FIRST-YEAR GROUP THEORY
SOLUTIONS TO THE EXERCISES FOR SECTIONS 4 AND 5

1. Set \( G = D_3 \) and let the elements of \( G \) be denoted by \( e, a, a^2, b, ab, a^2b \) as usual as in Example 1.21. Then \( G \) has one cyclic subgroup of order 1, which consists of \( e \) only. There are three elements of order 2, namely \( b, ab, a^2b \) and each of these together with \( e \) gives a cyclic subgroup of order 2. The remaining elements \( a \) and \( a^2 \) both have order 3, and together with \( e \) they form the unique cyclic subgroup of order 3. No element of \( G \) has order 6, so that \( G \) itself is a non-cyclic subgroup of \( G \).

2. (1) 1, 3, 5, 9, 11, 13.
(2) Working in \( Z/14Z \) we have \( 3^2 = 9, 3^3 = 27 = -1, 3^4 = 3^3.3 = -1.3 = -3, 3^5 = -9, 3^6 = -27 = 1. \) Therefore \( 3 \) has order 6 as an element of \( U_{14} \), and because \( U_{14} \) has order 6 it follows that \( U_{14} \) is cyclic and is generated by the element 3.

3. Set \( G = U_{15} \). The elements of \( G \) are 1, 2, 4, 7, 8 = -7, 11 = -4, 13 = -2, 14 = -1. Thus \( G \) has order 8. Working in \( Z/15Z \) we have \( 2^4 = 16 = 1 = (-2)^4; 4^2 = 16 = 1 = (-4)^2; 7^2 = 49 = 4 \) so that \( 7^4 = 16 = 1 = (-7)^4. \) Hence \( G \) has no element of order 8, so that \( G \) is not cyclic.

4. 0, 4, 8.

5. Set \( G = (Q, +) \), and suppose that \( G \) is cyclic with generator \( x \). Then every element of \( G \) is of the form \( nx \) for some integer \( n \). In particular \( x/2 \in G \), so that \( x/2 = nx \) for some \( n \in Z \). Because every rational number is of the form \( kx \) for some integer \( k \), we have \( x \neq 0 \). This, together with \( x/2 = nx \) for some \( n \in Z \), is a contradiction.

6. Without loss of generality we can suppose that the operations in \( H \) and \( K \) are multiplication. Let \( x \) be a generator of the cyclic group \( G \). We have \( x = (a, b) \) for some \( a \in H, b \in K \). Let \( h \in H \) and set \( g = (h, e) \), where \( e \) is the identity element of \( K \). Because \( g \in G \) and \( G \) is generated by \( x \), we have \( g = x^n \) for some \( n \in Z \). Thus \( (h, e) = (a, b)^n = (a^n, b^n) \), so that \( h = a^n \). Therefore every element of \( H \) is a power of \( a \), so that \( H \) is cyclic with generator \( a \).

7. Let \( z \) be the element of \( H \) of order 2, and let \( Z_2 \) consist of the integers 1 and \(-1\) under multiplication. Then there are three elements of order 2 in \( H \times Z_2 \), namely \((z, 1), (z, -1), (e, -1)\) where \( e \) is the identity element of \( H \). The other 12 non-identity elements of \( H \times Z_2 \) have order 4.