Problems I.

1. For each of the following pairs of polynomials $f$ and $g$, (i) find the quotient and remainder on dividing $g$ by $f$; (ii) use the Euclidean Algorithm to find the highest common factor $h$ of $f$ and $g$; (iii) find polynomials $a$ and $b$ with the property that $h = af + bg$.
   (a) $g = t^7 - t^3 + 5$, $f = t^3 + 7$ over $\mathbb{Q}$;
   (b) $g = 4t^3 - 17t^2 + t - 3$, $f = 2t + 5$ over $\mathbb{Q}$.

2. For each of the following pairs of polynomials $f$ and $g$, (i) find the quotient and remainder on dividing $g$ by $f$; (ii) use the Euclidean Algorithm to find the highest common factor $h$ of $f$ and $g$; (iii) find polynomials $a$ and $b$ with the property that $h = af + bg$.
   (a) $g = t^3 + 2t^2 - t + 3$, $f = t + 2$ over $\mathbb{F}_5$;
   (b) $g = t^7 - 4t^6 + t^3 - 4t + 6$, $f = 2t^3 - 2$ over $\mathbb{F}_7$.

3. (a) Show that $t^3 + 3t + 1$ is irreducible in $\mathbb{Q}[t]$.
   (b) Suppose that $\alpha$ is a root of $t^8 + 3t + 1$ in $\mathbb{C}$. Express $\alpha^{-1}$ and $(1 + \alpha)^{-1}$ as linear combinations, with rational coefficients, of $1$, $\alpha$ and $\alpha^2$.
   (c) Is it possible to express $(1 + \alpha)^{-1}$ as a linear combination, with rational coefficients, of $1$ and $\alpha$? Justify your answer.

4. (a) Show that the polynomial $t^2 + t + 1$ is irreducible in $\mathbb{F}_2[t]$.
   (b) Give a complete list of the coset representatives of the quotient ring $\mathbb{F}_2[t]/(t^2 + t + 1)$.
   (c) For each of the non-zero elements $\alpha$ of $\mathbb{F}_2[t]/(t^2 + t + 1)$, determine the least integer $n$ (if one exists) for which $\alpha^n = 1$.

5. Suppose that $L : K$ is a field extension, and that $K_1$ and $K_2$ are two intermediate fields between $K$ and $L$ satisfying the condition that $L = K(K_1, K_2)$. Show that $[L : K] \leq [K_1 : K][K_2 : K]$. 