Analytic Number Theory
Lecture Notes
Taught Course Centre 2007

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Part A
Basic Topics in Analytic Number Theory

1 Introduction

1.1 What is Analytic Number Theory?

A basic human instinct is to count interesting objects. For example:

- Prime numbers less than some bound
$$\pi(x) := \#\{p \in \mathbb{N} : p \text{ prime and } p \leq x\}.$$
- What is
$$R_{s,d}(N) := \#\{(x_1, \dots, x_s) \in \mathbb{N}^s : x_1^d + \cdots + x_s^d = N\} \ ?$$
- For which s, d do we have
$$R_{s,d}(N) > 0 \ ?$$
- Lagrange (1770):
$$R_{1,2}(N) + R_{2,2}(N) + R_{3,2}(N) + R_{4,2}(N) > 0 \quad \text{for all } N \in \mathbb{N}.$$

Often 'non-exact' answers are sufficient, which is where analysis enters the picture. It is not always clear why or how analysis should help with theorems like:

- Every odd number greater than 10^{1347} is a sum of three primes (Vinogradov).
- The sequence of primes contains arbitrarily long arithmetic progressions (Green and Tao).

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[*] All errors are the responsibility of the typesetter. In particular there are some arguments which, as an exercise for the typesetter, have been 'fleshed out' or re-interpreted, possibly incorrectly. Tim's lectures were neater and more concise. Corrections would be gratefully received at *sean.prendiville@bristol.ac.uk*

1.2 Notation

Let $f, g : \mathbb{R} \to \mathbb{C}$ with g non-negative. Then we say:

(1) $f = O(g)$, or $f \ll g$, if there exists x_0 and $C > 0$ such that for all $x \geq x_0$

$$|f(x)| \leq Cg(x).$$

(2) $f \gg g$ if f is also non-negative and $g \ll f$.

(3) $f \sim g$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

(4) $f \asymp g$ if $f \ll g \ll f$.

(5) $f = o(g)$ if for any $\varepsilon > 0$ there exists $X \in \mathbb{R}$ such that for all $x \geq X$

$$|f(x)| \leq \varepsilon g(x).$$

So $f = o(g)$ certainly when

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

2 Arithmetic Functions

An **arithmetic funciton** is any function $f : \mathbb{N} \to \mathbb{C}$. It is said to be **multiplicative** if it is not identically zero and for any $m, n \in \mathbb{N}$ with $(m, n) = 1$

$$f(mn) = f(m)f(n). \tag{1}$$

f is said to be **completely multiplicative** if (1) holds even when m and n are not coprime.

Lemma 1. *If f, g are multiplicative, then $f * g : \mathbb{N} \to \mathbb{C}$ defined by*

$$f * g(n) := \sum_{d|n} f(d)g(n/d).$$

is also multiplicative.

Proof. Let $(m, n) = 1$. Then every divisor $d|mn$ can be written as a product $d = ab$ with $a|m$ and $b|n$. In particular

$$(a, b) = \left(\frac{m}{a}, \frac{n}{b}\right) = 1.$$

It can be checked that this induces a bijection

$$\{d \in \mathbb{N} : d|mn\} \to \{a \in \mathbb{N} : a|m\} \times \{b \in \mathbb{N} : b|n\},$$

given by $d = ab \mapsto (a, b)$. Hence

$$\begin{aligned} f * g(mn) &= \sum_{\substack{a|m \\ b|n}} f(ab) g\left(\frac{mn}{ab}\right) \\ &= \sum_{a|m} \sum_{b|n} f(a) g\left(\frac{m}{a}\right) f(b) g\left(\frac{n}{b}\right) \\ &= \big(f * g(m)\big)\big(f * g(n)\big). \end{aligned}$$

□

Remark. *The operation* $*$ *is called* **Dirichlet convolution**.

In fact the set of all multiplicative arithmetic functions, equipped with $*$, forms an abelian group with identity I defined by

$$I(n) = \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note. *If f is multiplicative then*

$$f(1) = 1,$$

and if $p_1, \ldots, p_r$ are distinct primes, then for any non-negative integers e_i

$$f\left(p_1^{e_1} \cdots p_r^{e_r}\right) = f\left(p_1^{e_1}\right) \cdots f\left(p_r^{e_r}\right).$$

Let's have some examples:

2.0.1 The Möbius Function

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ where } p_1, \ldots, p_r \text{ are distinct primes.} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly μ is multiplicative. A key property of μ is:

Lemma 2.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \textit{if } n = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Suppose $n > 1$. Then there exist distinct primes $p_1, \ldots, p_r$ and positive integers $e_1, \ldots, e_r$ such that $n = p_1^{e_1} \cdots p_r^{e_r}$.

$$\begin{aligned} \sum_{d|n} \mu(d) &= 1 + \mu(p_1) + \mu(p_2) + \cdots + \mu(p_r) + \mu(p_1 p_2) + \ldots \\ &= 1 + \binom{r}{1}(-1)^1 + \binom{r}{2}(-1)^2 + \cdots + \binom{r}{r}(-1)^r \\ &= (1-1)^r = 0. \end{aligned}$$

□

Lemma 3 (Möbius Inversion). *Let f be an arithmetic function and define*

$$F(n) := \sum_{d|n} f(d).$$

Then

$$f = F * \mu.$$

Proof.

$$\begin{aligned}
F * \mu(n) &= \sum_{ab=n} F(a)\mu(b) \\
&= \sum_{ab=n} \sum_{cd=a} f(c)\mu\left(\frac{n}{cd}\right) \\
&= \sum_{cd|n} f(c)\mu\left(\frac{n}{cd}\right) \\
&= \sum_{c|n} f(c) \sum_{d|\frac{n}{c}} \mu\left(\frac{n/c}{d}\right) \\
&= f(n), && \text{by Lemma 2.}
\end{aligned}$$

□

2.0.2 The Euler Totient Function

$$\begin{aligned}
\phi(n) &:= \sum_{d\leq n} I\Big((d,n)\Big) \\
&= \#\{d \leq n : d \text{ and } n \text{ coprime}\}.
\end{aligned}$$

Lemma 2 implies that

$$\begin{aligned}
\phi(n) &= \sum_{d\leq n} \sum_{c|(d,n)} \mu(c) \\
&= \sum_{c|n} \mu(c) \times \#\{d \leq n : c|d\} \\
&= \sum_{c|n} \mu(c)\frac{n}{c} \\
&= \mu * h(n),
\end{aligned}$$

where $h(x) := x$ for all $x \in \mathbb{N}$. Hence, by Lemma 1, ϕ is a multiplicative arithmetic function. In fact:

$$\begin{aligned}
\phi(n) &= \phi\left(p_1^{e_1} \cdots p_r^{e_r}\right) \\
&= \prod_i \phi\left(p_i^{e_i}\right) \\
&= \prod_i \left(\sum_{d|p_i^{e_i}} \mu(d) \frac{p_i^{e_i}}{d} \right) \\
&= \prod_i \left(p_i^{e_i} - p_i^{e_i-1}\right) \\
&= n \prod_{p|n} \left(1 - \frac{1}{p}\right).
\end{aligned}$$

2.0.3 The Divisor Function

$$d(n) := \sum_{d|n} 1 = u * u(n),$$

where $u(x) := 1$ for all $x \in \mathbb{N}$. Hence d is another example of a multiplicative function. More generally:

$$\begin{aligned}
d_k(n) &:= \sum_{a_1 \cdots a_k = n} 1 \\
&= \sum_{a_k|n} \sum_{a_1 \cdots a_{k-1} = \frac{n}{a_k}} 1 \\
&= (u * d_{k-1})(n).
\end{aligned}$$

It follows by a simple induction and the fact that $d_1 = u$ (or $d_2 = d$) that d_k is multiplicative for all $k \in \mathbb{N}$.

