Galois Theory problems, #2.

1. Let \( \mathbb{Q} \subset \mathbb{Q}(\alpha) \) be a simple extension where the minimal polynomial of \( \alpha \) is \( t^4 + 2t - 2 \). Calculate the minimal polynomials of \( \alpha - 1 \) and \( \alpha^2 + 1 \) over \( \mathbb{Q} \) and express their inverses in \( \mathbb{Q}(\alpha) \) in the form \( a + b\alpha + c\alpha^2 \) with \( a, b, c \in \mathbb{Q} \).

2. Is \( \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) \) a simple extension of \( \mathbb{Q} \)?

3. Find all irreducible quadratics over \( \mathbb{Z}_3 \) and then construct all possible extensions of \( \mathbb{Z}_3 \) by an element with quadratic minimal polynomial. How many elements do these extensions have? How many isomorphism classes of extensions does one get in this way?

4. Construct a field with 4 elements and a field with 8 elements.

5. (Quadratic extensions)

(a) Let \( K \) be a field of characteristic not equal to 2 and let \( m \) be a quadratic polynomial over \( K \). Show that \( m \) has a zero in a simple extension \( K(\alpha) \) of \( K \) where \( \alpha^2 = k \in K \). Thus allowing ‘square roots’ \( \sqrt{k} \) enables us to solve all quadratic equations over \( K \).

(b) Show that for fields of characteristic 2 there exist quadratic equations which cannot be solved by adjoining square roots of elements of the field. (Hint: try \( \mathbb{Z}_2 \)).

(c) Show that we can solve all quadratic equations over a field of characteristic 2 if we allow ourselves not only to adjoin square roots of elements but ‘generalised’ square roots \( \sqrt{k} \) defined to be solutions of the equation \( t^2 + t = k \).

6. Suppose \( L = K(\alpha, \beta) \), where the degrees of the minimal polynomials of \( \alpha \) and \( \beta \) over \( K \) are relatively prime integers \( m \) and \( n \). Show that \( [L : K] = mn \).

7. Suppose \( L \supset K \) and \( \alpha \) and \( \beta \) are elements of \( L \) such that \( \alpha \beta \) and \( \alpha + \beta \) are algebraic over \( K \). Deduce that \( \alpha \) and \( \beta \) are algebraic over \( K \).

8. (a) Show that if \( p \) is a prime number, then for every positive integer \( n \), the polynomial \( x^n - p \) is irreducible over \( \mathbb{Q} \).

(b) By making the substitution \( y = x - 1 \), or otherwise, show that when \( p \) is a prime number, the polynomial \( t^{p-1} + t^{p-2} + \ldots + t + 1 \) is irreducible over \( \mathbb{Q} \).