetilde{\Omega}\cap\{x_0<y_0\}} \mathbf{E}\left([c_{S'(t)}(t)]^{2n} \cdot [\zeta_{S'(t)}(t) - \eta_{S'(t)}(t)]^{2k} \,|\, \underline{\eta}'(0) = \underline{x},\, \underline{\zeta}'(0) = \underline{y}\right) \times$$

$$\times\, \mathrm{d}\underline{\widehat{\mu}}(\underline{x},\, \underline{y})\Bigg]^{\frac{1}{2}} \cdot \Bigg[\int\limits_{\widetilde{\Omega}\cap\{x_0<y_0\}} \frac{\mu(x_0,\, y_0)}{\widehat{\mu}(x_0,\, y_0)} \cdot \left[\sum_{n=0}^{\infty} E_n(\underline{x},\, \underline{y})\right]^2 \cdot \mathrm{d}\underline{\mu}(\underline{x},\, \underline{y})\Bigg]^{\frac{1}{2}} =$$

$$= \left[\mathbf{E}\left([c_{S'(t)}(t)]^{2n} \cdot [\zeta_{S'(t)}(t) - \eta_{S'(t)}(t)]^{2k}\right)\right]^{\frac{1}{2}} \times$$

$$\times \Bigg[\int\limits_{\widetilde{\Omega}\cap\{x_0<y_0\}} \frac{\mathbf{E}(\zeta_0 - \eta_0)}{y_0 - x_0} \cdot \left[\sum_{n=0}^{\infty} E_n(\underline{x},\, \underline{y})\right]^2 \cdot \mathrm{d}\underline{\mu}(\underline{x},\, \underline{y})\Bigg]^{\frac{1}{2}}$$

by (55). The first factor of the last display is finite by the first part of the proof. Using the definition of the indicator E_n, the second factor can be bounded from above by

$$[\mathbf{E}(\zeta_0 - \eta_0)]^{\frac{1}{2}} \cdot \Bigg[\int\limits_{\widetilde{\Omega}\cap\{x_0<y_0\}} \left[\sum_{n=0}^{\infty} (2n+1) \cdot E_n(\underline{x},\, \underline{y})\right] \cdot \mathrm{d}\underline{\mu}(\underline{x},\, \underline{y})\Bigg]^{\frac{1}{2}} =$$

$$= [\mathbf{E}(\zeta_0 - \eta_0)]^{\frac{1}{2}} \cdot \left[\sum_{n=0}^{\infty} (2n+1) \cdot \mu\{\eta_0 = \zeta_0\}^n \cdot \mu\{\eta_0 < \zeta_0\}\right]^{\frac{1}{2}}$$

using the product property of $\underline{\mu}$, and is again finite.

□

Using the rates for the S-particle to move, we can prove the following bound for the moments of $S(t)$:

Proposition 18.2. *For $n \in \mathbb{Z}^+$,*

$$\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right) < K(n) < \infty \tag{59}$$

for all large t.

Proof. For this proof, we denote the jumping rates (57) for the S-particle by $r^{S\,\mathrm{left}}$ and $r^{S\,\mathrm{right}}$, respectively. For $t > 0$, we consider the derivative of the quantity above, using these rates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right) = -\frac{n}{t^{n+1}}\mathbf{E}\left(|S(t)|^n\right)+\frac{1}{t^n}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\left(|S(t)|^n\right) = -\frac{n}{t^{n+1}}\mathbf{E}\left(|S(t)|^n\right)+$$
$$+\frac{1}{t^n}\mathbf{E}\left[r^{S\,\mathrm{left}}\cdot\left(|S(t)-1|^n-|S(t)|^n\right)+r^{S\,\mathrm{right}}(\cdot|S(t)+1|^n-|S(t)|^n)\right].$$

For $|S(t)| \geq 1$, we can bound our expressions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right) \leq -\frac{n}{t^{n+1}}\mathbf{E}\left(|S(t)|^n\right)+\frac{2^n}{t^n}\mathbf{E}\left((r^{S\,\mathrm{right}}+r^{S\,\mathrm{left}})\cdot|S(t)|^{n-1}\right).$$

We continue by using Hölder's inequality on the right-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right) \leq \tag{60}$$
$$\leq -\frac{n}{t}\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right)+\frac{2^n}{t}\left\{\mathbf{E}\left[(r^{S\,\mathrm{right}}+r^{S\,\mathrm{left}})^n\right]\right\}^{\frac{1}{n}}\cdot\left\{\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right)\right\}^{\frac{n-1}{n}}.$$

Recall that

$$r^{S\,\mathrm{right}}(t)+r^{S\,\mathrm{left}}(t) = c_{S(t)}(t)\cdot[\zeta_{S(t)}(t)-\eta_{S(t)}(t)]^{-1},$$

hence lemma 18.1 is applicable with $k = -n$ to show that

$$\mathbf{E}\left[(r^{S\,\mathrm{right}}(t)+r^{S\,\mathrm{left}}(t))^n\right]$$

is bounded in time. Therefore, (60) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right) \leq -\frac{n}{t}\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right)+\frac{K'(n)}{t}\cdot\left\{\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\right)\right\}^{\frac{n-1}{n}}$$

with some positive constant $K'(n)$. This means that $\mathbf{E}(|S(t)|^n/t^n)$ is bounded from above by a solution of the differential equation

$$\dot{y}(t) = -\frac{n}{t}\,y(t)+\frac{K'(n)}{t}\cdot y(t)^{\frac{n-1}{n}}.$$

Observe that the right-hand side is negative whenever

$$y(t) > \left(\frac{K'(n)}{n}\right)^n,$$

hence assuming $y(t_0) < \infty$ for some $t_0 > 0$, $y(t)$ is bounded (for all $t > t_0$), which gives the proof. □

Now we show law of large numbers for $s(t)$, and then we can show law of large numbers for $S(t)$. For what follows, $\mathbf{E}'$ stands for the expectation values according to the distribution of $\underline{\eta}$, $\underline{\zeta}$, $\{U^{(m)}\}_{m\in\mathbb{Z}}$, i.e. our background process which determine $m_i(t)$, also. Let $\mathcal{F}(t)$ denote the σ-field containing all information about these quantities at time t. Then $\mathcal{F}(t)$ contains all randomness except for the random permutations on $\big(s^{(n)}\big)_{n\in\mathbb{Z}}$. With (58), we also introduce the notations

$$\begin{aligned}
&C_i(t): \quad = (m_i(t) - m_{i-1}(t))^2 \cdot c_i(t),\\
&p\,(y,\,t) := \mathbf{P}(s(t) = y \mid \mathcal{F}(t)),\\
&\text{and}\\
&A_i(t): \quad = \max_{m_{i-1}(t)<y\le m_i(t)} p\,(y,\,t) - \min_{m_{i-1}(t)<y\le m_i(t)} p\,(y,\,t)
\end{aligned} \tag{61}$$

if $m_i(t) - m_{i-1}(t) > 1$, and $A_i(t) := 0$ otherwise.

Lemma 18.3.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(|s(t)|) \le \sqrt{\mathbf{E}' \sum_{i=-\infty}^{\infty} A_i(t)} \cdot \sqrt{\mathbf{E}' \sum_{j=-\infty}^{\infty} A_j(t)\, C_j^2(t)}. \tag{62}$$

Proof. We use convention that the empty sum equals zero.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(|s(t)|) &= \lim_{\varepsilon\to 0} \frac{\mathbf{E}(|s(t+\varepsilon)|) - \mathbf{E}(|s(t)|)}{\varepsilon} =\\
&= \lim_{\varepsilon\to 0} \sum_{z=-\infty}^{\infty} \frac{\mathbf{P}(s(t+\varepsilon) = z)\cdot |z| - \mathbf{P}(s(t) = z)\cdot |z|}{\varepsilon} =\\
= \lim_{\varepsilon\to 0} \mathbf{E}' \sum_{i=-\infty}^{\infty} &\sum_{z=m_{i-1}(t)+1}^{m_i(t)} \frac{\mathbf{P}(s(t+\varepsilon) = z \mid \mathcal{F}(t))\cdot |z| - \mathbf{P}(s(t) = z \mid \mathcal{F}(t))\cdot |z|}{\varepsilon}.
\end{aligned}$$

We know that uniform random permutation on the indices present at site S happens at each jump of second class particles from i at time t. The basic idea is that this permutation makes the probabilities $p\,(y,\,t) = \mathbf{P}(s(t) = y \mid \mathcal{F}(t))$ equalized between $y = m_{i-1}(t)+1 \ldots m_i(t)$. This jump happens with rate $c_i(t)$ defined in (58), hence for a site i with at least one second class particle and for $m_{i-1}(t)+1 \le z \le m_i(t)$,

$$\begin{aligned}
\mathbf{P}(s(t+\varepsilon) = z \mid \mathcal{F}(t)) = (1 - \varepsilon\, c_i(t))\, \mathbf{P}(s(t) = z \mid \mathcal{F}(t)) +&\\
+ \varepsilon\, c_i(t) \sum_{y=m_{i-1}(t)+1}^{m_i(t)} \frac{\mathbf{P}(s(t) = y \mid \mathcal{F}(t))}{m_i(t) - m_{i-1}(t)} &+ \mathfrak{o}(\varepsilon).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(|s(t)|) = \mathbf{E}' \sum_{i=-\infty}^{\infty} c_i(t) \sum_{z=m_{i-1}(t)+1}^{m_i(t)} \Bigg(\sum_{y=m_{i-1}(t)+1}^{m_i(t)} \frac{p\,(y,\,t)}{m_i(t) - m_{i-1}(t)}\, |z| -&\\
- p\,(z,\,t)\cdot |z| \Bigg)&.
\end{aligned}$$

There exists a π_i permutation of the numbers $\{m_{i-1}(t)+1\dots m_i(t)\}$, for which

$$\sum_{z=m_{i-1}(t)+1}^{m_i(t)}\sum_{y=m_{i-1}(t)+1}^{m_i(t)}\frac{p\,(y,\,t)}{m_i(t)-m_{i-1}(t)}\,|z|\le\sum_{z=m_{i-1}(t)+1}^{m_i(t)}p\,(z,\,t)\cdot|\pi_i(z)|$$

holds (by permuting higher values of $|z|$ on higher weights), and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(|s(t)|)\le\mathbf{E}'\sum_{i=-\infty}^{\infty}c_i(t)\sum_{z=m_{i-1}(t)+1}^{m_i(t)}p\,(z,\,t)\cdot(|\pi_i(z)|-|z|)=$$

$$=\mathbf{E}'\sum_{i=-\infty}^{\infty}c_i(t)\sum_{z=m_{i-1}(t)+1}^{m_i(t)}\Big(p\,(z,\,t)-\min_{m_{i-1}(t)<y\le m_i(t)}p\,(y,\,t)\Big)\cdot(|\pi_i(z)|-|z|)\le$$

$$\le\mathbf{E}'\sum_{i=-\infty}^{\infty}c_i(t)\sum_{z=m_{i-1}(t)+1}^{m_i(t)}\Big(\max_{m_{i-1}(t)<y\le m_i(t)}p\,(y,\,t)-$$

$$-\min_{m_{i-1}(t)<y\le m_i(t)}p\,(y,\,t)\Big)\cdot(m_i(t)-m_{i-1}(t))=$$

$$=\mathbf{E}'\sum_{i\,:\,m_i(t)>m_{i-1}(t)+1}c_i(t)\Big(\max_{m_{i-1}(t)<y\le m_i(t)}p\,(y,\,t)-$$

$$-\min_{m_{i-1}(t)<y\le m_i(t)}p\,(y,\,t)\Big)\cdot(m_i(t)-m_{i-1}(t))^2=\mathbf{E}'\sum_{i=-\infty}^{\infty}A_i(t)\,C_i(t)$$

with definitions (61). Finally, we use Schwartz and Cauchy's inequality (for simplicity we do not denote time-dependence of the quantities below):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(|s(t)|)\le\mathbf{E}'\sum_{i=-\infty}^{\infty}A_i\,C_i=\mathbf{E}'\sum_{i=-\infty}^{\infty}\sqrt{A_i}\,\sqrt{A_i}\,C_i\le$$

$$\le\mathbf{E}'\left[\sqrt{\sum_{i=-\infty}^{\infty}A_i}\cdot\sqrt{\sum_{j=-\infty}^{\infty}A_j\,C_j^2}\right]\le\sqrt{\mathbf{E}'\sum_{i=-\infty}^{\infty}A_i}\cdot\sqrt{\mathbf{E}'\sum_{j=-\infty}^{\infty}A_j\,C_j^2}.$$

□

Lemma 18.4. *The expression*

$$\sqrt{\mathbf{E}'\sum_{j=-\infty}^{\infty}A_j(t)\,C_j^2(t)},$$

which is the second factor on the right-hand side of (62)*, is a bounded function of time.*

Proof. Due to definitions (61), A_i can be bounded from above by

$$A_i(t)\le\max_{m_{i-1}(t)<y\le m_i(t)}p\,(y,\,t)\le\sum_{y=m_{i-1}(t)+1}^{m_i(t)}p\,(y,\,t)\;=\;\mathbf{P}(S(t)=i\,|\,\mathcal{F}(t)),$$

the probability that our S-particle is at site i. Hence

$$\sqrt{\mathbf{E}' \sum_{j=-\infty}^{\infty} A_j(t)\, C_j^2(t)} \leq \sqrt{\mathbf{E}' \sum_{j=-\infty}^{\infty} \mathbf{P}(S(t) = j \mid \mathcal{F}(t)) \cdot C_j^2(t)} =$$

$$= \sqrt{\mathbf{E}' \left[\mathbf{E} \left(C_{S(t)}^2(t) \mid \mathcal{F}(t) \right) \right]} = \sqrt{\mathbf{E} \left(C_{S(t)}^2(t) \right)}.$$

The expectation in the last term is bounded in time by lemma 18.1 with $n = 2,\ k = 4$, since

$$C_i = (m_i - m_{i-1})^2 \cdot c_i = c_i \cdot (\zeta_i - \eta_i)^2 .$$

□

As we know, for any site i, the probabilities $p\,(y,\,t)$ can only change by equalizing between $y = m_{i-1}(t) + 1 \ldots m_i(t)$, and the initial distribution is concentrated on $\{s(t=0) = 0\}$. Therefore, at every moment t, the function $y \to p\,(y,\,t)$ is unimodal. This is clearly the initial situation, and it stays true after each change of this function. By the equalizing property of the $(p\,(y,\,t))_{y\in\mathbb{Z}}$ process at a jump of second class particle from site i,

$$\max_{m_{i-1}(t) < z \leq m_i(t)} p\,(z,\,t)$$

can never increase. Hence the global maximum $\max_{z\in\mathbb{Z}} p\,(z,\,t)$ is also a non-increasing function of t, and it is bounded as well. Thus its limit exists, which we denote by P. It is believed that $P = 0$ but we cannot prove this, and this is not necessary for our arguments.

