Number Theory and Group Theory
Exercise Sheet 3

October 24, 2012

1. Let \( a \) and \( b \) be integers which are not relatively prime. Show that there is a prime number \( p \) such that \( p \) divides both \( a \) and \( b \).

2. Show that any two consecutive perfect squares are relatively prime, i.e. that if \( n \) is a positive integer then \( n^2 \) and \( (n+1)^2 \) are relatively prime.

3. (Cataldi 1548–1626) Let \( n \) be a positive integer with \( n \neq 1 \). Suppose that \( n \) is composite (i.e. that \( n \) is not a prime number). Show that there is a prime number \( p \) such that \( p \) divides \( n \) and \( p \leq \sqrt{n} \).

4. Use Question 3 and a calculator to determine whether or not 1763 and 1777 are prime numbers.

5. Find all prime numbers \( p \) such that \( 7^p + 4 \) is a perfect square and justify your answer.

6. Let \( n \) be a positive integers such that \( 2^n - 1 \) is a prime number. Show that \( n \) is a prime number. (Hint: Suppose that \( a \) and \( b \) are positive integers with \( ab = n \). Use the identity

\[
x^b - 1 = (x - 1)(x^{b-1} + x^{b-2} + \cdots + x^2 + x + 1)
\]

\( to show that 2^n - 1 \) divides \( 2^n - 1 \).

7. Use a calculator to show that \( 2^{11} - 1 \) is not a prime number. Thus the converse of the result in Question 6 is not true in general, i.e. there is a prime number \( p \) such that \( 2^p - 1 \) is not a prime number.