Lemma 4.

$$d_k(n) \leq d(n)^{k-1}.$$

Proof. Notice that when $c|n$ then $d_k(c) \leq d_k(n)$. Hence for all $k \geq 2$:

$$\begin{aligned}
d_k(n) &\leq d_{k-1}(n) \sum_{c|n} 1 \\
&= d_{k-1}(n) d(n).
\end{aligned}$$

The result now follows by induction. ☐

3 Partial Summation

The behaviour of arithmetic functions can be quite erratic. For example, $d(n)$ takes the value 2 infinitely often, but can also be extremely large (see the exercises). However, one still thinks of $d(n)$ as a 'small' function:

Lemma 5. *For all $\varepsilon > 0$*

$$d(n) \ll_\varepsilon n^\varepsilon. \tag{2}$$

Proof. If $n = \prod_p p^e$ then

$$\frac{d(n)}{n^\varepsilon} = \prod_p \frac{e+1}{p^{\varepsilon e}} = \prod_p f_p(e).$$

where each $f_p(e) := \frac{e+1}{p^{\varepsilon e}}$ is a non-negative function which tends to 0 as $e \to \infty$. In particular, f_p has some maximum $e_p \in \mathbb{N} \cup \{0\}$. Notice that

$$f_p(e_p) \geq f_p(e_p \pm 1).$$

Re-arranging, this implies

$$\frac{1}{p^\varepsilon - 1} \geq e_p \geq \frac{1}{p^\varepsilon - 1} - 1. \tag{3}$$

Hence $e_p = \left[\frac{1}{p^\varepsilon - 1}\right]$, unless $e_p = \frac{1}{p^\varepsilon - 1} - 1$. In the latter case, reversing our previous re-arrangement gives $f_p(e_p) = f_p(e_p + 1)$, so $\left[\frac{1}{p^\varepsilon - 1}\right] = e_p + 1$ is again a maximum of f_p. We may therefore assume $e_p = \left[\frac{1}{p^\varepsilon - 1}\right]$ for all p. For all $n \in \mathbb{N}$ we thus have that

$$\frac{d(n)}{n^\varepsilon} \leq \prod_p f_p(e_p) := C(\varepsilon).$$

Notice that $e_p = 0$ if $p > 2^{1/\varepsilon}$, so $C(\varepsilon)$ is certainly finite. □

Note. *$C(\varepsilon)$ is in fact the best possible implied constant in (2) (equality can be achieved by taking $n = \prod_p p^{e_p}$).*

It is often more revealing to study the mean value of an arithmetic function f:

$$\frac{1}{x} \sum_{n \leq x} f(n) \qquad \text{as } x \to \infty.$$

An alternative reason for studying these averages is when f is an indicator function for an interesting set, e.g.

$$\pi(x) := \sum_{n \leq x} \delta_n,$$

where

$$\delta_n = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$$

Part A of this course will culminate in a proof of the Prime Number Theorem (de la Vallée-Poussin, Hadamard; 1896), which states:

$$\pi(x) \sim \frac{x}{\log x} \qquad \text{as } x \to \infty.$$

Lemma 6 (Partial Summation)**.** *Suppose* $a_n \in \mathbb{C}$ $(n \in \mathbb{N})$ *and that* f *is continuously differentiable on* $[x, y]$*. Define*

$$A(t) := \sum_{n \leq t} a_n.$$

Then

$$\sum_{x<n\leq y} a_n f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)dt.$$

Proof. By defining $f(t) := f(x) + f'(x)(t - x)$, for $t < x$, we can extend f to a continuously differentiable function on $[1, y]$. Then

$$\int_x^y A(t)f'(t)dt = \int_1^y A(t)f'(t)dt - \int_1^x A(t)f'(t)dt.$$

We'll partition $[1, y]$ into $[y]$ sub-intervals by defining $y_i := i$ for $i = 1, \ldots, [y]$ and $y_i := y$ for $i = [y] + 1$. Then

$$\begin{aligned}\int_1^y A(t)f'(t)dt &= \sum_{i\leq y} \int_{y_i}^{y_{i+1}} A(t)f'(t)dt \\ &= \sum_{i\leq y} \sum_{n\leq i} a_n \int_{y_i}^{y_{i+1}} f'(t)dt \\ &= \sum_{n\leq y} \sum_{i\geq n} a_n \int_{y_i}^{y_{i+1}} f'(t)dt \\ &= \sum_{n\leq y} a_n \int_n^y f'(t)dt.\end{aligned}$$

Similarly

$$\int_1^x A(t)f'(t)dt = \sum_{n\leq x} a_n \int_n^x f'(t)dt.$$

Therefore

$$\begin{aligned}\int_x^y A(t)f'(t)dt &= \sum_{n\leq y} a_n \left(f(y) - f(n)\right) - \sum_{n\leq x} a_n \left(f(x) - f(n)\right) \\ &= A(y)f(y) - A(x)f(x) - \sum_{x<n\leq y} a_n f(n).\end{aligned}$$

□

Lemma 7.

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

for some constant $\gamma > 0$.

Proof. Let $a_n = u(n)$ and $f(x) = x^{-1}$ in Lemma 6. Then

$$\begin{aligned} A(x) = [x] &= x - \{x\} \\ &= x + O(1). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n \le x} \frac{1}{n} &= 1 + \sum_{1 < n \le x} \frac{1}{n} \\ &= 1 + \frac{[x]}{x} - \frac{[1]}{1} + \int_1^x \frac{[t]}{t^2} dt \\ &= 1 + O\left(\frac{1}{x}\right) + \int_1^x \frac{1}{t} dt - \int_1^x \frac{\{t\}}{t^2} dt \\ &= \log x + \gamma + O(x^{-1}), \end{aligned}$$

where

$$\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2} dt.$$

☐

Here $\gamma = \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1}{n} - \log x \right) \simeq 0.5772$ is **Euler's constant**.