Lemma 18.5. *Assume $P > 0$. Then the set*

$$\{x \in \mathbb{Z} \,:\, p\,(x,\,t) \geq P\}$$

is always contained in the interval $[-1/P,\, 1/P]$.

Proof. The statement clearly holds initially. For a discrete interval $[x,\,y]$ (with possibly $x = y$ as well), we introduce the block-average

$$B_{[x,\,y]}(t) := \frac{1}{y - x + 1} \sum_{z=x}^{y} p\,(z,\,t),$$

and we say that $[x,\,y]$ is a *good block*, if $B_{[x,\,y]}(t) \leq \min_{z\in[x,\,y]} 1/|z|$ (for site $z = 0$, we can write 1 instead of $1/|z|$). Any interval is a good block initially. We show this for any time t as well. More precisely, fix $x \leq y$, and assume that at a moment t, an equalization in the interval $[u,\,v]$ happens:

$$p(z,\,t+0) = B_{[u,\,v]}(t)$$

for each $z \in [u,\,v]$. If each finite interval is a good block at t, then we show that $[x,\,y]$ is also a good block after this step, at $t + 0$. There are four cases.

(i) If $[u,\,v]$ and $[x,\,y]$ are disjoint or $[u,\,v] \subset [x,\,y]$, then the block-average of $[x,\,y]$ does not change by this step, hence it keeps on being a good block.

(ii) If $[x, y] \subset [u, v]$, then $B_{[x, y]}(t+0) = B_{[u, v]}(t+0) = B_{[u, v]}(t)$, and $[u, v]$ was a good block at time t, hence $[x, y]$ is also a good block after this step.

(iii) In case $[x, y] \setminus [u, v] \neq \emptyset$, $[u, v] \setminus [x, y] \neq \emptyset$ and $B_{[u, v]}(t) \geq B_{[x, y] \setminus [u, v]}(t)$ before the step, then

$$B_{[u, v]}(t+0) = B_{[u, v]}(t) \geq B_{[x, y] \setminus [u, v]}(t) = B_{[x, y] \setminus [u, v]}(t+0),$$

hence $B_{[x, y]}(t+0) \leq B_{[x, y] \cup [u, v]}(t+0)$. The latter does not change by the step, thus $[x, y] \cup [u, v]$ keeps on being a good block, which shows that $[x, y]$ is also a good block after the step.

(iv) In case $[x, y] \setminus [u, v] \neq \emptyset$, $[u, v] \setminus [x, y] \neq \emptyset$ and $B_{[u, v]}(t) < B_{[x, y] \setminus [u, v]}(t)$ before the step, then by unimodality, $B_{[u, v]}(t) \leq B_{[u, v] \cap [x, y]}(t)$, since the function $z \to p(z, t)$ has no local minimum. This means that $B_{[u, v] \cap [x, y]}$ does not increase:

$$B_{[u, v] \cap [x, y]}(t+0) = B_{[u, v]}(t+0) = B_{[u, v]}(t) \leq B_{[u, v] \cap [x, y]}(t).$$

Since $B_{[x, y] \setminus [u, v]}$ does not change, $B_{[x, y]}$ can not increase either, and $[x, y]$ was a good block before the step, thus it keeps on being a good block.

Applying this result shows the interval containing any single point z to be a good block, i.e. $p(z, t) < 1/P$ for $z \notin [-1/P, 1/P]$, which completes the proof. □

Lemma 18.6. *Assume* $\lim_{t\to\infty} \max_{z\in\mathbb{Z}} p(z, t) = P > 0$. *Then there are* z, y *neighboring sites in the interval* $[-1/P - 1, 1/P + 1]$ *and a time* $T > 0$*, such that the second class particles indexed by* z *and* y *cannot be at the same site after* T*:* $U^{(z)}(t) \neq U^{(y)}(t)$ $(\forall t > T)$.

Proof. Let

$$A := \left\{ z \in \mathbb{Z} : \limsup_{t\to\infty} p(z, t) = P \right\} \neq \emptyset.$$

By the previous lemma, $A \subset [-1/P, 1/P]$, and any index $z_{\max}(t)$, for which $p(z_{\max}(t), t)$ is maximal (and hence larger than or equals to P), is also contained in $[-1/P, 1/P]$ for any t. With fixed $P_1 < P$ large enough, there exists a moment T, such that $p(x, t) < P_1$ for any $x \notin A$ and for all $t > T$. Hence by $p(z_{\max}(t), t) \geq P$, all indices $z_{\max}(t) \in A$ for all $t > T$. Let us fix $z \in A$ and $y \notin A$ neighbors, and $y' \notin A$ the other neighbor of A. Then infinitely often for $t > T$, $p(z, t) \geq P > P_1 > p(y, t)$ and $P_1 > p(y', t)$ happens. In this situation, assume that $p(z, t)$ decreases due to equalization with its neighbors in A. Would the result of this step be $p(z, t) < P$, all indices $z_{\max}(t)$ would be included in this step by unimodality, hence $p(z_{\max}(t+0), t+0) < P$ would follow, a contradiction. Thus we see that $p(z, t) \geq P$ can only be violated by an equalization including y or y'. If this equalization also includes all indices $z_{\max}(t)$, then the result must be $p(y, t) \geq P$ or $p(y', t) \geq P$ by $p(z_{\max}(t+0), t+0) \geq P$, pulling out at least $P - P_1$ probability from the set A. If this step does not include all $z_{\max}$ indices, then it includes indices all with probability at least P, hence pulling out at least $(P - P_1)/2$ probability from the set A. Since $t > T$, $z_{\max}(t) \in A$, and hence by unimodality, the joint probability of the set A can only decrease. We conclude that assuming equalizing of probabilities between $z \in A$ and y or $y' \notin A$ infinitely often results in decreasing the joint probability of the finite set A infinitely often by a positive constant, which contradicts $P \leq p(z_{\max}(t), t)$ and $z_{\max}(t) \in A$. □

Now we can prove law of large numbers for the index $s(t)$ of the second class particles carrying our S-particle:

Proposition 18.7.

$$(\forall \delta > 0) \quad \lim_{t\to\infty} \mathbf{P}\left(\left|\frac{s(t)}{t}\right| > \delta\right) = 0.$$

Proof. By the previous lemma, we see that for $P > 0$ there exists a neighboring pair $(z, y) \in [-1/P - 1, 1/P + 1]$ of second class particles which will never meet after some T. After T, the process $s(t)$ can not cross such a pair (z, y). By translation invariance, it follows a.s. that such pairs appear with positive density on $\mathbb{Z}$ in this case, thus $s(t)$ is bounded a.s. and the statement is true. Hence we assume $P = 0$ for the rest of the proof. By unimodality,

$$\sum_{i=-\infty}^{\infty} A_i(t) = \sum_{i\,:\,m_i(t)>m_{i-1}(t)+1} \Big(\max_{m_{i-1}(t)<y\le m_i(t)} p\,(y,\,t) - \\ - \min_{m_{i-1}(t)<y\le m_i(t)} p\,(y,\,t)\Big) \le 2\,\max_{z\in\mathbb{Z}} p\,(z,\,t). \tag{63}$$

Indirectly let's assume

$$(\exists \delta > 0)\ (\exists K > 0)\ (\forall T > 0)\ (\exists t > T)\ :\ \mathbf{P}\left(\left|\frac{s(t)}{t}\right| > \delta\right) > K.$$

Then it follows that

$$\mathbf{E}(|s(t)|) > K\,\delta\,t \tag{64}$$

for infinitely many and arbitrarily large $t > 0$. By (63) and $P = 0$,

$$\sum_{i=-\infty}^{\infty} A_i(t) \to 0,$$

thus by dominated convergence theorem

$$\sqrt{\mathbf{E}' \sum_{i=-\infty}^{\infty} A_i(t)} \to 0.$$

Hence by lemma 18.3

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(|s(t)|) \le \sqrt{\mathbf{E}' \sum_{i=-\infty}^{\infty} A_i(t)} \cdot \sqrt{\mathbf{E}' \sum_{j=-\infty}^{\infty} A_j(t)\, C_j^2(t)} \to 0$$

when $t \to \infty$, as

$$\sqrt{\mathbf{E}' \sum_{j=-\infty}^{\infty} A_j(t)\, C_j^2(t)}$$

is bounded by lemma 18.4. That means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(|s(t)|)$$

tends to zero as $t \to \infty$, which contradicts (64). □

Now we show the law of large numbers for $S(t)$, the random walk on the background process $\underline{\eta}$ with parameter θ_1 and $\underline{\zeta}$ with parameter θ_2.

Proposition 18.8. *Let*

$$c(\theta_1, \theta_2) := 2 \frac{\cosh(\theta_2) - \cosh(\theta_1)}{\mathbf{E}_{\theta_2}(\zeta_0) - \mathbf{E}_{\theta_1}(\eta_0)} \tag{65}$$

for BL models, and

$$c(\theta_1, \theta_2) := \frac{e^{\theta_2} - e^{\theta_1}}{\mathbf{E}_{\theta_2}(\zeta_0) - \mathbf{E}_{\theta_1}(\eta_0)} \tag{66}$$

for the ZR process. Then for every $\delta > 0$

$$\lim_{t\to\infty} \mathbf{P}\left(\left|\frac{S(t)}{t} - c(\theta_1, \theta_2)\right| > \delta\right) = 0. \tag{67}$$

Proof. We show the proposition for BL models, the modification for the ZR process is straightforward. By the coupling rules, if a second class particle jumps from i to $i+1$ then the column g_i of $\underline{\zeta}$ between sites i and $i+1$ increases by one. If one jumps from $i+1$ to i then the column h_i of $\underline{\eta}$ increases by one. Hence for the current $J_i^{(2^{\mathrm{nd}})}$ of second class particles defined earlier in this subsection,

$$J_i^{(2^{\mathrm{nd}})}(t) = (g_i(t) - g_i(0)) - (h_i(t) - h_i(0)),$$

i.e. it is the difference between the growth of columns i of $\underline{\zeta}$ and of $\underline{\eta}$ until time t. Due to separate ergodicity of each $\underline{\zeta}$ and $\underline{\eta}$, we have law of large numbers for $g_i(t) - g_i(0)$ and for $h_i(t) - h_i(0)$, since each of these models is distributed according to its ergodic stationary measure. Hence with the expectation of the column growth rates, we have

$$\lim_{t\to\infty} \frac{J_i^{(2^{\mathrm{nd}})}(t)}{t} = \mathbf{E}_{\theta_2}\left(f(\zeta_0) + f(-\zeta_0)\right) - \mathbf{E}_{\theta_1}\left(f(\eta_0) + f(-\eta_0)\right) = \\ = 2\left(\cosh(\theta_2) - \cosh(\theta_1)\right) \quad a.s. \tag{68}$$

We extend definition (56) for $x \in \mathbb{R}$:

$$m_x(t) := m_{\lfloor x \rfloor}(t) = \max\{m \ : \ U^{(m)}(t) \leq x\}$$

Obviously, $m_x(t) = m_x(0) - J_{\lfloor x \rfloor}^{2^{\mathrm{nd}}}(t)$. If $K \in \mathbb{R}$ then

$$\lim_{v\to\infty} \frac{m_{(Kv)}(0)}{K\,v} = \mathbf{E}(\zeta_0) - \mathbf{E}(\eta_0) =: p \ \text{ a.s.}$$

since at $t = 0$, the starting distribution of the number of second class particles at different sites is a product measure.

$$(69)\quad \lim_{t\to\infty}\mathbf{P}\left(\left|\frac{m_{(Kt)}(t)}{t}+(c(\theta_1,\,\theta_2)-K)p\right|>\varepsilon\right)=$$

$$=\lim_{t\to\infty}\mathbf{P}\left(\left|\frac{m_{(Kt)}(t)}{t}-Kp+2\cosh(\theta_2)-2\cosh(\theta_1)\right|>\varepsilon\right)=$$

$$=\lim_{t\to\infty}\mathbf{P}\left(\left|\frac{m_{(Kt)}(0)-J^{(2^{\mathrm{nd}})}_{\lfloor Kt\rfloor}(t)}{t}-Kp+2\cosh(\theta_2)-2\cosh(\theta_1)\right|>\varepsilon\right)=$$

$$=\lim_{t\to\infty}\mathbf{P}\left(\left|-\frac{J^{(2^{\mathrm{nd}})}_{\lfloor Kt\rfloor}(t)}{t}+2\cosh(\theta_2)-2\cosh(\theta_1)\right|>\varepsilon\right)=$$

$$=\lim_{t\to\infty}\mathbf{P}\left(\left|-\frac{J^{(2^{\mathrm{nd}})}_{0}(t)}{t}+2\cosh(\theta_2)-2\cosh(\theta_1)\right|>\varepsilon\right)=0$$

by translation-invariance and by (68), for any $\varepsilon>0$. Recall that $S(t)$ is the position of the zeroth S-particle, i.e. the position of the $s(t)$-th second class particle: $S(t)=U^{(s(t))}(t)$. Hence

$$(70)\quad \mathbf{P}\left(\left|\frac{S(t)}{t}-c(\theta_1,\,\theta_2)\right|>\delta\right)=\mathbf{P}\left(\left|\frac{U^{(s(t))}(t)}{t}-c(\theta_1,\,\theta_2)\right|>\delta\right)=$$
$$=\mathbf{P}\left(U^{(s(t))}(t)>c(\theta_1,\,\theta_2)\,t+\delta\,t\right)+\mathbf{P}\left(U^{(s(t))}(t)<c(\theta_1,\,\theta_2)\,t-\delta\,t\right).$$

