QUANTITATIVE BOUNDS FOR SETS LACKING HOMOGENEOUS POLYNOMIAL PROGRESSIONS

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Abstract. We obtain quantitative bounds in a special case of the polynomial Szemerédi theorem of Bergelson and Leibman, provided the polynomials are homogeneous and of the same degree. Such configurations include arithmetic progressions with common difference equal to a $k$th power.

1. Introduction

The polynomial Szemerédi’s theorem of Bergelson and Leibman [BL96] states that for polynomials $P_1, \ldots, P_n \in \mathbb{Z}[x]$ with zero constant term, a set $A \subset [N] := \{1, 2, \ldots, N\}$ lacking configurations of the form

$$x, x + P_1(y), \ldots, x + P_n(y) \quad \text{with} \quad y \in \mathbb{Z} \setminus \{0\}$$

satisfies the size bound $|A| = o_P(N)$. Gowers [Gow01a, Gow01b] has posed the problem of obtaining quantitative bounds for the $o_P(N)$ term appearing in this theorem. The purpose of this note is to prove a special case of such a result.

Theorem 1.1. Let $c_1, \ldots, c_n \in \mathbb{Z}$. If $A \subset [N]$ lacks configurations of the form

$$x, x + c_1y^k, \ldots, x + c_ny^k \quad \text{with} \quad y \in \mathbb{Z} \setminus \{0\}$$

then $A$ satisfies the size bound

$$|A| \ll_{c,k} N (\log \log N)^{-c(n,k)},$$

where $c(n,k)$ is a positive absolute constant dependent only on $n$ and $k$.

This has a seemingly more general consequence.

Corollary 1.2. Let $P_1, \ldots, P_n \in \mathbb{Z}[y_1, \ldots, y_m]$ be homogeneous polynomials of degree $k$, and let $K$ denote a finite union of proper subspaces of $\mathbb{R}^m$. If $A \subset [N]$ lacks configurations of the form

$$x, x + P_1(y), \ldots, x + P_n(y) \quad \text{with} \quad y \in \mathbb{Z}^m \setminus K$$

then $A$ satisfies the size bound

$$|A| \ll_{P,K} N (\log \log N)^{-c(n,k)}.$$
the methods of Gowers [Gow01b]. The use of van der Corput’s inequality allows us to control the size of the coefficients \(a_i\).

The main technical difficulty in our approach is that the common difference \(y\) in the resulting linear configuration is constrained to lie in a much shorter interval than the shift parameter \(x\). Unfortunately, the current inverse theory for the Gowers norms can only handle parameters \(x\) and \(y\) ranging over similarly sized intervals. Our strategy, heuristically at least, is to decompose \(y\) into a difference of smaller parameters \(y = y_2 - y_1\). Changing variables in the shift \(x\), we transform the configuration \(x, x + a_1y, \ldots, x + a_dy\) into one of the form

\[
x + b_1y_1, \ x + c_2y_2, \ x + b_3y_1 + c_3y_2, \ldots, \ x + b_dy_1 + c_dy_2.
\]

For each fixed value of \(x\), one can view this as a shift of the linear configuration

\[
b_1y_1, \ c_2y_2, \ b_3y_1 + c_3y_2, \ldots, \ b_dy_1 + c_dy_2.
\]

Crucially, in this linear configuration the parameters \(y_1\) and \(y_2\) range over the same interval. To each of these shifted ‘short’ configurations we apply Gowers’s local inverse theorem for the \(U^d\)-norm [Gow01b], which yields a density increment on an even shorter subprogression.

Unfortunately, we cannot use Gowers’s local inverse theorem as a black box. Instead, we must modify the theorem in a standard way to deliver a density increment on an arithmetic progression with common difference equal to a \(k\)th power. The proof of this modification requires extra information on small fractional parts of polynomials to be incorporated into a few of the initial lemmas of [Gow01b]. This slight twist must then be carried through the remainder of the paper, and appears to require re-running the majority of the arguments. We devote a separate note [Pre] to an exposition of this modified local inverse theorem.

To the author’s knowledge, the only existing quantitative bound for a non-linear configuration of the form (3) and with length greater than two is a result of Green [Gre02], which deals with the configuration

\[
x, \ x + y, \ x + 2y \quad \text{where} \quad y = a^2 + b^2.
\]

In a technical tour de force, Green employs the Gowers \(U^3\)-norm to attack this configuration, whereas a naive application of the method in our present paper would require the \(U^7\)-norm. In a future paper, we show how one can in fact use the \(U^2\)-norm\(^1\) to deal with this configuration, and consequently obtain good quantitative bounds.