We're now ready to study the mean value for the divisor function. Our first attempt proceeds as follows:

$$\begin{aligned} D(X) := \sum_{n \le X} d(n) &= \sum_{n \le X} \sum_{d \mid n} 1 \\ &= \sum_{d \le X} \sum_{\substack{n \le X \\ d \mid n}} 1 \\ &= \sum_{d \le X} \left[\frac{X}{d} \right] \\ &= X \log X + O(X), \end{aligned}$$

by Lemma 7. The sum $D(X) = \sum_{\substack{d,e \\ de \le X}} 1$ counts the number of points in the lattice $\mathbb{N} \times \mathbb{N}$ which lie on or below the hyperbola $xy = X$. Dirichlet significantly improved the error term above by utilising the symmetry in the inequality $ab \le X$:

$$\begin{aligned} \sum_{ab \le X} 1 &= \sum_{\substack{ab \le X \\ a \le \sqrt{X}}} 1 + \sum_{\substack{ab \le X \\ b \le \sqrt{X}}} 1 - \sum_{\substack{ab \le X \\ a,b \le \sqrt{X}}} 1 \\ &= 2 \sum_{\substack{ab \le X \\ a \le \sqrt{X}}} 1 - \left[\sqrt{X} \right]^2 \\ &= 2 \sum_{a \le \sqrt{X}} \left[\frac{X}{a} \right] - [\sqrt{X}]^2. \end{aligned}$$

Here we have fewer terms of summation than in our first attempt. This trick is called the **Dirichlet hyperbola method**. By Lemma 7

$$2 \sum_{a \le \sqrt{X}} \left[\frac{X}{a} \right] - [\sqrt{X}]^2 = 2X \left(\tfrac{1}{2} \log X + \gamma + O\left(X^{-1/2} \right) \right) + O\left(X^{1/2} \right) - X.$$

Dividing the latter by X we obtain the following lemma.

Lemma 8.

$$\frac{1}{x} \sum_{n \le x} d(n) = \log x + (2\gamma - 1) + O\left(x^{-1/2} \right).$$

Remark. *Let* $\Delta(x) := D(x) - (x \log x + (2\gamma - 1)x)$*. The problem of understanding the behaviour of* $\Delta(x)$ *is called the* **Dirichlet divisor problem** *and is still actively investigated. We saw in Lemma 8*

$$\Delta(x) \ll x^{1/2}$$

and in fact we know that

$$\Delta(x) = \Omega\left(x^{1/4} \right) \qquad \textit{(Hardy).}$$

(this means there are infinitely many values of x *for which* $\Delta(x)$ *grows faster than* $x^{1/4}$*). The current record is*

$$\Delta(x) \ll x^{\frac{131}{416} + \varepsilon} \qquad \textit{(Huxley).}$$

We now define three functions which play an important role in prime number theory. The **von Mangoldt function**:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

and the **Chebyshev functions**:

$$\theta(x) = \sum_{p \le x} \log p, \qquad \psi(x) = \sum_{n \le x} \Lambda(n).$$

How do these relate to $\pi(x)$? We'll show that

$$\pi(x) \sim \frac{x}{\log x} \iff \theta(x) \sim x \tag{4}$$

$$\iff \psi(x) \sim x. \tag{5}$$

In order to establish the above equivalences we utilise the following: $f \sim g$ if and only if $f(x) = g(x)\Big(1 + o\big(\mathbf{1}(x)\big)\Big)$, where $\mathbf{1}(x) = 1$ for all x.

Applying partial summation we have

$$\theta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt.$$

Supposing $\pi(x) \sim \frac{x}{\log x}$

$$\begin{aligned}\int_2^x \frac{\pi(t)}{t} dt &= \int_2^x \frac{1 + o\big(\mathbf{1}(t)\big)}{\log t} dt \\ &= O\left(\int_2^x \frac{1}{\log t} dt\right).\end{aligned}$$

Therefore

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1 + \lim_{x\to\infty} O\left(\frac{1}{x}\int_2^x \frac{1}{\log t} dt\right).$$

The right-hand side above is equal to one, since

$$\begin{aligned}\int_2^x \frac{dt}{\log t} &= \int_2^{\sqrt{x}} \frac{dt}{\log t} + \int_{\sqrt{x}}^x \frac{dt}{\log t} \\ &\ll \sqrt{x} + \frac{x}{\log x}.\end{aligned}$$

For the converse, we once again begin by using partial summation to deduce:

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2}$$

So if $\theta(x) \sim x$, then

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1 + \lim_{x\to\infty} O\left(\frac{1}{x}\int_2^x \frac{1}{(\log t)^2} dt\right).$$

The last expression is equal to one by a similar argument to that given before. Thus (4) holds. To prove (5), note that

$$\begin{aligned}0 \le \psi(x) - \theta(x) &= \sum_{2\le m\le \frac{\log x}{\log 2}} \sum_{\substack{p \\ p^m \le x}} \log p \\ &= \sum_{2\le m\le \frac{\log x}{\log 2}} \theta\left(x^{1/m}\right) \\ &\le \sum_{2\le m\le \frac{\log x}{\log 2}} x^{1/m} \log\left(x^{1/m}\right) \\ &\le \frac{\log x}{\log 2} x^{1/2} \log\left(x^{1/2}\right) \\ &\ll \sqrt{x}(\log x)^2.\end{aligned}$$

Hence it follows that

$$\frac{\psi(x)}{x} = \frac{\theta(x)}{x} + O\left(\frac{(\log x)^2}{\sqrt{x}}\right).$$

which establishes (5).

To prove the Prime Number Theorem (PNT), we begin by showing that $\theta(x)$ has the right order of magnitude.

Lemma 9. *We have* $x \ll \theta(x) \ll x$.

Proof. In view of $\theta(x) = \psi(x) + O\left(\sqrt{x}(\log x)^2\right)$ it suffices to establish the result for ψ. An easy application of partial summation implies that

$$\begin{aligned} T(x) &:= \sum_{n \le x} \log n \\ &= x \log x - x + O(\log x). \end{aligned}$$

Taking logs in the prime factorisation of n, it's easy to see that

$$\log n = \sum_{d|n} \Lambda(d).$$

Thus it follows that

$$T(x) = \sum_{n \le x} \sum_{d|n} \Lambda(d) = \sum_{d \le x} \Lambda(d) \left[x/d\right],$$

and so

$$T(x) - 2T(x/2) = \sum_{d \le x} \Lambda(d) \left(\left[\frac{x}{d}\right] - 2\left[\frac{x}{2d}\right] \right).$$

Now $\left[\frac{x}{d}\right] - 2\left[\frac{x}{2d}\right] \in [0,1]$ and is equal to 1 when $\frac{x}{2} < d \le x$. Hence

$$\psi(x) - \psi(x/2) \le T(x) - 2T(x/2) \le \psi(x),$$

whilst $T(x) - 2T(x/2) = x \log 2 + O(\log x)$. In particular, $\psi(x) \gg x$ and furthermore

$$\begin{aligned} \psi(x) &\le \psi(x/2) + x \log 2 + O(\log x) \\ &\le \psi(x/4) + (1 + 1/2) x \log 2 + O(2 \log x) \\ &\vdots \\ &\le \psi(x/2^r) + (1 + 1/2 + \cdots + 1/2^{r-1}) x \log 2 + O(r \log x), \end{aligned}$$

for any $r \ge 1$. Choose r such that $2^r \le x < 2^{r+1}$. Then

$$\psi(x) \le (2 \log 2) x + O\left((\log x)^2\right).$$

□

Euler observed that 'the sum of the reciprocals of the primes is like the logarithm of the harmonic series'. He proved

$$\sum_{p \le x} \frac{1}{p} \longrightarrow \infty \quad \text{as } x \longrightarrow \infty.$$