In case

$$U^{(s(t))}(t)>c(\theta_1,\,\theta_2)\,t+\delta\,t$$

it follows by definitions of $m_x(t)$ and of p that

$$s(t)>m_{(c(\theta_1,\,\theta_2)\,t+\delta\,t)}(t)\quad\text{, hence}$$

$$\mathbf{P}\left(U^{(s(t))}(t)>c(\theta_1,\,\theta_2)\,t+\delta\,t\right)\le\mathbf{P}\left(\frac{s(t)}{t}>\frac{m_{(c(\theta_1,\,\theta_2)\,t+\delta\,t)}(t)}{t}\right)\le$$
$$\le\mathbf{P}\left(\frac{s(t)}{t}>\frac{\delta}{2}\,p\right)+\mathbf{P}\left(\frac{m_{(c(\theta_1,\,\theta_2)\,t+\delta\,t)}(t)}{t}<\frac{\delta}{2}\,p\right).$$

As time goes on, the first term goes to zero due to proposition 18.7, and so does the second term by (69) (with $K=c(\theta_1,\,\theta_2)+\delta$).
In case

$$U^{(s(t))}(t)<c(\theta_1,\,\theta_2)\,t-\delta\,t$$

it follows that

$$s(t)\le m_{(c(\theta_1,\,\theta_2)\,t-\delta\,t)}(t)\quad\text{, hence}$$

$$\mathbf{P}\left(U^{(s(t))}(t)<c(\theta_1,\,\theta_2)\,t-\delta\,t\right)\le\mathbf{P}\left(\frac{s(t)}{t}\le\frac{m_{(c(\theta_1,\,\theta_2)\,t-\delta\,t)}(t)}{t}\right)\le$$
$$\mathbf{P}\left(\frac{s(t)}{t}\le-\frac{\delta}{2}\,p\right)+\mathbf{P}\left(\frac{m_{(c(\theta_1,\,\theta_2)\,t-\delta\,t)}(t)}{t}>-\frac{\delta}{2}\,p\right).$$

The first term again goes to zero due to proposition 18.7, and so does the second term by (69) (with $K=c(\theta_1,\,\theta_2)-\delta$). Thus we see that both terms on the right-hand side of (70) tend to zero as $t\to\infty$. □

18.3 Coupling the defect tracer to the S-particles

We fix the model $\underline{\omega}$ in stationary distribution $\underline{\mu}_\theta$ with the defect tracer $Q(t)$ started from the origin. We prove theorem 14.6 for BL and (totally asymmetric) ZR models. A natural idea would be to couple the defect tracer Q to the second class particles, present at the same site Q. The problem is that, either to the left or to the right, the rate for any jump of second class particles form the site Q may be higher than the rate for Q to jump. On the other hand, one second class particle always stays at site i after one jump from i, in case more than one of them were present at i. The solution is to couple the defect tracer to the S-particle, for which the desired conditions are already proven by propositions 18.2 and 18.8. For simplicity reasons, in case of the ZR process we let $f(-z) := 0$ for $z > 0$, and hence $\mu(-z)$ of ZR is also zero in these cases.

The upper bound for Q.

First, we identify $\underline{\eta}$ distributed according to $\underline{\mu}_{\theta_1}$ with $\underline{\omega}$ possessing the defect tracer Q, therefore we set $\theta_1 := \theta < \theta_2$. We have then $\omega_i(t) \le \zeta_i(t)$ for all t according to the basic coupling, and recall that $Q(0) = 0 \le S(0)$. In what follows, we are going to couple the random permutations of the S-particles, thus the random walk $S(t)$ of the zeroth S-particle, with the defect tracer $Q(t)$. We only couple them in case $Q(t) = S(t)$. The basic observation we use is that the rates (57) for the jump of the S-particle can be compared to the rates for the jump of the defect tracer $Q(t)$. As we have seen at the introduction of BL models, it is enough to consider the "effect of bricklayers" standing at each position i. That is to say, we are allowed to consider the ω_i-dependent parts of $r(\omega_{i-1}, \omega_i)$ and $r(\omega_i, \omega_{i+1})$ only, since the ω_i-dependent parts are added to the ω_{i-1}-dependent or to the ω_{i+1}-dependent parts in these rates. In the rest of the paper, we describe couplings by giving rates of bricklayers standing at each site i. This observation also holds for the zero range process (by saying rate for a particle to jump instead of saying rate for bricklayers to lay bricks).

In tables 4 and 3, $h_i \uparrow$ means that the column of the model $\underline{\omega}$ between i and $i+1$ has increased by one, $g_i \uparrow$ means that this column of $\underline{\zeta}$ has increased by one, $\curvearrowright$ means the jump to the right from i, $\curvearrowleft$ means the jump to the left from i.

with rate	$h_i \uparrow$	$g_i \uparrow$	$Q \curvearrowright$	$S \curvearrowright$	a second class particle
$\frac{\zeta_i - \omega_i - 1}{\zeta_i - \omega_i} \times$ $\times [f(\zeta_i) - f(\omega_i)]$		•			$\curvearrowright$
$\frac{f(\zeta_i) - f(\omega_i)}{\zeta_i - \omega_i} -$ $- [f(\omega_i + 1) - f(\omega_i)]$		•		•	$\curvearrowright$
$f(\omega_i + 1) - f(\omega_i)$		•	•	•	$\curvearrowright$
$f(\omega_i)$	•	•			

Table 3: Rates for Q and S to step right and for bricklayers at site $i = S = Q$ to lay brick on their right

with rate	$h_{i-1} \uparrow$	$g_{i-1} \uparrow$	$Q \curvearrowleft$	$S \curvearrowleft$	a second class particle
$f(-\omega_i - 1) - f(-\zeta_i)$	•				$\curvearrowleft$
$[f(-\omega_i) - f(-\omega_i - 1)]-$ $- \frac{f(-\omega_i)-f(-\zeta_i)}{\zeta_i-\omega_i}$	•		•		$\curvearrowleft$
$\frac{f(-\omega_i)-f(-\zeta_i)}{\zeta_i-\omega_i}$	•		•	•	$\curvearrowleft$
$f(-\zeta_i)$	•	•			

Table 4: Rates for Q and S to step left and for bricklayers at site $i = S = Q$ to lay brick on their left

Note that by $i = S = Q$, $\zeta_i \geq \omega_i + 1$. The rates are non negative due to monotonicity of f and convexity condition 14.5. By summing the rates corresponding to any column of the tables, one can verify that each $\underline{\omega}$ and $\underline{\zeta}$ evolves according its original rates, Q has the jump rates according to the basic coupling described in table 2, and S also has the appropriate rates (57). We see that once being at position S, the defect tracer Q can't move right without moving S with it and S can't move left without moving Q with it. Hence our rules preserve the condition $Q \leq S$.

We have so far the upper bound $Q(t) \leq S(t)$, and we have the law of large numbers (67) with speed $c(\theta, \theta_2)$ defined in either (65) or in (66) for any $\theta_2 > \theta$, and the n-th moment condition (59) for this $S(t)$ process.

The lower bound for Q.

Now we show a similar coupling which results in a lower bound for Q. The natural idea would be to identify $\underline{\zeta}$ with $\underline{\omega}$, and couple Q to the S-particle. The rates for Q and S to jump with would allow $Q(t) \geq S(t)$. However, this coupling can not be realized in a similar way that the coupling described above: there is no way for Q and S to step together, since only one brick can be laid at a time to a column.

Therefore, we need to modify the initial distribution of the models as follows. Let $\mu(x, y)$ be, as before, a two dimensional distribution giving probability zero to $x > y$, and having marginals μ_{θ_1} and μ_{θ_2}, respectively. Fix the pair $(\underline{\eta}, \underline{\zeta})$, as before, with the product of $\mu(x, y)$ for different sites as initial distribution. Define

$$\mu'(y, x) := \mu(x, y) \cdot \frac{\mu_{\theta_2}(y-1)}{\mu_{\theta_2}(y)}. \tag{71}$$

Fix the pair $(\underline{\eta}', \underline{\zeta}')$, with the product of $\mu(x, y)$ for each site $i \neq 0$ and of $\mu'(x, y)$ for the site $i = 0$ as initial distribution. Then $\eta'_i(0) \leq \zeta'_i(0)$ holds a.s. for each site i, hence the basic coupling is applicable for this pair of models. We have second class particles between $\underline{\eta}'$ and $\underline{\zeta}'$, and we introduce the S'-particles as well, starting S'_0 from the first site on the left-hand side of the origin:

$$S'(0) = S'_0(0) := \max\{i \leq 0 \,:\, \zeta'_i(0) > \eta'_i(0)\}.$$

Assume now that the S-particle of $(\underline{\eta},\,\underline{\zeta})$ is also started from the first site on the left-hand side of the origin, instead of starting it from the right-hand side of the origin:

$$S(0) = S_0(0) := \max\{i \le 0 \,:\, \zeta_i(0) > \eta_i(0)\}.$$

Then it is clear, that propositions 18.2 and 18.8 also hold for this S-particle. Now we derive these statements for S' as well. Since initially $(\underline{\eta}',\,\underline{\zeta}')$ only differs from $(\underline{\eta},\,\underline{\zeta})$ by the distribution at the origin, the conditional expectations

$$\mathbf{E}\left(S'(t)\,|\,\eta_0'(0) = x,\ \zeta_0'(0) = y\right) = \mathbf{E}\left(S(t)\,|\,\eta_0(0) = x,\ \zeta_0(0) = y\right) \tag{72}$$

agree. This is the basic idea of the following

Lemma 18.9. *The moment condition* (59) *and the law of large numbers* (67) *hold for S' as well.*

Proof. By the use of (72) and Cauchy's inequality in a similar way than in the proof of lemma 18.1,

$$\begin{aligned}
\mathbf{E}\left(\frac{|S'(t)|^n}{t^n}\right) &= \sum_{x\le y}\mathbf{E}\left(\frac{|S'(t)|^n}{t^n}\,\Big|\,\eta_0'(0)=x,\,\zeta_0'(0)=y\right)\cdot\mu'(x,\,y) = \\
&= \sum_{x\le y}\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\,\Big|\,\eta_0(0)=x,\,\zeta_0(0)=y\right)\cdot\sqrt{\mu(x,\,y)}\cdot\frac{\mu'(x,\,y)}{\sqrt{\mu(x,\,y)}} \le \\
&\le \left[\sum_{x\le y}\left(\mathbf{E}\left(\frac{|S(t)|^n}{t^n}\,\Big|\,\eta_0(0)=x,\,\zeta_0(0)=y\right)\right)^2\cdot\mu(x,\,y)\right]^{\frac12}\times \\
&\times\left[\sum_{x\le y}\frac{\mu'(x,\,y)}{\mu(x,\,y)}\cdot\mu'(x,\,y)\right]^{\frac12} \le \left[\mathbf{E}\left(\frac{S(t)^{2n}}{t^{2n}}\right)\right]^{\frac12}\cdot\left[\sum_{x\le y}\frac{\mu'(x,\,y)}{\mu(x,\,y)}\cdot\mu'(x,\,y)\right]^{\frac12}.
\end{aligned}$$

The first factor of the display is bounded by proposition 18.2. For the second factor, by (71) and (23) we write

$$\begin{aligned}
&\sum_{x\le y}\frac{\mu'(x,\,y)}{\mu(x,\,y)}\cdot\mu'(x,\,y) = \sum_{x\le y}\frac{\mu_{\theta_2}(y-1)}{\mu_{\theta_2}(y)}\cdot\mu'(x,\,y) = \\
&= \sum_{y\in\mathbb{Z}}\frac{\mu_{\theta_2}(y-1)}{\mu_{\theta_2}(y)}\cdot\mu_{\theta_2}(y-1) = \sum_{y\in\mathbb{Z}}\frac{f(y)}{\mathrm{e}^{\theta_2}}\cdot\frac{\mathrm{e}^{\theta_2(y-1)}}{f(y-1)!}\cdot\frac{1}{Z(\theta_2)} = \frac{1}{\mathrm{e}^{2\theta_2}}\mathbf{E}_{\theta_2}(f(y)^2),
\end{aligned}$$

which is again finite. Hence (59) holds for S' as well.