Unfortunately, we have not been able to devise a way to directly apply our methods to inhomogeneous configurations, such as

\[
x, \ x + y, \ x + y^2.
\]

The difficulty lies in our application of the local inverse theorem for the Gowers norms. This gives us a density increment on a subprogression whose modulus can be much larger than its length, and this poses a difficulty for the inhomogeneous density increment strategy. In another future paper, we employ the global \(U^3\)-inverse theorem of Green and Tao [GT08] in order to obtain quantitative bounds for sets lacking the configuration (4).

The structure of our argument should be standard to those familiar with the density increment method, and is hopefully ordered in a self-explanatory manner. For those unfamiliar with this strategy see Green [Gre05] or Gowers [Gow01a] for an overview.

We end this introduction by showing how Corollary 1.2 follows from Theorem 1.1.

\(^1\)More precisely, classical linear Fourier analysis.
Proof that Theorem 1.1 $\implies$ Corollary 1.2. Suppose that $A \subset [N]$ lacks configurations of the form (3). Since $K$ is a union of proper subspaces of $\mathbb{R}^m$, there exists $z \in \mathbb{Z}^m \setminus K$. Notice that we must have $yz \notin K$ for all $y \in \mathbb{Z} \setminus \{0\}$. Let us define $c_i := P_i(z)$ for $i = 1, \ldots, n$. Then by homogeneity, the set $A$ lacks configurations of the form $x, x + c_1y^k, \ldots, x + c_ny^k$ with $y \in \mathbb{Z} \setminus \{0\}$. The result now follows on employing Theorem 1.1. \hfill $\square$

2. Configuration-free sets have large balanced count

Given a set $A \subset [N] := \{1, 2, \ldots, N\}$ with $|A| = \delta N$, define the balanced function of $A$ with respect to $[N]$ by

$$f_A := 1_A - \delta 1_{[N]}.$$ 

The function $f_A$ has mean zero. Hence, if $A$ were a random set, one would expect the statistics of $f_A$ to be close to zero when averaged along polynomial configurations. Our first lemma shows that if $A$ lacks configurations of the form (2), then this non-randomness is witnessed by such arithmetic average.

In the remainder of the paper we assume that $c_1, \ldots, c_n$ are distinct non-zero integers. Notice one may assume this in Theorem 1.1 by reducing $n$ if necessary.

Lemma 2.1 (Large balanced count). Given distinct non-zero integers $c_1, \ldots, c_n$ there exists an absolute constant $B = B(c, k)$ such that if $N \geq B\delta^{-kn}$ and $A \subset [N]$ lacks a configuration of the form

$$x, x + c_1y^k, \ldots, x + c_ny^k \quad \text{with} \quad y \in \mathbb{Z} \setminus \{0\},$$ (5)

then there exist functions $f_0, f_1, \ldots, f_n \in \{ \delta 1_{[N]}, f_A \}$ with $f_n = f_A$ and distinct non-zero integers $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n$ of magnitude at most $B$ such that

$$\left| \sum_{x,y} f_0(x)f_1(x + \tilde{c}_1y^k) \cdots f_n(x + c_ny^k) \right| \geq \frac{1}{B} \delta^{n+1} N^{1+\frac{k}{2}}. \quad (6)$$

Proof. Since $A$ lacks configurations of the form (5), we have

$$\sum_{x,y} 1_A(x)1_A(x + c_1y^k) \cdots 1_A(x + c_ny^k) = |A| = \delta N.$$ 

Making the substitution $1_A = \delta 1_{[N]} + f_A$ and expanding, we deduce that there exist $f_0, f_1, \ldots, f_n$ satisfying

$$\{f_A\} \subset \{f_0, f_1, \ldots, f_n\} \subset \{f_A, \delta 1_{[N]}\}$$

and

$$\delta N + (2^{n+1} - 1) \left| \sum_{x,y} f_0(x)f_1(x + c_1y^k) \cdots f_n(x + c_ny^k) \right| \geq \delta^{n+1} \sum_{x,y} 1_{[N]}(x)1_{[N]}(x + c_1y^k) \cdots 1_{[N]}(x + c_ny^k). \quad (7)$$

Choosing $B = O_{c,k}(1)$ sufficiently large, the assumption $N \geq B$ yields

$$\sum_{x,y} 1_{[N]}(x)1_{[N]}(x + c_1y^k) \cdots 1_{[N]}(x + c_ny^k) \gg_{c,k} N^{1+\frac{k}{2}}.$$
Since $N \geq B\delta^{-kn}$ we can subtract $\delta N$ from either side of (7) and deduce that
\[
\left| \sum_{x,y} f_0(x)f_1(x+c_1y^k) \cdots f_n(x+c_ny^k) \right| \gg c_{e,k} \delta^{n+1} N^{1+\ell}.
\] (8)

