Number Theory and Group Theory
Exercise Sheet 2

October 17, 2012

1. One way to find a solution in integers of $3x + 4y = 5$ is to find a solution of $3x + 4y = 1$ and then multiply it by 5. Show by an example that not every solution of $3x + 4y = 5$ can be obtained in this way.

2. In each of the following cases find the general solution in integers of the given equation.
   (a) $15x + 12y = 7$.
   (b) $15x + 12y = 9$.
   (c) $463x + 46y = 4$.

3. Find all positive integers $a, b, c$ with $a + b + c = 20$ and $6a + 4b + c = 44$.

4. Let $a$ and $b$ be integers which are not both zero. Show that every common factor of $a$ and $b$ divides $(a, b)$.

5. A competition has 100 prizes each worth £100, £5 or £1.25. The total value of the prizes is £1000. How many prizes are there of each value?

6. Let $a, b, n$ be positive integers such that $(a, b) = 1$ and
   \[ n \geq (a - 1)(b - 1). \]
   Show that there are NON-NEGATIVE integers $x$ and $y$ such that $ax + by = n$.
   
   (Hint: Because $(a, b) = 1$ we know that there are integers $x_0$ and $y_0$ such that $ax_0 + by_0 = n$. Try showing that $x_0$ and $y_0$ can be ‘adjusted’ to ensure that they are both non-negative by showing that there is an integer $k$ with $x_0 + bk > -1$ and $y_0 - ak > -1$.)

   Note: This result shows for instance that if you have a supply of 3p and 5p stamps then you can make up any amount from 8p upwards, and in this case also note that you can not make up 7p in this way.