For the law of large numbers, we know that for any $\delta > 0$,

$$\begin{aligned}
0 &= \lim_{t\to\infty}\mathbf{P}\left(\left|\frac{S(t)}{t} - c(\theta_1,\,\theta_2)\right| > \delta\right) = \\
&= \lim_{t\to\infty}\sum_{x\le y}\mathbf{P}\left(\left|\frac{S(t)}{t} - c(\theta_1,\,\theta_2)\right| > \delta\,\Big|\,\eta_0(0)=x,\,\zeta_0(0)=y\right)\cdot\mu(x,\,y) = \\
&= \lim_{t\to\infty}\sum_{x\le y}\mathbf{P}\left(\left|\frac{S'(t)}{t} - c(\theta_1,\,\theta_2)\right| > \delta\,\Big|\,\eta_0'(0)=x,\,\zeta_0'(0)=y\right)\cdot\mu(x,\,y),
\end{aligned}$$

hence (67) follows for S' as well by absolute continuity of μ' w.r.t. μ. □

In order to obtain lower bound for Q of $\underline{\omega}$ distributed according to $\underline{\mu}_\theta$, set $\theta_2 = \theta > \theta_1$. The marginal distribution of $\zeta_0(0)$ is the second marginal of μ', namely, $\mu_{\theta_2}(y-1) = \mu_\theta(y-1)$. Hence it is possible to fix the pair $(\underline{\eta}', \underline{\zeta}')$ defined above with

$$\underline{\zeta}'(t) = \underline{\omega}(t) + \underline{\delta}_{Q(t)}, \qquad Q(0) = 0,$$

i.e. $\underline{\omega}$ is coupled to $\underline{\zeta}'$ with the defect tracer Q between them. Note that $S'(0) \le 0 = Q(0)$. We show the coupling that preserves $S'(t) \le Q(t)$ for all later times. We only couple Q to the random permutations acting on S' in case $Q = S'$ for a site i. For tables 6 and 5, $h_i' \uparrow$ means that the column of the model $\underline{\eta}'$ between i and $i+1$ has increased by one, $g_i' \uparrow$ means that this column of $\underline{\zeta}'$ has increased by one. Note that by $i = S' = Q$, $\zeta_i' \ge \eta_i' + 1$. As at the coupling for the upper bound, the rates are non negative due to monotonicity of f and convexity condition 14.5. By summing the rates corresponding to any column of the tables, one can verify that each $\underline{\eta}'$ and $\underline{\zeta}'$ evolves according its original rates, Q has the jump rates according to the basic coupling described in table 2 (hence $\underline{\omega}$ also evolves according its original rates), and S' also has the appropriate rates (57). We see that once being at position S', the defect tracer Q can't move left without moving S' with it and S' can't move right without moving Q with it. Hence our rules preserve the condition $Q \ge S'$.

with rate	$h_i' \uparrow$	$g_i' \uparrow$	$Q \curvearrowright$	$S' \curvearrowright$	a second class particle
$f(\zeta_i'-1) - f(\eta_i')$		•			$\curvearrowright$
$[f(\zeta_i') - f(\zeta_i'-1)] - \frac{f(\zeta_i')-f(\eta_i')}{\zeta_i'-\eta_i'}$		•	•		$\curvearrowright$
$\frac{f(\zeta_i')-f(\eta_i')}{\zeta_i'-\eta_i'}$		•	•	•	$\curvearrowright$
$f(\eta_i')$	•	•			

Table 5: Rates for Q and S' to step right and for bricklayers at site $i = S' = Q$ to lay brick on their right

with rate	$h_{i-1}' \uparrow$	$g_{i-1}' \uparrow$	$Q \curvearrowleft$	$S' \curvearrowleft$	a second class particle
$\frac{\zeta_i'-\eta_i'-1}{\zeta_i'-\eta_i'} \times [f(-\eta_i') - f(-\zeta_i')]$	•				$\curvearrowleft$
$\frac{f(-\eta_i')-f(-\zeta_i')}{\zeta_i'-\eta_i'} - [f(-\zeta_i'+1) - f(-\zeta_i')]$	•			•	$\curvearrowleft$
$f(-\zeta_i'+1) - f(-\zeta_i')$	•		•	•	$\curvearrowleft$
$f(-\zeta_i')$	•	•			

Table 6: Rates for Q and S' to step left and for bricklayers at site $i = S' = Q$ to lay brick on their left

Proof of theorem 14.6. By the upper bound and the lower bound above, we have

$$S(t) \geq Q(t) \geq S'(t)$$

and for any $\theta_2 > \theta > \theta_1$, we have weak law of large numbers for S with $c(\theta, \theta_2)$, and for S' with $c(\theta_1, \theta)$, respectively. Hence taking the limits $\theta_1 \nearrow \theta$ and $\theta_2 \searrow \theta$ completes the proof of the law of large numbers (27) by computing

$$\lim_{\theta_1 \nearrow \theta} c(\theta_1, \theta) = \lim_{\theta_2 \searrow \theta} c(\theta, \theta_2) = C(\theta)$$

both for BL and ZR models. Moreover, for any $n \in \mathbb{Z}^+$, we have n-th moment condition (59) for both S and S', hence not only (28), but the n-th moment condition follows as well for Q. This also shows L^n-convergence of $Q(t)/t$ for any $n \in \mathbb{Z}^+$. □

18.4 Strict monotonicity of $C(\theta)$

As a consequence of the type of coupling methods shown above, we are able to show strict convexity of the function $\mathcal{H}(\varrho)$ of (25). First we refer to the coupling which shows (non strict) convexity, and then we complete the proof of strict convexity by some analytic arguments.

Remark 18.10. *Let $\underline{\omega}$, $\underline{\omega}'$ be two copies of a model (either BL or ZR model) possessing condition 14.5, with the defect tracers $Q(t)$ and $Q'(t)$, respectively. Assume that for each site i and for time $t = 0$*

$$\omega_i(0) \leq \omega_i'(0) \quad \textit{and} \quad Q(0) \leq Q'(0).$$

Then it is possible to couple such way that for all $t \geq 0$ and any $i \in \mathbb{Z}$,

$$\omega_i(t) \leq \omega_i'(t) \quad \textit{and} \quad Q(t) \leq Q'(t) \quad \textit{a.s.}$$

is satisfied.

This coupling is very similar to the ones shown in this subsection, we do not give the details here. The pair $(\underline{\omega}, \underline{\omega}')$ is coupled according to the basic coupling, and we can apply this proposition for the case when their joint distribution has marginals $\underline{\mu}_\theta$ and $\underline{\mu}_{\theta'}$, respectively. Then we simply see that the motion of the defect tracer of a model has a monotonicity in the parameter θ of the model's stationary distribution. In the introduction we saw that this implies convexity of the function $\mathcal{H}(\varrho)$. We prove now strict convexity of this function:

Proof of proposition 14.7. First note that by the form (23) of the measure μ_θ, we have

$$\begin{aligned} \varrho(\theta) = \quad & \mathbf{E}_\theta(\omega) \quad = \tfrac{\mathrm{d}}{\mathrm{d}\theta} \log\left(Z(\theta)\right), \\ \mathbf{E}_\theta\left(\widetilde{\omega}^2\right) = \quad & \tfrac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{E}_\theta\left(\omega\right) \ > 0, \\ \mathbf{E}_\theta\left(\widetilde{\omega}^3\right) = \ & \tfrac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{E}_\theta\left(\widetilde{\omega}^2\right) = \tfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathbf{Var}_\theta(\omega)\right), \end{aligned}$$

where tilde stands for the centered variable. For the BL model, we need to show strict convexity of the function

$$\mathcal{H}(\varrho) = \mathbf{E}_{\theta(\varrho)}(r) = \mathrm{e}^{\theta(\varrho)} + \mathrm{e}^{-\theta(\varrho)}.$$

We compute its derivative

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}\mathcal{H}(\varrho) = \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathrm{e}^{\theta}+\mathrm{e}^{-\theta}\right)}{\frac{\mathrm{d}\varrho}{\mathrm{d}\theta}} = \frac{\left(\mathrm{e}^{\theta}-\mathrm{e}^{-\theta}\right)}{\mathbf{E}_{\theta}\left(\widetilde{\omega}^{2}\right)},$$

and, similarly, the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2}\mathcal{H}(\varrho) = \frac{1}{\left[\mathbf{E}_{\theta}\left(\widetilde{\omega}^{2}\right)\right]^3}\left[\left(\mathrm{e}^{\theta}+\mathrm{e}^{-\theta}\right)\,\mathbf{E}_{\theta}\left(\widetilde{\omega}^{2}\right)-\left(\mathrm{e}^{\theta}-\mathrm{e}^{-\theta}\right)\,\mathbf{E}_{\theta}\left(\widetilde{\omega}^{3}\right)\right].$$

Hence (strict) positivity of

$$\left[\left(\mathrm{e}^{\theta}+\mathrm{e}^{-\theta}\right)\,\mathbf{E}_{\theta}\left(\widetilde{\omega}^{2}\right)-\left(\mathrm{e}^{\theta}-\mathrm{e}^{-\theta}\right)\,\mathbf{E}_{\theta}\left(\widetilde{\omega}^{3}\right)\right] \tag{73}$$

on an interval of θ is equivalent to (strict) convexity of $\mathcal{H}(\varrho)$ on the corresponding interval of $\varrho(\theta)$. (73) contains derivatives of $\log\left(Z\left(\theta\right)\right)$, which is by definition analytic, hence (73) is also an analytic function of θ. Moreover, by the previous remark, we know convexity of $\mathcal{H}(\varrho)$, hence non-negativity of (73). Since this function is strictly positive at $\theta = 0$ by symmetry properties of μ_θ, there are at most countably many isolated points at which this analytic function is not strictly positive, hence we have at most countably many isolated points at which the second derivative of $\mathcal{H}(\varrho)$ is not strictly positive. This completes the proof for the BL models.

As for the ZR process, similar computation leads to

$$\left[\mathrm{e}^{\theta}\,\mathbf{E}_{\theta}\left(\widetilde{\omega}^{2}\right)-\mathrm{e}^{\theta}\,\mathbf{E}_{\theta}\left(\widetilde{\omega}^{3}\right)\right]$$

in place of (73). As we know non-negativity of this function by convexity of $\mathcal{H}(\varrho)$, we only need to show $\mathbf{E}_{\theta}\left(\widetilde{\omega}^{2}\right) \neq \mathbf{E}_{\theta}\left(\widetilde{\omega}^{3}\right)$ for some θ, then the previous analytic argument leads to strict convexity.

Indirectly, assume

$$\mathbf{E}_{\theta}\left(\widetilde{\omega}^{2}\right) = \mathbf{E}_{\theta}\left(\widetilde{\omega}^{3}\right) \tag{74}$$

for all $\theta < \bar{\theta}$. Since the right-hand side is the derivative of the left-hand side, it follows that

$$\mathbf{E}_{\theta}\left(\widetilde{\omega}^{2}\right) = A\cdot\mathrm{e}^{\theta}$$

for some $A > 0$. Integrating this we have

$$\mathbf{E}_{\theta}(\omega) = A\cdot\mathrm{e}^{\theta}$$

(the additive constant is zero as can be seen by taking the limit $\theta \to -\infty$). Integrating again we have

$$\begin{aligned} \log\left(Z(\theta)\right) &= A\cdot\mathrm{e}^{\theta}+K, \qquad \text{i.e.}\\ Z(\theta) &= K'\cdot\mathrm{e}^{A\cdot\mathrm{e}^{\theta}}, \qquad \text{i.e.}\\ \sum_{z=0}^{\infty}\frac{\mathrm{e}^{\theta z}}{f(z)!} &= K'\cdot\sum_{z=0}^{\infty}\frac{A^z\cdot\mathrm{e}^{\theta z}}{z!} \end{aligned}$$

for all $\theta < \bar{\theta}$, which leads to $f(z)! = z!/A^z$, $f(z) = z/A$. Hence we see that if at least for one $z \geq 1$ value we have $f(z+1) - f(z) > f(z) - f(z-1)$, then (74) is not true for some θ, and then strict convexity of $\mathcal{H}(\varrho)$ holds. We also see linearity of $\mathcal{H}(\varrho)$ when f is linear. □

Acknowledgement

The author is grateful to Bálint Tóth for initiating the study of bricklayers' models, for giving some of the basic ideas of this paper, and for helping him in many questions. Correcting some mistakes is also acknowledged to him, as well as to Benedek Valkó. The author is also grateful to Pablo A. Ferrari and to Frank H. J. Redig for extremely helpful comments on some of the proofs.

This work was partially supported by the Hungarian National Scientific Research Fund, grant no. OTKA T037685 and by the Hungarian Academy of Sciences, TKI Stochastics@TUB.

Part IV

Existence of the zero range process and a deposition model with superlinear growth rates

19 Introduction

In [2], Balázs introduces the *bricklayers' process*, which is a kind of stochastic deposition models. It can be represented by neighboring columns of bricks, each growing with a rate which depends on the neighboring columns' relative heights. There are other representations, showing close connection to interacting particle systems; the details will be given later. He finds a shock-like product (time-) stationary measure in this model, provided the dependence of the growth rates on the relative heights is exponential. As the model is not rigorously constructed, the natural question of existence of dynamics arises here.

In the area of interacting particle systems, there are two main situations where construction methods are available. One of them applies when the rate with which the configuration changes at a site is bounded. As described in Liggett [15], the construction can be carried out in this case via functional analysis properties of the infinitesimal generator and via the Hille-Yosida theorem. This is the way how existence of dynamics is usually proved for stochastic Ising models, the voter model, contact processes, simple- and K-exclusion processes.

The other situation is when the growth rates are unbounded, but satisfy a *sublinear* growth condition. This means that the growth rates are bounded from above by a linear function of the local state space. The famous example is the zero range process, where there is a nonnegative number ω_i of particles at each site i, and with rate $r(\omega_i)$ depending on the number of these particles, one of them jumps to another site. The sublinear condition mentioned above is formulated here by $|r(k+1) - r(k)| \leq K$ for any $k \geq 0$ and some $K > 0$. Under this condition, it is possible to compare the model to the so-called *multi-type branching process*, or to consider some differential equation arguments, and hence give stochastic bounds on the states realized by the process. This is the way Andjel [1] constructs the process, generalizing the earlier work of Liggett [13]. The method can be extended to more complicated systems, but sublinearity is still an essential condition in the proof of existence.

None of these methods fit to the bricklayers' process with exponential rates in [2]. Although a sublinear growth condition would make it possible to use the arguments mentioned above (see Booth [4] or Quant [21]), e.g. [2] sets up a claim to a proof of existence for models with *superlinear* growth rates. Not superlinearity, but convexity considerations also play an important role in hydrodynamical and second class particle-related arguments, see e.g. Balázs [3] or Rezakhanlou [23], and convexity of the growth rates in some cases may imply superlinearity of them.

In the present paper we consider the bricklayers' process, where the jump rates are unbounded, and we *do not* require sublinear growth conditions. We will use attractivity of the system instead (see below) to construct it. The construction is carried out via coupling considerations, and makes use of auxiliary systems. All our arguments are also valid for the zero range process, hence this model is also constructed for any monotone increasing rate function.

