FIRST YEAR GROUP THEORY
SOLUTIONS FOR SECTION 1

1. Let $x, y, z \in Z$. Then $x * y = x + y + 1 \in Z$, so that $Z$ is closed under $*$. For associativity we must show that $(x * y) * z = x * (y * z)$. We have $(x * y) * z = (x + y + 1) * z = x + y + z + 1 = x + y + z + 2$, and $x * (y * z) = x * (y + z + 1) = x + (y + z + 1) + 1 = x + y + z + 2 = (x * y) * z$. Therefore $*$ is associative. For an identity element $e$ we need an integer $e$ such that $x * e = x = e * x$ for all $x \in Z$, i.e. $x + e + 1 = x$, i.e. $e = -1$. For all $x \in Z$ we have $x * (-1) = x + (-1) + 1 = x = (-1) * x$, so that $-1$ is the identity element. Given $x \in Z$, the inverse $a$ of $x$ should be an integer $a$ such that $x * a = e$, i.e. $x + a + 1 = -1$, i.e. $a = -2 - x$. Therefore the inverse of an integer $x$ with respect to $*$ is $-2 - x$.

2. (1) The associative law fails, e.g. $1 - (2 - 3) = 2$ but $(1 - 2) - 3 = -4$.

(2) Closure fails because $2$ is a non-zero element of $Z/4Z$ but $2.2 = 0$ in $Z/4Z$.

(3) The set of odd integers is not closed with respect to addition, e.g. $1 + 1$ is not odd.

3. (1) Suppose that $xy = e$. Let $a$ be the inverse of $x$ in $G$. Then $xa = e$. Hence $xy = xa$, and using the cancellation law (1.31) gives $y = a$ as required. The case $yx = e$ is similar.

(2) We have $x^2(x^{-1})^2 = x xx^{-1}x^{-1} = xex^{-1} = x e^{-1} = e$. Therefore $(x^{-1})^2$ is the inverse of $x^2$, by (1).

(3) For all $x, y \in G$ we have $xy(y^{-1}x^{-1}) = x(yy^{-1})x^{-1} = xex^{-1} = x e^{-1} = e$. Therefore $y^{-1}x^{-1}$ is the inverse of $xy$, by (1).

4. Because $G$ is closed we must have either $a^2 = e$ or $a^2 = a$. Suppose that $a^2 = a$. Then $a = e$ by 1.32, which is a contradiction because $e$ and $a$ are distinct. Therefore $a^2 = e$.

5. (1) Because $G$ is closed we know that $ab$ must be one of the three elements $e, a, b$. Suppose that $ab = a$. Then $ab = ae$ so that cancellation gives $b = e$, which is a contradiction. Similarly $ab = b$ would lead to the contradiction $a = e$. Therefore $ab = e$.

(2) By closure we know that $a^2$ is one of the elements $e, a, b$. Suppose that $a^2 = e$. Then by (1) we have $a^2 = ab$ so that $a = b$, which is a contradiction. Suppose that $a^2 = a$. Then $a = e$ by 1.32, which is a contradiction. Therefore $a^2 = b$.

(3) Combining (1) and (2), we have $e = ab = a, a^2 = a^3$. Continued/...
6. Working in $Z/11Z$ we have for instance that $10 = -1, 9 = -2$, etc.

(1) 1 is its own inverse; 10 = -1 is its own inverse; 2 and 6 are inverses for each other because $26 = 12 = 1$ in $Z/11Z$; 3 and 4 are inverses for each other; 5 and 9 are inverses for each other; 7 and 8 are inverses for each other.

(2) Take $w = 2$. Then $w = 2, w^2 = 4, w^3 = 8 = -3, w^4 = -6 = 5, w^5 = 10 = -1, w^6 = -2 = 9, w^7 = -4 = 7, w^8 = -8 = 3, w^9 = 6, w^{10} = 12 = 1$.

NOTE: $w = 2$ is the simplest thing to try, and it works in this case. If 2 does not work for this purpose in $Z/pZ$ for some other prime number $p$, try -2, and if that does not work try 3, etc. This “trial and error” approach is the only method we have for this sort of problem.

7. In view of 1.34, all we need to do in each case is to list those positive integers from 1 to $n$ which are relatively prime to $n$.

(1) 1 and 3.
(2) 1 and 5.
(3) 1, 2, 4, 5, 7, 8.
(4) 1, 3, 7, 9.

Thus $U_{10}$ consists of the four elements 1, 3, 7 = -3, and 9 = -1. But $U_{10}$ is not the Klein 4-group (Definition 1.26) because in $Z/10Z$ we have $3^2 = 9 \neq 1$.

8. There are four symmetries of a non-square rectangle: the identity transformation; rotation through 180 degrees; the two reflections about the lines joining mid-points of opposite sides. The group is the Klein 4-group.

9. In effect we are given in the question that every element of $G$ is its own inverse. Let $x, y \in G$. Then $x^{-1} = x$ and $y^{-1} = y$. Also $(xy)^{-1} = xy$, i.e. $y^{-1}x^{-1} = xy$, i.e. $yx = xy$. Therefore $G$ is Abelian.