Theorem A 1 (Mertens). *There exists a constant* B *such that*

$$\sum_{p \le x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x}\right).$$

We begin the proof with:

Lemma 10.

$$\sum_{p\le x}\frac{\log p}{p}=\log x+O\left(1\right).$$

Proof.

$$\sum_{p\le x}\frac{\log p}{p}=\sum_{n\le x}\frac{\Lambda(n)}{n}-\sum_{p\le x}\log p\sum_{2\le k\le\frac{\log x}{\log p}}\frac{1}{p^k}.$$

We have

$$\begin{aligned}\sum_{p\le x}\log p\sum_{2\le k\le\frac{\log x}{\log p}}\frac{1}{p^k}&\le\sum_{p\le x}\log p\ \frac{p^{-2}}{1-p^{-1}}\\&\ll\sum_{p\le x}\frac{\log p}{p^2}\\&\le\sum_{n=1}^{\infty}\frac{\log n}{n^2}\ll 1.\end{aligned}$$

Therefore

$$\sum_{p\le x}\frac{\log p}{p}=\sum_{n\le x}\frac{\Lambda(n)}{n}+O\left(1\right).$$

By Lemma 9 and partial summation

$$\begin{aligned}\sum_{n\le x}\frac{\Lambda(n)}{n}&=x^{-1}\sum_{n\le x}\Lambda(n)\left[\frac{x}{n}\right]+x^{-1}\sum_{n\le x}\Lambda(n)\left\{\frac{x}{n}\right\}\\&=x^{-1}\sum_{n\le x}\log n+x^{-1}O\left(\psi(x)\right)\\&=\log x-1+x^{-1}O\left(\log x\right)+x^{-1}O\left(x\right)\\&=\log x+O\left(1\right).\end{aligned}$$

□

Proof of A 1. Let

$$a_n=\begin{cases}\frac{\log n}{n} & \text{if } n \text{ is prime}\\ 0 & \text{otherwise.}\end{cases}$$

By partial summation (Lemma 6):

$$\begin{aligned}
\sum_{p\le x}\frac{1}{p} &= \frac{1}{2}+\sum_{2<n\le x}\frac{a_n}{\log n}\\
&= \frac{1}{2}+\left(\sum_{n\le x}a_n\right)\frac{1}{\log x}-\frac{1}{2}+\int_2^x\left(\sum_{n\le t}a_n\right)\frac{1}{t(\log t)^2}dt\\
&= \big(\log x+O(1)\big)\frac{1}{\log x}+\int_2^x\big(\log t+O(1)\big)\frac{1}{t(\log t)^2}dt\\
&= 1+O\left(\frac{1}{\log x}\right)+\log\log x-\log\log 2+O\left(\frac{1}{\log x}-\frac{1}{\log 2}\right)\\
&= \log\log x+B+O\left(\frac{1}{\log x}\right) \qquad \text{for some } B.
\end{aligned}$$

□

4 Dirichlet Series

A **Dirichlet series** is an infinite series of the form

$$F(s):=\sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

where a_n will be an arithmetic function and $s=\sigma+it$ is a complex variable. They are ubiquitous in analytic number theory, forming an important class of generating functions.

Lemma 11. *Suppose $F(s_0)$ converges for some value $s_0=\sigma_0+it_0$. Then $F(s)$ converges uniformly on every compact subset of the half-plane $\sigma>\sigma_0$.*

Proof. Let K be a compact subset of the half-plane $\sigma>\sigma_0$. Then K is bounded, so there is some B such that $|s|\le B$ for all $s\in K$ and it can be checked that there exists $\varepsilon>0$ such that for all $s\in K$, $\sigma\ge\sigma_0+\varepsilon$. We apply partial summation (Lemma 6), taking $\frac{a_n}{n^{s_0}}$ $(n\in\mathbb{N})$ as our sequence and $f(t)=t^{-(s-s_0)}$ as our continuously differentiable function, to get that

$$\begin{aligned}
\left|\sum_{a<n\le b}\frac{a_n}{n^s}\right| &= \left|\frac{1}{b^{s-s_0}}\sum_{n\le b}\frac{a_n}{n^{s_0}}-\frac{1}{a^{s-s_0}}\sum_{n\le a}\frac{a_n}{n^{s_0}}+(s-s_0)\int_a^b\left(\sum_{n\le t}\frac{a_n}{n^{s_0}}\right)\frac{dt}{t^{s-s_0-1}}\right|\\
&\le S\left(\frac{1}{b^{\varepsilon}}+\frac{1}{a^{\varepsilon}}+\frac{|s-s_0|}{\sigma-\sigma_0}\left(\frac{1}{b^{\sigma-\sigma_0}}+\frac{1}{a^{\sigma-\sigma_0}}\right)\right)\\
&\le \frac{2S(\varepsilon+B+|s_0|)}{\varepsilon a^{\varepsilon}}.
\end{aligned}$$

Where $S=\sup_{N\in\mathbb{N}}\left|\sum_{n=1}^{N}\frac{a_n}{n^{s_0}}\right|<\infty$. Hence Cauchy's condition for uniform convergence completes the proof. □

Define the **abscissa of convergence** of a Dirichlet series $F(s)$ to be

$$\sigma_c := \inf\{\sigma \in \mathbb{R} : F(\sigma + it) \text{ converges}\}.$$

Similarly the **abscissa of absolute convergence** is given by

$$\sigma_a := \inf\left\{\sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \text{ converges}\right\}.$$

Example. *We define the* **Riemann zeta function** *to be*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{(for } \sigma > 1\text{)}.$$

It's clear that $\zeta(s)$ is absolutely convergent for $\sigma > 1$ and diverges for $s = 1$. Hence (in this case) $\sigma_c = \sigma_a = 1$.