19.1 Formal description of the bricklayers' process

The state space of our process is $\Omega = \mathbb{Z}^{\mathbb{Z}}$, i.e. for each site $i \in \mathbb{Z}$ there is an integer $\omega_i \in \mathbb{Z}$. Let $r : \mathbb{Z} \to \mathbb{R}_+$ be a positive function, with the property

$$r(z) \cdot r(1 - z) = 1 \tag{75}$$

for each $z \in \mathbb{Z}$. Given a configuration

$$\underline{\omega} = \{\omega_i \in \mathbb{Z} : i \in \mathbb{Z}\} \in \Omega,$$

we define $\underline{\omega}^{(i,\,i+1)}$ by

$$\left(\underline{\omega}^{(i,\,i+1)}\right)_j = \begin{cases} \omega_j & \text{for } j \neq i,\, i+1, \\ \omega_i - 1 & \text{for } j = i, \\ \omega_{i+1} + 1 & \text{for } j = i+1. \end{cases}$$

Conditioned on $\underline{\omega}$, the jump $\underline{\omega} \to \underline{\omega}^{(i,\,i+1)}$ happens independently for each site i with rate $r(\omega_i) + r(-\omega_{i+1})$. We *do not assume any growth condition on* r, only monotonicity. The formal infinitesimal generator of the process acts on a $\varphi : \Omega \to \mathbb{R}$ finite cylinder function (i.e. a function depending only on a finite number of values of ω_i) as

$$(L\varphi)(\underline{\omega}) = \sum_{i \in \mathbb{Z}} [r(\omega_i) + r(-\omega_{i+1})] \cdot \left[\varphi(\underline{\omega}^{(i,\,i+1)}) - \varphi(\underline{\omega})\right].$$

The process can be represented by a wall consisting of columns of bricks. ω_i is the "negative discrete gradient" of the wall at site i, i.e. the difference between the height of the column on the left-hand side and on the right-hand side of i. The jumps of the process can be considered as growth of a column, see figure 4. As the growth rate for a fixed column consists of two additive parts depending on ω of the left-hand side and on ω of the right-hand side, respectively, the process can also be represented by conditionally independent bricklayers standing at each site i, laying bricks on their right with rate $r(\omega_i)$, and on their left with rate $r(-\omega_i)$.

We also define the *heights of the columns*, playing an essential role when coupling the models later on, as follows. As initial data we fix an integer h_0, and let

$$h_i := \begin{cases} h_0 - \sum_{j=1}^{i} \omega_j & \text{for } i > 0, \\ h_0 + \sum_{j=i+1}^{0} \omega_j & \text{for } i < 0. \end{cases} \tag{76}$$

Whenever the jump $\underline{\omega} \to \underline{\omega}^{(i,\,i+1)}$ happens for some i, we increase h_i by one. Doing so, the quantities $h_i(t)$ satisfy (76) for all later times t, and $h_i(t)$ represents

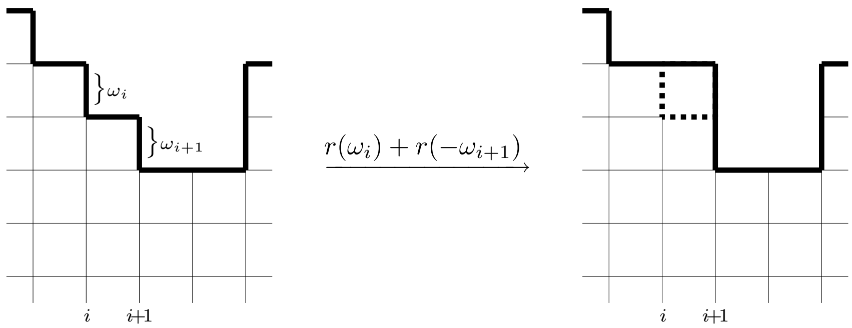

Figure 4: A possible move

the height of the column between sites i and $i+1$. For simplicity we fix the initial condition $h_0(0) := 0$, i.e. the growth of the column right to the origin starts from zero height.

Throughout the construction, we will use attractivity of the processes, which means monotonicity of the function r. In words, this means that the higher neighbors a column has, the faster it grows.

As in Balázs [2], for $n \in \mathbb{Z}^+$ we define

$$r(0)! := 1, \qquad r(n)! := \prod_{y=1}^{n} r(y).$$

Let

$$\bar{\theta} := \log\left(\liminf_{n\to\infty} (r(n)!)^{1/n}\right) = \lim_{n\to\infty} \log(r(n)) \quad , \tag{77}$$

which is strictly positive by (75) and by monotonicity of r, and can even be infinite. With a generic real parameter $\theta \in \left(-\bar{\theta},\, \bar{\theta}\right)$, we define

$$Z(\theta) := \sum_{z=-\infty}^{\infty} \frac{\mathrm{e}^{\theta z}}{r(|z|)!}$$

and the measure

$$\mu^{(\theta)}(z) := \frac{1}{Z(\theta)} \cdot \frac{\mathrm{e}^{\theta z}}{r(|z|)!}. \tag{78}$$

This measure has the property that

$$\frac{\mu^{(\theta)}(z+1)}{\mu^{(\theta)}(z)} = \frac{\mathrm{e}^{\theta}}{r(z+1)} = \mathrm{e}^{\theta} \cdot r(-z)$$

for any $z \in \mathbb{Z}$. Based on this equation, formal computations indicate that the product measure $\underline{\mu}^{(\theta)}$ with marginals

$$\underline{\mu}^{(\theta)}\{\underline{\omega} : \omega_i = z\} := \mu^{(\theta)}(z)$$

is stationary for the process. For details, see Balázs [2]. Throughout this paper, by an arbitrary θ, we mean any parameter $\theta \in \big(-\bar{\theta}, \bar{\theta}\big)$.

19.2 The zero range process

In fact, bricklayers' process is in close connection to interacting particle systems. Considering ω_i as the number of particles at site i leads to a nearest neighbor process with particles ($\omega_i > 0$) which jump to the right and antiparticles ($\omega_i < 0$) jumping to the left, and annihilating when particles and antiparticles meet.

The zero range process is very much similar to the bricklayers' model. In this process $\Omega = \mathbb{N}^{\mathbb{Z}}$, i.e. $\omega_i \geq 0$ for all $i \in \mathbb{Z}$. The formal infinitesimal generator is

$$(L\varphi)(\underline{\omega}) = \sum_{i\in\mathbb{Z}} r(\omega_i) \cdot \Big[\varphi(\underline{\omega}^{(i,\,i+1)}) - \varphi(\underline{\omega})\Big] .$$

Instead of (75), we require $r(0) = 0$, hence $\omega_i \geq 0$ can never be violated. The measure $\underline{\mu}^{(\theta)}$, which is now a product of marginal measures (78) on $\mathbb{N}$, is (formally) stationary for the process, where $\theta \in (-\infty, \bar{\theta})$ with $\bar{\theta}$ of (77). This measure also has the property

$$\frac{\mu^{(\theta)}(z+1)}{\mu^{(\theta)}(z)} = \frac{\mathrm{e}^{\theta}}{r(z+1)} \qquad (z \geq 0).$$

Throughout the paper we show the arguments for the bricklayers' process. However, these arguments are word by word valid for the zero range process. Formally we obtain the definitions, statements and proofs by simply neglecting all terms $r(-\omega_i)$, $r(-\zeta_i)$, $r(-\xi_i)$, $r(-\eta_i)$, $\mathrm{e}^{-\theta}$, $\mathrm{e}^{-\theta_1}$, $\mathrm{e}^{-\theta_2}$.

20 The process on a finite number of sites

20.1 The monotone process

First we show that the process on a finite number of sites does exist. Fix $n \in \mathbb{N}$, and define the infinitesimal generator $L^{(n)}$ acting on functions of $\underline{\omega}$:

$$\Big(L^{(n)}\varphi\Big)(\underline{\omega}) = \sum_{i=-n}^{n-1} [r(\omega_i) + r(-\omega_{i+1})] \cdot \Big[\varphi(\underline{\omega}^{(i,\,i+1)}) - \varphi(\underline{\omega})\Big] . \tag{79}$$

This is well defined for any $\underline{\omega} \in \Omega$. For this finite site-process, the jump $\underline{\omega} \to \underline{\omega}^{(i,\,i+1)}$ happens with rate $r(\omega_i) + r(-\omega_{i+1})$, independently for different sites i, but only for $-n \leq i \leq n-1$. For columns not in this interval, nothing happens.

Since even this finite site process has an infinite state space, we need to show that for each initial configuration $\underline{\omega}(0)$, the state $\underline{\omega}(t)$ evolving according to these rules is stochastically dominated. This is equivalent to showing that the heights $h_i(t)$ defined by (76) are stochastically dominated. We only need upper bound on them, since these are non-decreasing quantities. Let

$$H(t) := \max_{-n\leq j\leq n-1} h_j(t), \quad \text{and} \quad J := \{-n \leq j \leq n-1 \,:\, h_j(t) = H(t)\}.$$

Then we have two possibilities:

(i) $H(t) \leq h_{-n-1}(0) = h_{-n-1}(t)$ or $H(t) \leq h_n(0) = h_n(t)$ (these are the heights of the closest columns to the origin which do not grow), or

(ii) for each $j \in J$, $\omega_j \leq 0$ and $\omega_{j+1} \geq 0$, since $h_j(t)$ is by definition maximal.

In the first case, $H(t)$ is trivially dominated, while in the second case, by monotonicity of the rates r, for each $j \in J$ the column between sites j and $j+1$ has a growth rate dominated by $r(0) + r(0)$. This implies that whenever $H(t) \geq \max(h_{-n-1}(0), h_n(0))$, it grows according to a continuous time jump process with rate dominated by $2n \cdot (r(0)+r(0))$. Since $H(t)$ is maximal between the heights of the growing columns, this means that the growth of any column is dominated by this jump process.

We call this process defined on a finite number of sites *the n-monotone process*. Note that this type of bounds can not be used when we want to pass to the limit $n \to \infty$.

20.2 The stable process

We now define a slightly different model on a finite range of sites. For an integer $n > 0$ and a parameter θ, let us consider the generator

$$(80) \quad \left(G^{(n,\theta)}\varphi\right)(\underline{\omega}) = \sum_{i=-n}^{n-1} [r(\omega_i) + r(-\omega_{i+1})] \cdot \left[\varphi(\underline{\omega}^{(i,\,i+1)}) - \varphi(\underline{\omega})\right] +$$

$$+[\mathrm{e}^{\theta} + r(-\omega_{-n})] \cdot [\varphi(\underline{\omega}^{(-n-1,\,-n)}) - \varphi(\underline{\omega})] + [\mathrm{e}^{-\theta} + r(\omega_n)] \cdot [\varphi(\underline{\omega}^{(n,\,n+1)}) - \varphi(\underline{\omega})].$$

The difference between this model and the monotone process is that the leftmost bricklayer also has a (modified) rate for laying bricks on his left, and so has the rightmost one for laying bricks on his right. Their rates only depend on ω at their position. What we are doing here is simply replacing the effect of the $-n-1$ th and $n+1$ th bricklayers by the $\mu^{(\theta)}$-expectations of their rates. We can repeat the argument shown in section 20.1: the growth of any column is dominated by either $h_{n-2}(0)$, or $h_{n+1}(0)$, or by a Poisson process growing with rate $2n \cdot (r(0) + r(0)) + 2r(0) + \mathrm{e}^{\theta} + \mathrm{e}^{-\theta}$. We call this model the *(n, θ)-stable process*. The following proposition gives reason for this name:

Proposition 20.1. *The product-measure*

$$(81) \qquad \underline{\mu}^{(n,\theta)}(\underline{\omega}) := \prod_{i=-n}^{n} \mu^{(\theta)}(\omega_i)$$

is stationary for the (n, θ)-stable process, where the marginals are of the form (78).