If $f_A = f_i$ for some $i > 0$ then, after re-labelling the indices of the $f_j$, we take $\tilde{c}$ to be any permutation of $c$ which ensures that $f_n = f_A$. If $f_0 = f_A$ we shift the $x$ variable in (8) by $-c_n y^k$ and take
\[\tilde{c} := (c_1-c_n, \ldots, c_{n-1} - c_n, -c_n).\]

\[\square\]

3. The linearisation process

In this section we show how the large non-linear arithmetic average (6) leads to a large linear arithmetic average, albeit over a longer configuration. This deduction proceeds via the method of van der Corput differencing.

Lemma 3.1 (van der Corput differencing). Let $g : \mathbb{Z} \to \mathbb{C}$ be a function supported on a finite set $S \subset \mathbb{Z}$. Given a finite set $H \subset \mathbb{Z}$, write $r_H(h)$ for the number of pairs $(h_1, h_2) \in H^2$ such that $h_1 - h_2 = h$. Then we have the estimate
\[
\left| \sum_y g(y) \right|^2 \leq \frac{|S - H|}{|H|^2} \sum_h r_H(h) \sum_y g(y + h)g(y).
\]

Proof. By a change of variables, for any $h \in \mathbb{Z}$ we have
\[
\sum_y g(y) = \sum_y g(y + h).
\]
Averaging over $h \in H$ and interchanging the order of summation gives
\[
\sum_y g(y) = \frac{1}{|H|} \sum_y \sum_{h \in H} g(y + h).
\]
The function
\[y \mapsto \sum_{h \in H} g(y + h)\]
is supported on the difference set $S - H$. Squaring and applying Cauchy–Schwarz, we deduce that
\[
\left| \sum_y g(y) \right|^2 \leq \frac{|S - H|}{|H|^2} \sum_{h_1, h_2 \in H} g(y + h_1)g(y + h_2)
\]
\[= \frac{|S - H|}{|H|^2} \sum_{h_1, h_2 \in H} \sum_y g(y + h_1 - h_2)g(y)
\]
\[= \frac{|S - H|}{|H|^2} \sum_h r_H(h) \sum_y g(y + h)g(y).
\]
\[\square\]

Lemma 3.2 (weak van der Corput). Suppose that $g : \mathbb{Z}^2 \to [-1, 1]$ is supported on $[N] \times [M]$ with $N, M \geq 1$. Let $1 \leq H \leq M$ and let $H_0 \subset \mathbb{Z}$ denote a set containing 0 of cardinality at most $H_0$. Then there exists $h \in [H] \setminus H_0$ such that
\[
\sum_x \left( \sum_y g(x, y) \right)^2 \leq 4NM^2H_0H^{-1} + 4M \sum_{x,y} g(x, y + h)g(x, y).
\]
Remark. Applying this result to the function \((x, y) \mapsto g(x, -y)\), we see that the same inequality holds for some \(h \in [H] \setminus \mathcal{H}_0\). We may therefore choose whether \(h\) is positive or negative.

Proof. Let us apply Lemma \([3.1]\) with \(g_x(y) := g(x, y)\), \(S := [M]\) and \(\mathcal{H} := [H]\). One can check that
\[
\frac{|S - \mathcal{H}|}{|\mathcal{H}|} \leq \frac{2M}{H},
\]
from which we deduce that
\[
\sum_{x} \left( \sum_{y} g(x, y) \right)^2 \leq \frac{2M}{H} \sum_{h} r_{|\mathcal{H}|(h)} \sum_{x,y} g(x, y + h)g(x, y).
\]

A change of variables gives the identity
\[
\sum_{x,y} g(x, y + h)g(x, y) = \sum_{x,y} g(x, y)g(x, y - h).
\]

Combining this with the fact that \(r_{|\mathcal{H}|}(0) = |\mathcal{H}|, r_{|\mathcal{H}|}(-h) = r_{|\mathcal{H}|}(h)\) and \(r_{|\mathcal{H}|}(h) = 0\) if \(|h| \geq H\), we have
\[
\sum_{h} \frac{r_{|\mathcal{H}|}(h)}{|\mathcal{H}|} \sum_{x,y} g(x, y + h)g(x, y) = \sum_{x,y} g(x, y)^2 + \sum_{h \in |\mathcal{H}| \setminus \mathcal{H}_0} \frac{2r_{|\mathcal{H}|}(h)}{|\mathcal{H}|} \sum_{x,y} g(x, y + h)g(x, y). \tag{9}
\]