In general we have $\sigma_c \leq \sigma_a \leq \sigma_c + 1$ for any Dirichlet series. $\sigma_c \leq \sigma_a$ is obvious. To see the second inequality suppose $F(s_0)$ converges for some s_0. It suffices then to show $F(s)$ converges absolutely for $\sigma > \sigma_0 + 1$. Since we must have $\frac{a_n}{n^{s_0}} \to 0$, there exists A such that for all $n \in \mathbb{N}$

$$\left|\frac{a_n}{n^{s_0}}\right| \leq A.$$

Hence for $\sigma > \sigma_0 + 1$

$$\sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right| \leq A \sum_{n=1}^{\infty} \frac{1}{n^{\sigma - \sigma_0}} < \infty.$$

Lemma 12. *Let*

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad , \quad G(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

be Dirichlet series with abscissae of absolute convergence $\sigma_a(F)$, $\sigma_a(G)$. Provided $\sigma > \max\{\sigma_a(F), \sigma_a(G)\}$, we have

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(a * b)(n)}{n^s}.$$

Proof.

$$\begin{aligned} F(s)G(s) &= \sum_{m,n=1}^{\infty} \frac{a(m)b(n)}{(mn)^s} \\ &= \sum_{k=1}^{\infty} \sum_{mn=k} \frac{a(m)b(n)}{k^s}. \end{aligned}$$

□

Example. *Recall that $u(n) = 1$ and $I(n) = [1/n]$ $(n \in \mathbb{N})$. Then we know $u * \mu = I$. Clearly the Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

has abscissa of absolute convergence $\sigma_a = 1$*. So for* $\sigma > 1$

$$\begin{aligned}\zeta(s)\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} &= \sum_{n=1}^{\infty}\frac{u*\mu(n)}{n^s}\\ &= \sum_{n=1}^{\infty}\frac{I(n)}{n^s}\\ &= 1.\end{aligned}$$

Thus we conclude that when $\sigma > 1$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \zeta(s)^{-1}. \tag{6}$$

Example. *Since* $d(n) = u * u(n)$

$$\zeta(s)^2 = \sum_{n=1}^{\infty}\frac{d(n)}{n^s} \qquad (\sigma > 1).$$

The following is a key property of Dirichlet series (sometimes called the 'analytic fundamental theorem of arithmetic'):

> When $a(n)$ is a multiplicative arithmetic function, $F(s)$ has an **Euler product**.

Lemma 13 (Euler Product Formula)**.** *Suppose* f *is a multiplicative function and* $F(s) := \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ *has abscissa of absolute convergence* σ_a*. Then provided* $\sigma > \sigma_a$

$$F(s) = \prod_p\left(\frac{f(1)}{1} + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right).$$

In particular, if f *is completely multiplicative, then*

$$F(s) = \prod_p\left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

Proof. Let $\sigma > \sigma_a$. Since $F(s)$ is absolutely convergent, so is

$$\sum_{m=0}^{\infty}\frac{f(p^m)}{p^{ms}},$$

for all primes p. We can therefore define

$$P(s;x) := \prod_{p\le x}\sum_{m=0}^{\infty}\frac{f(p^m)}{p^{ms}} \qquad (x \ge 2).$$

Let $r = \pi(x)$. Absolute convergence ensures the following identity is valid

$$\begin{aligned} P(s;x) &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{f\left(p_1^{m_1}\right) \cdots f\left(p_r^{m_r}\right)}{\left(p_1^{m_1} \cdots p_r^{m_r}\right)^s} \\ &= \sum_n \frac{f(n)}{n^s} \# \left\{(m_1, \ldots, m_r) : n = p_1^{m_1} \cdots p_r^{m_r}\right\}. \end{aligned}$$

By the fundamental theorem of arithmetic

$$\sum_n \frac{f(n)}{n^s} \# \left\{(m_1, \ldots, m_r) : n = p_1^{m_1} \cdots p_r^{m_r}\right\} = \sum_{\substack{n \\ p \mid n \Rightarrow p \leq x}} \frac{f(n)}{n^s}.$$

Therefore

$$|F(s) - P(s;x)| \leq \sum_{n>x} \frac{|f(n)|}{n^\sigma},$$

where the latter quantity $\to 0$ as $x \to \infty$. ☐

Example. *For* $\sigma > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{u(n)}{n^s}.$$

Hence

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The above example is the first piece of evidence for the following correspondence principal:

$$\{\text{Analytic properties of } \zeta(s)\} \longleftrightarrow \{\text{Properties of primes}\}$$

Exercise. *By using Lemma 13, show that*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \zeta(s)^{-1} \qquad (\sigma > 1).$$

One of the key properties of Dirichlet series lies in the following result.

Lemma 14 (Perron). *Let* $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ *be a Dirichlet series with abscissa of absolute convergence* σ_a. *Provided* $x \notin \mathbb{Z}$ *and* $\alpha > \max\{\sigma_a, 0\}$ *we have*

$$\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{F(s) x^s}{s} ds + O\left(\frac{x^\alpha}{T} \sum_{n=1}^{\infty} \frac{|f(n)| n^{-\alpha}}{\left|\log\left(\frac{x}{n}\right)\right|}\right).$$

Proof. It's an easy exercise in contour integration to show that for any $\alpha > 0$

$$\frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{u^s}{s} ds = \begin{cases} 1 + O\left(\frac{u^\alpha}{T|\log u|}\right) & \text{if } u > 1 \\ \frac{1}{2} + O\left(\frac{\alpha}{T}\right) & \text{if } u = 1 \\ O\left(\frac{u^\alpha}{T|\log u|}\right) & \text{if } u \in (0,1). \end{cases}$$

By Lemma 11 $\sum\limits_{n\le N} \frac{f(n)}{n^s}$ converges uniformly to $F(s)$ on the set $[\alpha-iT,\alpha+iT]:=\{\alpha+it:t\in[-T,T]\}$. We are therefore justified in swapping the order of integration and summation in the following:

$$\int_{\alpha-iT}^{\alpha+iT}\frac{F(s)x^s}{s}ds=\sum_{n=1}^{\infty}f(n)\int_{\alpha-iT}^{\alpha+iT}\frac{(x/n)^s}{s}ds$$
$$=\sum_{n\le x}\left(2\pi i f(n)+O\left(\frac{|f(n)|(x/n)^\alpha}{T|\log(x/n)|}\right)\right)+\sum_{n>x}O\left(\frac{|f(n)|(x/n)^\alpha}{T|\log(x/n)|}\right).$$

□

We end this section by establishing the analyticity of a Dirichlet series in an appropriate half-plane.

Lemma 15. $F(s)=\sum\limits_{n=1}^{\infty}\dfrac{f(n)}{n^s}$ *is holomorphic for* $\sigma>\sigma_c$ *with derivative*

$$F'(s)=-\sum_{n=1}^{\infty}\frac{f(n)\log n}{n^s}.$$

Proof. Let $H:=\{s\in\mathbb{C}:\sigma>\sigma_c\}$ and let $D\subset H$ be any compact disc with boundary ∂D. Since the partial sums $F_x(s):=\sum_{n\le x}f(n)n^{-s}$ are entire functions, Cauchy's integral formula tells us that for all x

$$F_x(a)=\frac{1}{2\pi i}\int_{\partial D}\frac{F_x(z)}{z-a}dz,\qquad \text{for any } a \text{ belonging to the interior of } D.$$

By uniform convergence (Lemma 11) we can pass to a limit under the integral sign, giving

$$F(a)=\frac{1}{2\pi i}\int_{\partial D}\frac{F(z)}{z-a}dz,$$

whence F is holomorphic inside D. The treatment of derivatives is similar[1]. □

5 The Riemann Zeta Function

We now explore some basic properties of $\zeta(s)$ as needed to establish the Prime Number Theorem.