Proof. Since the growth rates only depend on ω_i for $n \leq i \leq n$, it is sufficient to show that with the expectation w.r.t. $\underline{\mu}^{(n,\theta)}$ of (81),

$$\mathbf{E}^{(n,\theta)}\left((G^{(n,\theta)}\varphi)(\underline{\omega})\right) = 0$$

holds for any function φ depending on $\omega_{-n}, \omega_{-n+1}, \ldots, \omega_n$. We begin the computation of the left-hand side by changing variables in the (product-)expectation in order to obtain only $\varphi(\underline{\omega})$:

$$\mathbf{E}^{(n,\,\theta)}\left((G^{(n,\,\theta)}\varphi)(\underline{\omega})\right) =$$

$$= \mathbf{E}^{(n,\,\theta)}\Bigg\{\sum_{i=-n}^{n-1}[r(\omega_i)+r(-\omega_{i+1})]\cdot\Big[\varphi(\underline{\omega}^{(i,\,i+1)})-\varphi(\underline{\omega})\Big]+$$

$$+[\mathrm{e}^{\theta}+r(-\omega_{-n})]\cdot[\varphi(\omega_{-n}+1,\dots)-\varphi(\underline{\omega})]+[\mathrm{e}^{-\theta}+r(\omega_n)]\cdot[\varphi(\dots,\,\omega_n-1)-\varphi(\underline{\omega})]\Bigg\} =$$

$$= \mathbf{E}^{(n,\,\theta)}\Bigg\{\Bigg[\sum_{i=-n}^{n-1}\Big([r(\omega_i+1)+r(-\omega_{i+1}+1)]\cdot\frac{\mu^{(\theta)}(\omega_i+1)\cdot\mu^{(\theta)}(\omega_{i+1}-1)}{\mu^{(\theta)}(\omega_i)\cdot\mu^{(\theta)}(\omega_{i+1})}-$$

$$-r(\omega_i)-r(-\omega_{i+1})\Big)+$$

$$+[\mathrm{e}^{\theta}+r(-\omega_{-n}+1)]\cdot\frac{\mu^{(\theta)}(\omega_{-n}-1)}{\mu^{(\theta)}(\omega_{-n})}-[\mathrm{e}^{\theta}+r(-\omega_{-n})]+$$

$$+[\mathrm{e}^{-\theta}+r(\omega_n+1)]\cdot\frac{\mu^{(\theta)}(\omega_n+1)}{\mu^{(\theta)}(\omega_n)}-[\mathrm{e}^{-\theta}+r(\omega_n)]\Bigg]\cdot\varphi(\underline{\omega})\Bigg\}.$$

We can continue by using properties (78) of $\mu^{(\theta)}$ and then (75) of r:

$$\mathbf{E}^{(n,\,\theta)}\left((G^{(n,\,\theta)}\varphi)(\underline{\omega})\right) =$$

$$= \mathbf{E}^{(n,\,\theta)}\Bigg\{\Bigg[\sum_{i=-n}^{n-1}\Big([r(\omega_i+1)+r(-\omega_{i+1}+1)]\cdot\frac{r(\omega_{i+1})}{r(\omega_i+1)}-r(\omega_i)-r(-\omega_{i+1})\Big)+$$

$$+[\mathrm{e}^{\theta}+r(-\omega_{-n}+1)]\cdot\frac{r(\omega_{-n})}{\mathrm{e}^{\theta}}-[\mathrm{e}^{\theta}+r(-\omega_{-n})]+$$

$$+[\mathrm{e}^{-\theta}+r(\omega_n+1)]\cdot\frac{\mathrm{e}^{\theta}}{r(\omega_n+1)}-[\mathrm{e}^{-\theta}+r(\omega_n)]\Bigg]\cdot\varphi(\underline{\omega})\Bigg\}=$$

$$= \mathbf{E}^{(n,\,\theta)}\Bigg\{\Bigg[\sum_{i=-n}^{n-1}\Big(r(\omega_{i+1})+r(-\omega_i)-r(\omega_i)-r(-\omega_{i+1})\Big)+$$

$$r(\omega_{-n})+\mathrm{e}^{-\theta}-\mathrm{e}^{\theta}-r(-\omega_{-n})+r(-\omega_n)+\mathrm{e}^{\theta}-\mathrm{e}^{-\theta}-r(\omega_n)\Bigg]\cdot\varphi(\underline{\omega})\Bigg\}=0,$$

which completes the proof. □

20.3 Coupling the processes

By attractivity of the processes, we are able to couple them, as described below. Let $n > m$, and fix arbitrary initial configurations $\underline{\zeta}(0)$, $\underline{\omega}(0) \in \Omega$. The process $\underline{\zeta}$ evolves either according to the infinitesimal generator $L^{(n)}$ (79), or $G^{(n,\,\theta)}$ (80). The other one, $\underline{\omega}$ evolves according to $L^{(m)}$. We give the coupling between them via the following tables, rather than writing the complicated generator for the coupled process. In these tables, we assume generator $L^{(n)}$ for $\underline{\zeta}$, rather then $G^{(n,\,\theta)}$; the modification needed for $G^{(n,\,\theta)}$ is indicated before lemma 20.2.

As remarked in the introduction, it is enough to describe the possible moves and the associated rates for the bricklayers standing at a site i. Given a configuration $\underline{\zeta}$ and $\underline{\omega}$, the move of laying a brick to the left is independent of laying one to the right for each bricklayer, hence we give separate tables for these steps. We also distinguish between sites where both $\underline{\zeta}$ and $\underline{\omega}$ grow, and sites where only columns of $\underline{\zeta}$ grow. We denote the height of the column between i and $i+1$ by $g_i(t)$ (or $h_i(t)$) for the $\underline{\zeta}$ process (or the $\underline{\omega}$ process, respectively). We fix $g_0(0) = h_0(0) = 0$. We introduce the notation $d_i := \zeta_i - \omega_i$. As we are essentially interested in this quantity, we give separate columns in the tables as well to describe its behavior. Finally, let $\uparrow$ ($\downarrow$) mean that the corresponding quantity has increased (decreased, respectively) by one, e.g. $h_i \uparrow$ represents the move $\underline{\omega} \to \underline{\omega}^{(i,\, i+1)}$.

with rate	h_i	g_i	d_i	d_{i+1}
$r(\zeta_i)$		$\uparrow$	$\downarrow$	$\uparrow$

Table 7: Rate for bricklayers at sites $-n \le i < -m$ or $m \le i < n$ to lay brick on their right

with rate	h_{i-1}	g_{i-1}	d_{i-1}	d_i
$r(-\zeta_i)$		$\uparrow$	$\downarrow$	$\uparrow$

Table 8: Rate for bricklayers at sites $-n < i \le -m$ or $m < i \le n$ to lay brick on their left

with rate	h_i	g_i	d_i	d_{i+1}
$[r(\zeta_i) - r(\omega_i)]^+$		$\uparrow$	$\downarrow$	$\uparrow$
$[r(\omega_i) - r(\zeta_i)]^+$	$\uparrow$		$\uparrow$	$\downarrow$
$\min[r(\zeta_i),\, r(\omega_i)]$	$\uparrow$	$\uparrow$		

Table 9: Rates for bricklayers at sites $-m \le i < m$ to lay brick on their right

Note that considering the processes $\underline{\zeta}$ and $\underline{\omega}$ separately, each of them evolves according to its generator ($L^{(n)}$ and $L^{(m)}$, respectively). We say that d_i *number of (second class) particles are present at site* i if $d_i > 0$, and $-d_i$ *(second class) antiparticles are present at site* i if $d_i < 0$.

In case $\underline{\zeta}$ evolves according to $G^{(n,\,\theta)}$, then table 7 is also valid for $i = n$, but with rate $r(\zeta_n) + \mathrm{e}^{-\theta}$, and table 8 is valid for $i = -n$ with rate $r(-\zeta_n) + \mathrm{e}^{\theta}$.

Lemma 20.2. *For $n > m$ and for the coupled n-monotone or $(n,\,\theta)$-stable process $\underline{\zeta}$ and m-monotone process $\underline{\omega}$, started from initial configurations for which*

$$h_i(0) \le g_i(0)$$

holds initially for each $i \in \mathbb{Z}$, coupled as described above, $h_i(t) \le g_i(t)$ remains satisfied for any $i \in \mathbb{Z}$ and any $t \ge 0$.

with rate	h_{i-1}	g_{i-1}	d_{i-1}	d_i
$[r(-\omega_i) - r(-\zeta_i)]^+$	$\uparrow$		$\uparrow$	$\downarrow$
$[r(-\zeta_i) - r(-\omega_i)]^+$		$\uparrow$	$\downarrow$	$\uparrow$
$\min[r(-\omega_i),\, r(-\zeta_i)]$	$\uparrow$	$\uparrow$		

Table 10: Rates for bricklayers at sites $-m < i \le m$ to lay brick on their left

Proof. Observe that by monotonicity of r, $r(\zeta_i) > r(\omega_i)$ implies $\zeta_i > \omega_i$, and $r(\zeta_i) < r(\omega_i)$ implies $\zeta_i < \omega_i$. Hence the move given by the first line of table 9 can only happen for $\zeta_i > \omega_i$ i.e. $d_i > 0$. But this step reduces the number of particles at site i. It also increases the number of particles or reduces the number of antiparticles at site $i+1$. Thus this step describes either the jump of a particle from i to $i+1$ (in case $d_{i+1} \ge 0$), or the annihilation of a particle at i with an antiparticle at $i+1$ (in case $d_{i+1} < 0$). The same holds for the move according to the first line of table 10 with $i-1$ instead of $i+1$.

The step described in the second line of table 9 can only happen when $\zeta_i < \omega_i$, i.e. $d_i < 0$. As this step increases d_i by one and decreases d_{i+1} by one, it either describes the jump of an antiparticle from i to $i+1$, or annihilation of an antiparticle at i with a particle at $i+1$. The same holds for the second line of table 10 with $i-1$ instead of $i+1$. Hence we see that in the region $[-m \dots m]$ where both processes grow, particles or antiparticles are not created.

Now let us define

$$H_i(t) := g_i(t) - h_i(t),$$

which has the property $H_i(0) \ge 0$ initially for each i. Since for columns $-n-1 \le i < -m$ and $m \le i \le n$ only columns g of ζ can grow, $H_i(t) \ge 0$ holds for these indices. Consider now

$$\min_{-m \le i < m} H_i(t)$$

the minimum of H_i in the region where both processes grow. At $t = 0$, this minimum is clearly non-negative. Assume that at some time t this minimum is zero. For any site $-m \le i_{\min} < m$, where $H_{i_{\min}} = 0$ is achieved, we have

$$H_{i_{\min}-1} \ge H_{i_{\min}} \le H_{i_{\min}+1}, \quad \text{i.e.} \quad d_{i_{\min}} \ge 0 \ge d_{i_{\min}+1}.$$

Decrease of the minimum below zero would mean

$$H_{i_{\min}} \to H_{i_{\min}} - 1; \qquad d_{i_{\min}} \to d_{i_{\min}} + 1,\ d_{i_{\min}+1} \to d_{i_{\min}+1} - 1$$

for a site $i_{\min}$. Hence this would be a step by which a particle is created at site $i_{\min}$, and an antiparticle is created at site $i_{\min}+1$. As particle creation is not included in the coupling described in tables 9 and 10, and this pair is neither created according to tables 7 and 8, this step is simply not realized by the coupling rules. Hence $\min_i H_i(t) \ge 0$ is never violated, which shows $g_i(t) \ge h_i(t)$ for all i. ☐

21 The infinite volume limit

In this section we make connection between the monotone process and the stable process. While we have monotonicity described in lemma 20.2 for the monotone process, we have stationarity for the stable process, which gives us stochastic bounds for its growth. Since these bounds are independent of the size n of the stable process, our goal is now to construct and then to dominate the limit of the monotone processes by these bounds.

21.1 Starting from good distributions

First we consider the stable process, started from an appropriate initial distribution. We say that a measure $\underline{\pi}$ on Ω is a *good measure with parameters* θ_1 *and* θ_2, if there exist $-\bar{\theta} < \theta_1 < \theta_2 < \bar{\theta}$ such that the measure $\underline{\mu}^{(\theta_2)}$ dominates $\underline{\pi}$ and $\underline{\pi}$ dominates $\underline{\mu}^{(\theta_1)}$ stochastically. This is equivalent to saying that $\underline{\eta}$ distributed according to the product measure $\underline{\mu}^{(\theta_1)}$, $\underline{\zeta}$ distributed according to $\underline{\pi}$ and $\underline{\xi}$ distributed according to $\underline{\mu}^{(\theta_2)}$ can be coupled in such a way that

$$\eta_i \le \zeta_i \le \xi_i$$

holds for all $i \in \mathbb{Z}$. Note that if $\underline{\pi}$ is a product of marginals π_i on $\mathbb{Z}$, then this is equivalent to the corresponding stochastic dominations for the marginals at each site i.

Lemma 21.1. *Let $\underline{\zeta}(0)$ be distributed according to the good measure $\underline{\pi}$ with parameter θ_1 and θ_2, and let it evolve according to the $(n,\,\theta_1)$-stable evolution. Then the stochastic bounds*

$$\eta_i(t) \le \zeta_i(t) \le \xi_i(t)$$

hold for some random processes $\eta_i(t)$ and $\xi_i(t)$, having distributions $\mu^{(\theta_1)}$ and $\mu^{(\theta_2)}$, respectively.

Proof. By the coupling assumed in the definition of $\underline{\pi}$, we couple $\underline{\zeta}$ with the $(n,\,\theta_2)$-stable process $\underline{\xi}$ started from initial distribution $\underline{\mu}^{(\theta_2)}$, such that $\xi_i(0) \ge \zeta_i(0)$ holds initially for all $-n \le i \le n$. We show the coupling which preserves this inequality for all time $t > 0$. We denote the height of $\underline{\zeta}$ and $\underline{\xi}$ by g and f, respectively; the number of second class particles is $d_i := \xi_i - \zeta_i$. We do not have antiparticles when starting the processes. We rewrite tables 9 and 10 for inner sites to tables 11 and 12 with the present notations. For sites n and $-n$, where we modified the rates, the process is coupled according to tables 13 and 14.

with rate	g_i	f_i	d_i	d_{i+1}
$r(\xi_i) - r(\zeta_i)$		$\uparrow$	$\downarrow$	$\uparrow$
$r(\zeta_i)$	$\uparrow$	$\uparrow$		

Table 11: Rates for bricklayers at sites $-n \le i < n$ to lay brick on their right

These tables are valid while $\xi_i \ge \zeta_i$ holds for all $-n \le i \le n$. However, they preserve this condition as no antiparticles are created according to these tables. Decrease of d_i can only happen where $\xi_i > \zeta_i$, i.e. for sites where there is particle to jump from.

with rate	g_{i-1}	f_{i-1}	d_{i-1}	d_i
$r(-\zeta_i) - r(-\xi_i)$	$\uparrow$		$\uparrow$	$\downarrow$
$r(-\xi_i)$	$\uparrow$	$\uparrow$		

Table 12: Rates for bricklayers at sites $-n < i \le n$ to lay brick on their left

with rate	g_n	f_n	d_n
$r(\xi_n) - r(\zeta_n)$		$\uparrow$	$\downarrow$
$\mathrm{e}^{-\theta_1} - \mathrm{e}^{-\theta_2}$	$\uparrow$		$\uparrow$
$r(\zeta_n) + \mathrm{e}^{-\theta_2}$	$\uparrow$	$\uparrow$	

Table 13: Rates for the bricklayer at site n to lay brick on his right

Now, by the same method, we couple $\underline{\zeta}$ and $\underline{\eta}$, where the initial distribution of $\underline{\eta}$ is $\underline{\mu}^{(\theta_1)}$, and both models evolve according to the (n, θ_1)-stable generator. Writing η instead of ζ, ζ instead of ξ, and θ_1 in place of θ_2 as well makes us possible to repeat our arguments and to conclude $\zeta_i(t) \ge \eta_i(t)$ for all sites $-n \le i \le n$ and $t > 0$. Hence we see that $\eta_i(t) \le \zeta_i(t) \le \xi_i(t)$, where $\eta_i(t)$ and $\xi_i(t)$ have distributions $\mu^{(\theta_1)}$ and $\mu^{(\theta_2)}$, respectively, as these are processes started and evolving in their stationary distributions. $\square$

Lemma 21.2. *Let $\underline{\zeta}(0)$ be distributed according to the good measure $\underline{\pi}$ with parameter θ_1 and θ_2, and let it evolve according to the (n, θ_1)-stable evolution. Then we have*

$$\mathbf{E}\left[g_i(t) - g_i(0)\right] \le t \cdot \left(\mathrm{e}^{\theta_2} + \mathrm{e}^{-\theta_1}\right)$$

for its column's growth ($-n-1 \le i \le n$).

