Galois Theory problems, #3.

1. Show directly that the only subfields of \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \) are \( \mathbb{Q}, \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{6}) \), and \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \).

2. Describe the Galois groups of the following extensions \( L \supset K \):
   
   (a) \( \mathbb{Q}(\sqrt{2}) \supset \mathbb{Q} \).
   
   (b) \( \mathbb{Q}(\alpha) \supset \mathbb{Q} \), where \( \alpha \) is the real 5th root of 7.
   
   (c) \( \mathbb{Q}(\omega) \supset \mathbb{Q} \), where \( \omega = e^{2\pi i} \in \mathbb{C} \).
   
   (d) \( \mathbb{Z}_2(\alpha) \supset \mathbb{Z}_2 \), where \( \alpha \) has minimal polynomial \( t^3 + t + 1 \).

3. Let \( G \) be the Galois group of \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \supset \mathbb{Q} \), as calculated in class, and let \( \beta = \sqrt{2} + \sqrt{3} \). Multiply out the polynomial

\[
\prod_{g \in G} (t - g(\beta)).
\]

What do you notice?