It follows from Lemma 15 that $\zeta(s)$ is holomorphic on $\sigma>1$. It is clearly non-zero on $\sigma>1$ (see (6)). Assume $\sigma>1$. Then

$$\zeta'(s)=-\sum_{n=1}^{\infty}\frac{\log n}{n^s}.$$

---

[1] Calculate the derivative of the partial sum, then use Cauchy's integral formula for the first derivative, passing to the limit under the integral sign.

Hence by Lemma 12

$$\begin{aligned}\zeta(s)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} &= \sum_{n=1}^{\infty}\frac{u*\Lambda(n)}{n^s}\\ &= \sum_{n=1}^{\infty}\frac{\log(n)}{n^s}\\ &= -\zeta'(s).\end{aligned}$$

This establishes:

Lemma 16. *For $\sigma > 1$ we have*

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}.$$

Recall $\psi(x) = \sum_{n\le x}\Lambda(n)$. Letting $T\to\infty$ in Perron's Formula (Lemma 14) we get

$$\psi(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}ds \qquad (\alpha>1).$$

This is one of the most popular approaches to PNT and naturally leads to the study of analytic properties of ζ and ζ'. We shall follow an alternative path (due to Newman) but we will still need to make sense of $\zeta(s)$ for $\sigma\le 1$. An application of Lemma 6 gives

$$\sum_{n\le x} n^{-s} = \frac{[x]}{x^s} + s\int_1^x [t]t^{-s-1}dt \tag{7}$$

$$= x^{1-s} - \{x\}\,x^{-s} + \frac{s}{s-1}\left(1-x^{1-s}\right) - s\int_1^x \{t\}\,t^{-s-1}dt \tag{8}$$

for all $s\ne 1$. Letting $x\to\infty$ when $\sigma>1$ we obtain

Lemma 17. *For $\sigma > 1$*

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty}\frac{\{t\}}{t^{s+1}}dt. \tag{9}$$

Note. *For values of $\sigma > 0$, we have that*

$$\begin{aligned}\left|\int_1^{\infty}\frac{\{t\}}{t^{s+1}}dt\right| &\le \int_1^{\infty}\frac{1}{t^{\sigma+1}}dt\\ &\ll_{\sigma} 1.\end{aligned}$$

Lemma 17 therefore gives a formula for $\zeta(s)$ which continues to make sense for $\sigma\in(0,1]$.

Consider the sequence of functions

$$f_N(s) := \frac{s}{s-1} - s\int_1^N\frac{\{t\}}{t^{s+1}}dt \qquad (N\in\mathbb{N})$$

defined on the punctured half-plane $\Omega := \{s \in \mathbb{C} : s \neq 1 \text{ and } \sigma > 0\}$. Since $t \mapsto \{t\}$ is periodic modulo 1, a quick calculation establishes that each f_N is a finite sum of functions holomorphic on Ω, and therefore f_N is itself holomorphic on Ω. An argument similar to that given in Lemma 11 shows that the f_N converge uniformly to ζ (as defined by (9)) on all compact subsets of Ω. Using Cauchy's integral formula (as in Lemma 15) we can show that ζ is holomorphic on every open ball contained in Ω, and hence holomorphic in Ω. Clearly ζ has a simple pole at $s = 1$. To summarise:

> Formula (9) gives an **analytic continuation** of $\zeta(s)$ to $\sigma > 0$, whose only pole is a simple one at $s = 1$.

Since $\zeta(s)$ is not identically zero in the open connected set Ω, it follows that the zeroes of $\zeta(s)$ form a discrete subset of Ω. Moreover, we now also know that $\zeta'(s)$ is holomorphic in Ω with a pole of order 2 at $s = 1$. Thus $\dfrac{\zeta'(s)}{\zeta(s)}$ is meromorphic in the half-plane $\sigma > 0$ with a simple pole at $s = 1$.

Lemma 18. *(i) For all $\sigma \geq 1$ and $t \geq 2$*

$$|\zeta(s)| = O(\log t).$$

(ii) For all $\delta \in (0, 1)$, if $\sigma \geq \delta$ and $t \geq 1$ then

$$|\zeta(s)| = O_\delta\left(t^{1-\delta}\right).$$

Proof. Let $\sigma > 0$, $t \geq 1$, $x \geq 1$. Combining (9) and (8) gives

$$\zeta(s) - \sum_{n \leq x} \frac{1}{n^s} = \frac{1}{(s-1)x^{s-1}} + \frac{\{x\}}{x^s} - s \int_x^\infty \frac{\{u\}}{u^{s+1}} du.$$

Therefore

$$|\zeta(s)| \leq \sum_{n \leq x} \frac{1}{n^\sigma} + \frac{1}{t x^{\sigma-1}} + \frac{1}{x^\sigma} + |s| \int_x^\infty \frac{du}{u^{\sigma+1}}. \tag{10}$$

First suppose $\sigma \geq 1$ and $t \geq 2$. Take $x = t$ in (10) to deduce that

$$\begin{aligned} |\zeta(s)| &\ll \log t + \frac{1}{t^\sigma} + \frac{|\sigma + it|}{\sigma t^\sigma} \\ &\ll \log t + \frac{1}{t^\sigma} + \frac{1}{\sigma t^{\sigma-1}} \\ &\ll \log t + 1 \ll \log t. \end{aligned}$$

Next suppose $t \geq 1$ and $\sigma \geq \delta$ where $\delta \in (0, 1)$. Again taking $x = t$ in (10) we have

$$|\zeta(s)| \ll \log t + \left(\frac{1}{t} + \frac{1}{\sigma}\right) t^{1-\sigma} \ll_\delta t^{1-\delta}.$$

□

Taking $\delta = 1/2$ in Lemma 18.(ii) gives

$$\left|\zeta(\tfrac{1}{2} + it)\right| = O\left(t^{1/2}\right) \qquad (t \geq 1). \tag{11}$$

The **Lindelöf Hypothesis** predicts that for all $\varepsilon > 0$

$$\left|\zeta(\tfrac{1}{2} + it)\right| = O_\varepsilon\left(t^\varepsilon\right).$$

The record is $|\zeta(\frac{1}{2} + it)| = O\left(t^\theta\right)$ for $\theta = 0.156\ldots$ (M. Huxley). In fact the Riemann Hypothesis implies the Lindelöf Hypothesis.