Proof. Let

$$r_i(\underline{\zeta}) := \begin{cases} \mathrm{e}^{\theta_1} + r(-\zeta_{-n}) & \text{, for } i = -n-1, \\ r(\zeta_i) + r(-\zeta_{i+1}) & \text{, for } -n \le i < n, \\ \mathrm{e}^{-\theta_1} + r(\zeta_n) & \text{, for } i = n \end{cases} \tag{82}$$

be the growth rate in the (n, θ_1)-stable process $\underline{\zeta}$ for column i. For its columns' growth, consider

$$M_i(t) := g_i(t) - g_i(0) - \int_0^t r_i(\underline{\zeta}(s))\, \mathrm{d}s, \tag{83}$$

which is a martingale w.r.t. the filtration generated by $\left(\underline{\zeta}(s)\right)_{0 \le s \le t}$ with $M_i(0) = 0$. By the previous lemma, $\zeta_i(t)$ is bounded by $\eta_i(t)$ and $\xi_i(t)$, respectively. Due to monotonicity or r, this means

$$r_i(\underline{\zeta}) \le R_i(\underline{\eta}, \underline{\xi}) := \begin{cases} \mathrm{e}^{\theta_1} + r(-\eta_{-n}) & \text{, for } i = -n-1, \\ r(\xi_i) + r(-\eta_{i+1}) & \text{, for } -n \le i < n, \\ \mathrm{e}^{-\theta_1} + r(\xi_n) & \text{, for } i = n. \end{cases} \tag{84}$$

with rate	g_{-n-1}	f_{-n-1}	d_{-n}
$r(-\zeta_{-n}) - r(-\xi_{-n})$	$\uparrow$		$\downarrow$
$e^{\theta_2} - e^{\theta_1}$		$\uparrow$	$\uparrow$
$r(-\xi_{-n}) + e^{\theta_1}$	$\uparrow$	$\uparrow$	

Table 14: Rates for the bricklayer at site $-n$ to lay brick on his left

Hence by (83),

$$M_i(t) \geq g_i(t) - g_i(0) - \int_0^t R_i(\underline{\eta}(s), \underline{\xi}(s)) \, ds.$$

As $\eta_i(t)$ and $\xi_i(t)$ has (stationary) distributions $\mu^{(\theta_1)}$ and $\mu^{(\theta_2)}$, respectively, taking expectation value of this inequality leads to

$$0 \geq \mathbf{E}\left[g_i(t) - g_i(0)\right] - t \cdot \left(e^{\theta_2} + e^{-\theta_1}\right) \tag{85}$$

for columns $-n-1 \leq i \leq n$, where we used

$$\mathbf{E}^{(\theta_2)}(r(\xi_i)) = e^{\theta_2}, \quad \mathbf{E}^{(\theta_1)}(r(-\eta_i)) = e^{-\theta_1}.$$

□

For later use, we prove here a similar bound for the second moment of g. Much more detailed analysis is available for this quantity in equilibrium in [3].

Lemma 21.3. *Let $\underline{\zeta}(0)$ be distributed according to the good measure $\underline{\pi}$ with parameter θ_1 and θ_2, and let it evolve according to the (n, θ_1)-stable evolution. Then for all t large enough, we have*

$$\mathbf{E}\left(\left[g_i(t) - g_i(0)\right]^2\right) \leq \frac{9}{4}\mathbf{E}\left[R_i^2\right] \cdot (t - t_0)^2$$

for its column's growth ($-n-1 \leq i \leq n$) for some t_0, with R_i defined in (84).

Proof. We introduce

$$\widetilde{g}_i(t) := g_i(t) - g_i(0).$$

With the notations of (82), the quantity

$$N_i(t) := \widetilde{g}_i^2(t) - \int_0^t \left[2\widetilde{g}_i(s) \cdot r_i(s) + r_i(s)\right] ds$$

is a martingale. Hence

$$\mathbf{E}\left[\widetilde{g}_i^2(t)\right] = \int_0^t \left(2\mathbf{E}[\widetilde{g}_i(s) \cdot r_i(s)] + \mathbf{E}[r_i(s)]\right) ds \leq$$

$$\leq \int_0^t \left(2\sqrt{\mathbf{E}\left[\widetilde{g}_i^2(s)\right]} \cdot \sqrt{\mathbf{E}\left[r_i^2(s)\right]} + \mathbf{E}[r_i(s)]\right) ds.$$

Using the bounds given in (84) and stationarity of $\underline{\eta}$ and $\underline{\xi}$, the second moments of the rates can be dominated, and we can write

$$\mathbf{E}\left[\widetilde{g}_i^2(t)\right] \le 2\sqrt{\mathbf{E}\left[R_i^2\right]}\int_0^t \sqrt{\mathbf{E}\left[\widetilde{g}_i^2(s)\right]}\,\mathrm{d}s + t\cdot\mathbf{E}[R_i].$$

The quantity $\mathbf{E}\left[R_i^2\right]$ contains second moments of the rates w.r.t. $\mu^{(\theta_1)}$ or $\mu^{(\theta_2)}$ (78). The existence of these moments can easily be shown, but we do not obtain an explicit formula for them. The left-hand side of the previous equation can be bounded from above by a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = 2\sqrt{\mathbf{E}\left[R_i^2\right]}\cdot\sqrt{x(t)} + \mathbf{E}[R_i].$$

Whenever the increasing function $x(t)$ reaches value 1, its further evolution is dominated by a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = 3\sqrt{\mathbf{E}\left[R_i^2\right]}\cdot\sqrt{y(t)},\quad \text{i.e.}\quad y(t) = \frac{9}{4}\mathbf{E}\left[R_i^2\right]\cdot(t-t_0)^2$$

for some t_0. □

Theorem 21.4. *Let $\underline{\omega}(0)$ be distributed according to the good distribution $\underline{\pi}$ with parameter θ_1, θ_2, and let it evolve according to the n-monotone evolution. The height of column i at time t is denoted by $h_i^{(n)}(t)$, with the convention $h_0^{(n)}(0) = 0$. Then for all $i \in \mathbb{Z}$, $t > 0$, the limit*

$$h_i(t) := \lim_{n\to\infty} h_i^{(n)}(t)$$

exists for a.s. each $\underline{\omega}(0) \in \Omega$,

$$\mathbf{E}\left[h_i(t) - h_i(0)\right] \le t\cdot\left(\mathrm{e}^{\theta_2} + \mathrm{e}^{-\theta_1}\right)$$

for all t, and

$$\mathbf{E}\left(\left[h_i(t) - h_i(0)\right]^2\right) \le \frac{9}{4}\mathbf{E}\left[R_i^2\right]\cdot(t-t_0)^2$$

for all t large enough, with some parameter t_0.

Proof. By lemma 20.2, for each starting configuration of the n-monotone process $\underline{\omega}(t)$, there is a bounding (n, θ_1)-stable process $\underline{\zeta}(t)$ with column heights $g^{(n)}$, for which $\underline{\zeta}(0) = \underline{\omega}(0)$ and

$$h_i^{(n)}(0) = g_i^{(n)}(0), \qquad h_i^{(n)}(t) \le g_i^{(n)}(t)$$

holds ($-n \le i < n$). As $\underline{\zeta}(0)$ is distributed according to $\underline{\pi}$, lemma 21.2 leads to the inequality

$$\mathbf{E}\left[h_i^{(n)}(t) - h_i^{(n)}(0)\right] \le \mathbf{E}\left[g_i^{(n)}(t) - g_i^{(n)}(0)\right] \le t\cdot\left(\mathrm{e}^{\theta_2} + \mathrm{e}^{-\theta_1}\right). \tag{86}$$

Similarly, lemma 21.3 yields

$$\mathbf{E}\left(\left[h_i^{(n)}(t) - h_i^{(n)}(0)\right]^2\right) \le \frac{9}{4}\mathbf{E}\left[R_i^2\right]\cdot(t-t_0)^2 \tag{87}$$

for large t's. We can also use lemma 20.2 to show that $h_i^{(n)}(t)$ is monotone in n. Hence the limit

$$h_i(t) := \lim_{n\to\infty} h_i^{(n)}(t)$$

exists, and by taking liminf of (86) and (87) leads to similar bounds for $h_i(t)$ via Fatou's lemma. □

Proposition 21.5. *Let $\underline{\omega}(0)$ be the common initial state for the n-monotone processes $\underline{\omega}^{(n)}$, distributed according to the good distribution $\underline{\pi}$ with parameter θ_1, θ_2. Then the distribution $\underline{\pi}^t$ of the limit $\underline{\omega}(t)$ of the processes at time t is again a good measure having parameter θ_1, θ_2.*

Proof. Consider the coupling described in the proof of the previous theorem. The following inequalities hold:

$$\begin{aligned}(88)\quad & \mathbf{P}\{\omega_i(t) \neq \zeta_i^{(n)}(t)\} \leq \\ & \leq \mathbf{P}\{\omega_i(t) \neq \omega_i^{(n)}(t)\} + \mathbf{P}\{\omega_i^{(n)} \neq \zeta_i^{(n)}\} = \mathbf{P}\{\omega_i(t) \neq \omega_i^{(n)}(t)\} + \\ & + \mathbf{P}\{\text{there is second class particle at site } i \text{ between } \underline{\omega}^{(n)}(t) \text{ and } \underline{\zeta}^{(n)}(t)\}.\end{aligned}$$

Since $\omega_i^{(n)}(t) \to \omega_i(t)$ as $n \to \infty$, the first term tends to zero. As $\underline{\zeta}^{(n)}$ and $\underline{\omega}^{(n)}$ are started from the same initial state and their growth rates agree in $[-n,\, n-1] \subset \mathbb{Z}$, second class particles only come from the two ends of this interval. Uniformly positive probability of their arrival at site i as n grows would imply larger and larger jump rates for them, as they travel longer and longer distances by time t. But this would contradict the stochastic bounds obtained before for $\underline{\zeta}^{(n)}$ and $\underline{\omega}^{(n)}$, which were independent of n. Thus we see that the left hand-side of (88) tends to zero as $n \to \infty$. Hence $\zeta_i^{(n)}(t)$ also converges a.s. to $\omega_i(t)$. Now, by lemma 21.1 we know that $\underline{\zeta}^{(n)}$ is sandwiched by the n-stable processes $\underline{\eta}^{(n)}$ and $\underline{\xi}^{(n)}$ having marginals $\underline{\mu}^{(\theta_1)}$ and $\underline{\mu}^{(\theta_2)}$, respectively, which shows that $\omega_i(t)$ can be coupled to some random variables having the desired distributions. □

21.2 Starting from a fixed configuration

So far we are able to construct the bricklayers' process starting from good distributions. In this part, we refine the results in order to make the construction when the process starts from a deterministically given initial state $\underline{\omega}(0) \in \Omega = \mathbb{Z}^{\mathbb{Z}}$. Of course, we shall not allow all elements of this set. For $\underline{\omega}$ fixed, we define the set

$$A(\underline{\omega}) := \left\{(\underline{\zeta},\, g_0(0)) \in \Omega \times \mathbb{N} \,:\, g_i(0) \geq h_i(0) \text{ for all } i \in \mathbb{Z}\right\}$$

with columns

$$h_i := \begin{cases} h_0 - \sum_{j=1}^{i} \omega_j & \text{for } i > 0, \\ h_0 + \sum_{j=i+1}^{0} \omega_j & \text{for } i < 0 \end{cases}$$

of $\underline{\omega}$ and

$$g_i := \begin{cases} g_0 - \sum_{j=1}^{i} \zeta_j & \text{for } i > 0, \\ g_0 + \sum_{j=i+1}^{0} \zeta_j & \text{for } i < 0 \end{cases}$$

of another configuration $\underline{\zeta}$. We assume $h_0(0) = 0$ initially. Imagining the wall of bricks, a typical $\underline{\zeta}$ has larger negative gradient on the left-hand side than on the right-hand side of the origin.

For a measure $\underline{\Pi}$ on $\Omega \times \mathbb{N}$ we call the first marginal $\underline{\pi}$ on Ω, while the second marginal ν on $\mathbb{N}$. We define

$$\widetilde{\Omega} := \{\underline{\omega} \in \Omega \,:\, \text{there exists } \underline{\Pi} \text{ for which } \underline{\pi} \text{ is a good measure,} \\ \nu \text{ has finite second moment and } \underline{\Pi}\,\{A(\underline{\omega})\} > 0\}.$$

Note that one has many choice for $\underline{\Pi}$, it is not unique. $\widetilde{\Omega}$ is going to be the set of initial configurations for which we shall construct the process. Later we shall show that this set is large enough to include many configurations we are interested in. In particular, $\underline{\mu}^{(\theta)}\{\widetilde{\Omega}\} = 1$ for any $\underline{\theta} < \theta < \bar{\theta}$.