Define

$$\Phi(s) := \sum_p \frac{\log p}{p^s}$$

. Then:

Lemma 19. *For $\sigma \geq 1$ we have that $\zeta(s) \neq 0$ and $\Phi(s) - \frac{1}{s-1}$ is holomorphic.*

Remark. *We say f is holomorphic on the set X (where X is not necessarily open) if there exists an open set $\Omega \subset \mathbb{C}$ which contains X and on which f is holomorphic. So in Lemma 19 we claim that $\Phi(s) - \frac{1}{s-1}$ is holomorphic on some open set containing $\sigma \geq 1$.*

Proof. We've already seen that $\zeta(s) \neq 0$ for $\sigma > 1$. For $\sigma > 1$, Lemma 16 implies that

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} &= \sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^{ks}} \\ &= \sum_p \frac{\log p}{p^s - 1}. \end{aligned}$$

Since

$$\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x(x-1)},$$

we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_p \frac{\log p}{p^s\left(p^s - 1\right)}. \tag{12}$$

Notice that $\sum_p \frac{\log p}{p^s(p^s-1)}$ converges absolutely for $\sigma > \frac{1}{2}$. It therefore follows from Lemma 17 and the remarks after it, that $\Phi(s)$ has a meromorphic continuation to $\sigma > \frac{1}{2}$ with poles only at $s = 1$ and the zeros of ζ. To complete the proof of the lemma, it suffices to show that

$$\zeta(1 + it) \neq 0 \qquad \text{for all real values of } t. \tag{13}$$

Since if this holds then for each $t \in \mathbb{R}$ there exists $\varepsilon_t \in (0, 1)$ such that ζ is non-zero on $B_{\varepsilon_t}(1 + it)$. The set

$$\{z \in \mathbb{C} : \Re(z) > 1\} \cup \bigcup_t B_{\varepsilon_t}(1 + it)$$

is then open, contains $\Re(z) \geq 1$ and contains no zeros of ζ. It follows that $\Phi(s) - \frac{1}{s-1}$ is holomorphic on this set.

To prove (13), suppose ζ has a zero of order $\mu \in \mathbb{Z}$ at $s_0 := 1 + i\alpha$, where $\alpha \in \mathbb{R} \setminus \{0\}$. By which we mean, there is a function g which is non-zero at

s_0, holomorphic in a neighbourhood of s_0 and with $\zeta(s) = (s - s_0)^\mu g(s)$ in this neighbourhood. So a simple pole is a zero of order -1, for example. Similarly suppose $\zeta(s)$ has a zero of order ν at $s = 1 + i2\alpha$. Then certainly μ, $\nu \geq 0$ by Lemma 17 (the only pole of ζ in the half-plane $\sigma > 0$ is at $s = 1$). By (12) we have

$$\lim_{\varepsilon \to 0} \varepsilon \Phi(1 + \varepsilon \pm ri\alpha) = \begin{cases} 1 & r = 0 \\ -\mu & r = 1 \\ -\nu & r = 2. \end{cases}$$

Note that

$$\sum_{r=-2}^{2} \binom{4}{2+r} \Phi(1 + \varepsilon + ir\alpha) = \sum_p \frac{\log p}{p^{1+\varepsilon}} \left(p^{i\alpha/2} + p^{-i\alpha/2}\right)^4 \geq 0.$$

Multiplying through by ε and taking the limit as $\varepsilon \to 0$ we get

$$\lim_{\varepsilon \to 0} \left(\varepsilon\Phi(1+\varepsilon-i2\alpha) + 4\varepsilon\Phi(1+\varepsilon-i\alpha) + 6\varepsilon\Phi(1+\varepsilon) + 4\varepsilon\Phi(1+\varepsilon+i\alpha) + \varepsilon\Phi(1+\varepsilon+i2\alpha)\right)$$
$$= 6 - 8\mu - 2\nu \geq 0.$$

Therefore $\mu = 0$ and so there does not exist a zero of $\zeta(s)$ with $\sigma = 1$. ☐

Remark. *The above proof relies on a 'trick' which cannot be used for general L-functions: ζ is 'special'.*

6 The Prime Number Theorem

We now have everything in place to prove:

Theorem A 2. *$\theta(x) \sim x$ as $x \to \infty$.*

We saw prior to Lemma 9 that this is equivalent to

$$\pi(x) \sim \frac{x}{\log x}.$$

The main analytic tool is the following:

Lemma 20 (Newman's Analytic Lemma)**.** *For $t \geq 0$ let $f(t)$ be a bounded function which is locally integrable, i.e. integrable on any compact subset of $[0, \infty)$. Suppose*

$$g(z) := \int_0^\infty f(t) e^{-zt} dt \qquad (\Re(z) > 0)$$

extends holomorphically to $\Re(z) \geq 0$. Then

$$\int_0^\infty f(t) dt$$

exists and is equal to $g(0)$.

Proof. [2] When we say 'g extends holomorphically to $\Re(z) \geq 0$' we mean that there exists an open set $E \subset \mathbb{C}$ which contains the closed half-plane $\Re(z) \geq 0$ and on which g has an analytic extension. We will first show that for each $R > 0$ there exists $\delta = \delta(R) > 0$ such that the set

$$E_R := \{z \in \mathbb{C} : |z| < R + 1 \text{ and } \Re(z) > -2\delta\}$$

is contained in E. Suppose otherwise. Then for each $n \in \mathbb{N}$ there exists $z_n \in \mathbb{C} \setminus E$ with

$$|z_n| < R + 1 \quad \text{and} \quad \Re(z) > -\frac{1}{n}.$$

Since $(z_n)_n$ is a sequence in compact set $\overline{B}_{R+1}(0)$, it has a convergent subsequence $(z_{k(n)})_n$ with limit z_0. Then

$$z_0 \in \overline{B}_{R+1}(0) \cap \{z \in \mathbb{C} : \Re(z) \geq 0\} \subset E.$$

But $\mathbb{C} \setminus E$ is closed, so $z_0 \in \mathbb{C} \setminus E$ (a contradiction).

Next we define the closed contour C_R to be the boundary of the closed set

$$F_R := \{z \in \mathbb{C} : |z| \leq R \text{ and } \Re(z) \geq -\delta\} \subset E_R.$$

We also define:

$$\begin{aligned} C_{R,+} &:= \{z \in C_R : \Re(z) > 0\}, \\ C_{R,-} &:= \{z \in C_R : \Re(z) < 0\}. \end{aligned}$$

For each $T > 0$ let

$$g_T(z) := \int_0^T f(t) e^{-tz} dt \qquad (z \in \mathbb{C}).$$

Clearly each g_T is entire.

We wish to prove that

$$\lim_{T \to \infty} \big(g(0) - g_T(0)\big) = 0.$$

By Cauchy's residue theorem, we have that for all $T > 0$

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_{C_R} \frac{g(z) - g_T(z)}{z} \left(1 + \frac{z^2}{R^2}\right) e^{Tz} dz. \tag{14}$$

Since the left-hand side of (14) is independent of R, it suffices to show that for all large R

$$\limsup_{T \to \infty} \int_{C_R} I(z; R, T) dz \ll R^{-1}, \tag{15}$$

where

$$I(z; R, T) := \frac{g(z) - g_T(z)}{z} \left(1 + \frac{z^2}{R^2}\right) e^{Tz}.$$

---

[2] The proof presented here is based on that given by Dr Browning as well as that found in Lang's *Complex Analysis*. In the latter the author seems to have failed to remove a note to himself: '(Put the details as an exercise.)'. The profusion of details in this proof are a result of that exercise, performed inefficiently by the typesetter.

To establish this we will estimate the integral over $C_{R,+}$ and $C_{R,-}$ separately. Let $B := \sup|f(t)|$. For $z \in C_{R,+}$

$$\begin{aligned}
|I(z;R,T)| &= \left|\frac{1}{R}\left(\frac{R}{z}+\frac{z}{R}\right)e^{Tz}\int_T^\infty f(t)e^{-tz}dt\right| \\
&= \frac{2|\Re(z/R)|}{R}e^{T\Re(z)}\left|\int_T^\infty f(t)e^{-tz}dt\right| \\
&\le \frac{2\Re(z)}{R^2}e^{T\Re(z)}B\int_T^\infty e^{-t\Re(z)}dt \\
&= \frac{2\Re(z)}{R^2}e^{T\Re(z)}B\frac{e^{-T\Re(z)}}{\Re(z)} \\
&= \frac{2B}{R^2}.
\end{aligned}$$

Hence

$$\int_{C_{R,+}} I(z;R,T)dz \ll R^{-1}$$

Next, define

$$C'_{R,-} := \{z \in \mathbb{C} : |z| = R \text{ and } \Re(z) < 0\}.$$

Since

$$J(z;R,T) := \frac{g_T(z)}{z}\left(1+\frac{z^2}{R^2}\right)e^{Tz}$$

is analytic on simply connected domain $\mathbb{C} \setminus [0,\infty)$, by Cauchy's Theorem

$$\int_{C_{R,-}} J(z;R,T)dz = \int_{C'_{R,-}} J(z;R,T)dz.$$

For $z \in C'_{R,-}$

$$\begin{aligned}
|J(z;R,T)| &= \frac{1}{R}\left|\frac{R}{z}+\frac{z}{R}\right|\left|e^{Tz}\int_0^T f(t)e^{-tz}dt\right| \\
&\le \frac{2|\Re(z)|}{R^2}e^{T\Re(z)}\frac{B}{|\Re(z)|}\left(e^{-T\Re(z)}-1\right) \\
&\le \frac{2B}{R^2}.
\end{aligned}$$

Finally we estimate

$$K(z;R,T) := \frac{g(z)}{z}\left(1+\frac{z^2}{R^2}\right)e^{Tz}$$

on $C_{R,-}$. K is analytic on the intersection of E_R with the half-plane $\Re(z) < 0$. This intersection is in fact an open convex set, so by Cauchy's Theorem for convex domains, for each $\varepsilon \in (0, \delta(R)]$

$$\int_{\Gamma_\varepsilon} K(z;R,T)dz = \int_{C_{R,-}} K(z;R,T)dz,$$

where Γ_ε is the portion of the boundary of the region

$$\{z \in \mathbb{C} : |z| \leq R \text{ and } \Re(z) \geq -\varepsilon\} \subset F_R$$

which intersects $\Re(z) < 0$.

Let

$$S_R := \sup_{z \in F_R} |f(z)|$$

and

$$\gamma_\varepsilon := \{z \in \Gamma_\varepsilon : \Re(z) = -\varepsilon\}.$$

If $z \in \Gamma_\varepsilon \setminus \gamma_\varepsilon$ then

$$|K(z; R, T)| \leq \frac{2S_R}{R}.$$

Using elementary trigonometry, one can verify that the contour $\Gamma_\varepsilon \setminus \gamma_\varepsilon$ has length at most $\frac{2\pi^2\varepsilon}{R}$, hence

$$\left| \int_{\Gamma_\varepsilon \setminus \gamma_\varepsilon} K(z; R, T) dz \right| \leq \frac{4\pi^2 \varepsilon S_R}{R^2}.$$

So choosing $\varepsilon := \min\left\{\delta(R), \frac{1}{S_R}\right\}$ we have that when $R \geq 1$

$$\int_{\Gamma_\varepsilon \setminus \gamma_\varepsilon} K(z; R, T) dz \ll \frac{1}{R^2} \leq \frac{1}{R}.$$

Now for $z \in \gamma_\varepsilon$

$$\begin{aligned} |K(z; R, T)| &\leq S_R \frac{e^{T\Re(z)}}{|z|} \\ &\leq \frac{S_R}{\varepsilon} e^{-\varepsilon T}. \end{aligned}$$

So $K(z; R, T) \to 0$ uniformly on γ_ε as $T \to \infty$. (15) follows. ☐

We apply Newman's Lemma to study the integral

$$\int_1^\infty \frac{\theta(x) - x}{x^2}.$$

For $\sigma > 1$, Lemma 6 tells us

$$\begin{aligned} \Phi(s) = \sum_p \frac{\log p}{p^s} &= \lim_{x \to \infty} \sum_{p \leq x} \frac{\log p}{p^s} \\ &= \lim_{x \to \infty} \left(\frac{\theta(x)}{x^s} + s \int_1^x \frac{\theta(t)}{t^{s+1}} dt \right) \\ &= s \int_1^\infty \frac{\theta(t)}{t^{s+1}} dt \\ &= s \int_0^\infty \theta(e^u) e^{-su} du. \end{aligned}$$

The last equality is obtained by making the change of variables $t = e^u$. Hence for $\sigma > 1$

$$\Phi(s) = s \int_0^\infty e^{-su}\theta(e^u)du.$$

Let

$$\begin{aligned} f(t) &:= \theta(e^t)e^{-t} - 1, \\ g(z) &:= \frac{\Phi(z+1)}{z+1} - \frac{1}{z}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^\infty f(t)e^{-zt}dt &= \int_0^\infty \theta(e^t)e^{-t(z+1)}dt - \int_0^\infty e^{-zt}dt \\ &= g(z). \end{aligned}$$

By Lemma 9, $f(t)$ is bounded. By Lemma 19, $g(z) = \Phi(z+1)(z+1)^{-1} - z^{-1}$ extends holomorphically to $\Re(z+1) \geq 1$, i.e. $\Re(z) \geq 0$.

By Newman's Lemma

$$\int_0^\infty \big(\theta(e^t)e^{-t} - 1\big)$$

is a convergent integral and equal to

$$\int_1^\infty \frac{\theta(t) - t}{t^2}dt$$

(by a simple change of variables).

Assume that for some $\lambda > 1$, there exists arbitrarily large values of x such that

$$\theta(x) \geq \lambda x.$$

Since θ is non-decreasing, we have

$$\begin{aligned} \int_x^{\lambda x} \frac{\theta(t) - t}{t^2} &\geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}dt \\ &= \int_1^\lambda \frac{\lambda - t}{t^2}dt \\ &= (\lambda - 1 - \log\lambda) > 0. \end{aligned}$$

This contradicts the convergence of

$$\int_1^\infty \frac{\theta(t) - t}{t^2}dt.$$

Similarly, for each $\lambda < 1$ there can't exist arbitrarily large values of x for which

$$\theta(x) \leq \lambda x.$$

Therefore $\theta(x) \sim x$.