Theorem 21.6. *Let us fix $\underline{\omega}(0) \in \widetilde{\Omega}$ with $\underline{\Pi}$, of which the first marginal $\underline{\pi}$ is a good measure having parameters θ_1 and θ_2. Let $\underline{\omega}$ evolve according to the n-monotone evolution; the height of column i at time t is denoted by $h_i^{(n)}(t)$, with the convention $h_0^{(n)}(0) = 0$. Then for all $i \in \mathbb{Z}$, $t > 0$, the limit*

$$h_i(t) := \lim_{n\to\infty} h_i^{(n)}(t)$$

exists, and

$$\mathbf{E}\,[h_i(t)] \leq \frac{3 \cdot \sqrt{\mathbf{E}\,[R_i^2]} \cdot (t - t_0)}{2 \cdot \sqrt{\underline{\pi}\{A\}}} + \sqrt{\frac{\mathbf{E}\left([g_i(0)]^2\right)}{\underline{\Pi}\{A\}}}$$

for all large t and for some t_0, where $g_i(0)$ is the height of column i for $\underline{\zeta}(0)$ distributed according to $\underline{\pi}$, with $g_0(0)$ having distribution ν.

Proof. Given $\underline{\omega}(0)$, we drop $\underline{\zeta}(0)$ according to the good measure $\underline{\pi}$, and $g_0(0)$ according to ν. This determines $g_i(0)$ for all i, and leads either to the event A or $\bar{A}$. As $\underline{\omega}(0) \in \widetilde{\Omega}$, the ($\underline{\Pi}$-)probability of A is strictly larger than zero. Since $\underline{\omega}(0)$ is fixed and A only depends on the initial distribution of $\underline{\zeta}$ and g_0, the evolution of $\underline{\omega}$ does not depend on this event. Conditioned on A, we can use the coupling described in lemma 20.2 for the n-monotone process $\underline{\omega}$ and the (n, θ_1)-stable process $\underline{\zeta}$ for their columns $h^{(n)}$ and $g^{(n)}$, respectively:

$$\mathbf{E}\left(h_i^{(n)}(t)\right) = \mathbf{E}\left(h_i^{(n)}(t) \,\middle|\, A\right) \leq \mathbf{E}\left(g_i^{(n)}(t) \,|\, A\right) = \\ = \mathbf{E}\left(g_i^{(n)}(t) - g_i(0) \,|\, A\right) + \mathbf{E}\left(g_i(0) \,|\, A\right).$$

We continue by Cauchy's inequality:

$$\begin{aligned}
&\mathbf{E}\left(g_i^{(n)}(t)-g_i(0)\,|\,A\right)+\mathbf{E}\left(g_i(0)\,|\,A\right)=\\
&\qquad=\frac{\mathbf{E}\left(\left[g_i^{(n)}(t)-g_i(0)\right]\cdot\mathbf{1}\{A\}\right)}{\underline{\Pi}\{A\}}+\frac{\mathbf{E}\left([g_i(0)]\cdot\mathbf{1}\{A\}\right)}{\underline{\Pi}\{A\}}\le\\
&\qquad\le\frac{\sqrt{\mathbf{E}\left(\left[g_i^{(n)}(t)-g_i(0)\right]^2\right)\cdot\underline{\Pi}\{A\}}}{\underline{\Pi}\{A\}}+\frac{\sqrt{\mathbf{E}\left([g_i(0)]^2\right)\cdot\underline{\Pi}\{A\}}}{\underline{\Pi}\{A\}}=\\
&\qquad=\sqrt{\frac{\mathbf{E}\left(\left[g_i^{(n)}(t)-g_i(0)\right]^2\right)}{\underline{\Pi}\{A\}}}+\sqrt{\frac{\mathbf{E}\left([g_i(0)]^2\right)}{\underline{\Pi}\{A\}}}.
\end{aligned}$$

As $g_i(0)$ can be written as the sum of finitely many elements in L^2 (including the random variable $g_0(0)$), the second term is clearly finite. For the first term we can apply lemma 21.3 and finally write

$$\mathbf{E}\left(h_i^{(n)}(t)\right)\le\frac{\frac{3}{2}\sqrt{\mathbf{E}\left[R_i^2\right]}\cdot(t-t_0)}{\sqrt{\underline{\pi}\{A\}}}+\sqrt{\frac{\mathbf{E}\left([g_i(0)]^2\right)}{\underline{\Pi}\{A\}}}.$$

Note that the right-hand side does not depend on n if $n>|i|$. As in theorem 21.4, we can finish the proof by monotonicity of $h_i^{(n)}$ in n and by Fatou's lemma. □

Proposition 21.7. *Assume $\underline{\omega}(0)\in\widetilde{\Omega}$ with $\underline{\Pi}$, of which the first marginal is a good measure $\underline{\pi}^0$ having parameters $\theta_1<\theta_2$, and the second marginal is ν. Then letting $\underline{\omega}$ evolve according to the n-monotone rules and then taking the limit as $n\to\infty$, we have $\underline{\omega}(t)\in\widetilde{\Omega}$ with $\underline{\Pi}^t$, of which the first marginal is a good measure $\underline{\pi}^t$ having the same parameters $\theta_1<\theta_2$, and the second marginal is ν^t having finite second moment. Moreover, we have $\underline{\Pi}^t\left\{A(\underline{\omega}(t))\right\}\ge\underline{\Pi}^0\left\{A(\underline{\omega}(0))\right\}$.*

Proof. Given $\underline{\omega}(0)$ and $\underline{\Pi}^0$, we choose $\underline{\zeta}(0)$ and $g_0(0)$ according to $\underline{\Pi}^0$, and we let $\underline{\zeta}^{(n)}$ evolve by the n-monotone evolution, while $\underline{\omega}^{(m)}$ evolves according to the m-monotone rules. Due to the coupling shown in lemma 20.2 and the definition of $A(\underline{\omega})$, $(\underline{\zeta}(0),\, g_0(0))\in A(\underline{\omega}(0))$ implies $(\underline{\zeta}^{(n)}(t),\, g_0^{(n)}(t))\in A(\underline{\omega}^{(m)}(t))$ for the coupled pair $(\underline{\zeta}^{(n)},\,\underline{\omega}^{(m)})$ if $n>m$. By passing to the limit $n\to\infty$, we obtain the process $\underline{\zeta}$, which is by proposition 21.5 in a $\underline{\pi}^t$ good distribution, and for which $g_0(t)$ has finite second moment by theorem 21.4. As this process is the monotone limit of $\underline{\zeta}^{(n)}$, $(\underline{\zeta}(0),\, g_0(0))\in A(\underline{\omega}(0))$ implies $(\underline{\zeta}(t),\, g_0(t))\in A(\underline{\omega}^{(m)}(t))$ for all m. But this also shows that $(\underline{\zeta}(t),\, g_0(t))\in A(\underline{\omega}(t))$ after taking m to infinity, which completes the proof with $\underline{\Pi}^t$ generated by $(\underline{\zeta}(t),\, g_o(t))$. □

Proposition 21.8. *Fix $-\bar{\theta}<\theta_1<\theta_2<\bar{\theta}$ and $\mathbf{E}^{(\theta_1)}(z)<K_1<K_2<\mathbf{E}^{(\theta_2)}(z)$. Then*

$$\left\{\underline{\omega}\,:\,K_2>\limsup_{n\to-\infty}\frac{1}{|n|}\sum_{i=n+1}^{0}\omega_i\quad,\quad\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\omega_i>K_1\right\}\subset\widetilde{\Omega}.$$

Proof. Let $\underline{\pi}$ be the product of marginals

$$\pi_i(z) = \begin{cases} \mu^{(\theta_2)} \text{ , for } i \le 0, \\ \mu^{(\theta_1)} \text{ , for } i \ge 1, \end{cases}$$

and $g_0(0) = 0$. With a fixed $\underline{\omega}$ from the set described above, let $\underline{\zeta}$ have distribution $\underline{\pi}$. By the assumption on $\underline{\omega}$, there is a number $N > 0$ such that

$$\begin{cases} K_2 > \frac{1}{|n|} \sum_{i=n+1}^{0} \omega_i = \frac{h_n}{|n|} \text{ for } n < -N, \\ K_1 < \frac{1}{n} \sum_{i=1}^{n} \omega_i = -\frac{h_n}{n} \text{ for } n > N. \end{cases}$$

It is clear that with positive probability, $g_n \ge h_n$ happens for all $-N \le n \le N$. We show that, with positive probability, this also happens for $n < -N$ and $n > N$. This implies positive probability of the event A w.r.t. the measure $\underline{\pi} \times \delta_0$, hence the inclusion of $\underline{\omega}$ in $\widetilde{\Omega}$. For positive n's, due to the previous inequalities, it is enough to show that with positive probability, $g_n + K_1 \cdot n \ge 0$ for all $n > N$. But the latter is a drifted random walk, of which the increments $-\zeta_i + K_1$ are i.i.d. random variables having positive expectation by the the assumption on K_1. Moreover, by properties of μ^{θ_1} (78), the expectation

$$\mathbf{E}^{\theta_1} \left(e^{\lambda \cdot (-\zeta_i + K)} \right)$$

is finite for small λ's. Hence large deviation arguments and Borel-Cantelli is applicable to show that this random walk only hits zero finitely times a.s. thus has positive probability of never returning to zero. Similar argument works for $n < -N$ as well. □

Remark 21.9. *By the same arguments, for any $\underline{\theta} < \theta < \bar{\theta}$, one can find parameters θ_1, θ_2 such that the set in the previous proposition has $\underline{\mu}^{(\theta)}$ measure one, hence $\underline{\mu}^{(\theta)} \left\{ \widetilde{\Omega} \right\} = 1$ holds.*

Acknowledgement

The author is grateful to Bálint Tóth for extremely helpful conversations concerning the arguments in this paper.

References

[1] E. D. Andjel. Invariant measures for the zero range process. *The Annals of Prob.*, 10(3):325–547, 1982.

[2] M. Balázs. Microscopic shape of shocks in a domain growth model. *Journal of Stat. Phys.*, 105(3/4):511–524, 2001.

[3] M. Balázs. Growth fluctuations in a class of deposition models. *To appear in Annales de l'Institut Henri Poincaré*, 2003.

[4] L. Booth. *Random Spatial Structures and Sums.* PhD thesis, Utrecht University, 2002.

[5] A. De Masi, C. Kipnis, E. Presutti, and E. Saada. Microscopic structure at the shock in the asymmetric simple exclusion. *Stochastics*, 27:151–165, 1988.

[6] B. Derrida, J. L. Lebowitz, and E. R. Speer. Shock profiles for the asymmetric simple exclusion process in one dimension. *Journal of Stat. Phys.*, 89(1-2):135–167, 1997.

[7] P. A. Ferrari. Shock fluctuations in asymmetric simple exclusion. *Probab. Theory Relat. Fields*, 91:81–102, 1992.

[8] P. A. Ferrari and L. R. G. Fontes. Current fluctuations for the asymmetric simple exclusion process. *The Annals of Probability*, 22:820–832, 1994.

[9] P. A. Ferrari, L. R. G. Fontes, and Y. Kohayakawa. Invariant measures for a two-species asymmetric process. *Journal of Stat. Phys.*, 76:1153–1177, 1994.

[10] J. Gärtner and E. Presutti. Shock fluctuations in a particle system. *Ann. Inst. Henri Poincaré*, 53:1–14, 1990.

[11] J. Lamperti. *Probability: a survey of the mathematical theory.* Benjamin, 1966.

[12] T. M. Liggett. Existence theorems for infinite particle systems. *Trans. Am. Math. Soc.*, 165:471–481, 1972.

[13] T. M. Liggett. An infinite particle system with zero range interactions. *The Annals of Prob.*, 1(2):240–253, 1973.

[14] T. M. Liggett. Coupling the simple exclusion process. *Ann. Probab.*, 4:339–356, 1976.

[15] T. M. Liggett. *Interacting particle systems.* Springer-Verlag, 1985.

[16] T. M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes.* Springer-Verlag, 1999.

[17] J. Neveu. *Mathematical foundations of the calculus of probability.* Holden-Day, San Francisco, London, Amsterdam, 1965.

[18] C. Kipnis P. A. Ferrari. Second class particles in the rarefaction fan. *Ann. Inst. H. Poincaré Probab. Statist.*, 31(1):143–154, 1995.

[19] S. C. Port and C. J. Stone. Infinite particle systems. *Trans. Am. Math. Soc.*, 178:307–340, 1973.

[20] M. Prähofer and H. Spohn. Current fluctuations for the totally asymmetric simple exclusion process. In V. Sidoravicius, editor, *Progress of probab.: In and out equilibrium; probability with a physics flavor*, volume 51. Birkhäuser, 2002.

[21] C. Quant. *On the construction and stationary distributions of some spatial queueing and particle systems*. PhD thesis, Utrecht University, 2002.

[22] F. Rezakhanlou. Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. *Comm. Math. Phys.*, 140(3):417–448, 1991.

[23] F. Rezakhanlou. Microscopic structure of shocks in one conservation laws. *Ann Inst. H Poincaré Anal. Non Linéaire*, 12(2):119–153, 1995.

[24] T. Seppäläinen. Coupling the totally asymmetric simple exclusion process with a moving interface. *Markov Process. Related Fields*, 4(4):593–628, 1998.

[25] T. Seppäläinen. Existence of hydrodynamics for the totally asymmetric simple K-exclusion process. *Ann. Probab.*, 27(1):361–415, 1999.

[26] S. Sethuraman. On extremal measures for conservative particle systems. *Ann. Inst. H. Poincaré*, 37(2):139–154, 2001.

[27] J. Smoller. *Shock Waves and Reaction-Diffusion Equations.* Springer-Verlag, 1983.

[28] F. Spitzer. Interaction of markov processes. *Adv. in Math.*, 5:246–290, 1970.

[29] B. Tóth and B. Valkó. Between equilibrium fluctuations and eulerian scaling: perturbation of equilibrium for a class of deposition models. *Journal of Stat. Phys.*, 109(1/2):177–205, 2002.

[30] B. Tóth and W. Werner. Hydrodynamic equations for a deposition model. In V. Sidoravicius, editor, *Progress of probab.: In and out equilibrium; probability with a physics flavor*, volume 51. Birkhäuser, 2002.