Using the trivial estimates \(|\text{supp}(g)| \leq NM, r_{|\mathcal{H}|}(h) \leq |\mathcal{H}|\) and \(|\mathcal{H}_0 \cap |\mathcal{H}|| \leq H_0 - 1\), the right-hand side of \((9)\) is at most
\[
NM + 2(H_0 - 1)NM + \sum_{h \in |\mathcal{H}| \setminus \mathcal{H}_0} \frac{2r_{|\mathcal{H}|}(h)}{|\mathcal{H}|} \sum_{x,y} g(x, y + h)g(x, y)
\]

By the pigeon-hole principle and monotonicity of expectation, there exists \(h' \in [H] \setminus \mathcal{H}_0\) such that
\[
\sum_{h \in |\mathcal{H}| \setminus \mathcal{H}_0} \frac{r_{|\mathcal{H}|}(h)}{|\mathcal{H}|} \sum_{x,y} g(x, y + h)g(x, y) \leq H \sum_{x,y} g(x, y + h')g(x, y).
\]

The required inequality follows. \(\square\)

Lemma 3.3 (Linearisation step). Let \(f_0, f_1, \ldots, f_n : \mathbb{Z} \to [-1, 1]\) be 1-bounded functions supported on \([N]\), let \(I\) be an interval of at most \(M\) integers, let \(\mathcal{H}_0\) be a set containing 0 of size at most \(H_0\) and let \(P_1, \ldots, P_n : \mathbb{Z} \to \mathbb{Z}\). Suppose that
\[
\delta NM \leq \left| \sum_{x} \sum_{y \in I} f_0(x) f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) \right|.
\]

Then for \(M \geq 8H_0\delta^{-2}\) there exists \(h \in \mathbb{Z} \setminus \mathcal{H}_0\) with \(1 \leq h \leq 1 + 8H_0\delta^{-2}\) such that
\[
\frac{1}{8}\delta^2 NM \leq \sum_{x} \sum_{y \in I \cap (I - h)} \prod_{1 \leq i \leq n} f_i(x + P_i(y + \omega h) - P_i(y)).
\]

Remark. Again, the same conclusion can be drawn for some \(h\) with \(-1 - 8H_0\delta^{-2} \leq h \leq -1\), so that we may choose whether \(h\) is positive or negative.
Proof. Shifting the argument of the functions $P_i$, we may assume that $I = [M]$. Set $g(x, y) := f_1(x + P_1(y)) \cdots f_n(x + P_n(y))1_{[N]}(x)1_{[M]}(y)$ and let $H \in [1, M]$ (to be determined later). Then by the Cauchy–Schwarz inequality and Lemma 3.2 there exists $h \in [H] \setminus H_0$ such that

$$
(\delta NM)^2 \leq \left( \sum_{x, y \in I} f_0(x)f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) \right)^2 = \left( \sum_x f_0(x) \sum_y g(x, y) \right)^2
$$

$$
\leq \left( \sum_x f_0(x)^2 \right) \sum_x \left( \sum_y g(x, y) \right)^2
$$

$$
\leq 4N^2M^2H_0H^{-1} + 4NM \sum_{x,y} g(x, y + h)g(x, y)
$$

$$
= 4N^2M^2H_0H^{-1} + 4NM \sum_y \sum_x g(x - P_1(y), y + h)g(x - P_1(y), y).
$$

The $4N^2M^2H_0H^{-1}$ term in the above is sufficiently small, and our result follows, provided that we can take $H$ to satisfy

$$
8H_0\delta^{-2} \leq H \leq 8H_0\delta^{-2} + 1.
$$

That such an $H$ exists in the interval $[1, M]$ follows from our assumption that $M \geq 8H_0\delta^{-2}$. \hfill \Box

We iteratively apply this lemma, beginning with the configuration $x, x + c_1y^k, \ldots, x + c_ny^k$ and eventually obtaining a configuration of the form $x, x + a_1y, \ldots, x + a_dy$. The complexity of the intermediate configurations requires us to take an abstract approach. Moreover, each application of the linearisation step necessitates a number of technical assumptions whose purpose is to guarantee that the coefficients $a_i$ in our final linear configuration are non-zero and distinct.

Before proceeding to describe this argument in general, we illustrate the underlying ideas for the configuration $x, x + c_1y^2, x + c_2y^2$.

**Lemma 3.4 (Linearisation for square 3APs).** Let $f_0, f_1, f_2 : \mathbb{Z} \to [-1, 1]$ be supported on $[N]$ and let $c_1, c_2$ be distinct non-zero integers. Suppose that

$$
\left| \sum_x \sum_y f_0(x)f_1(x + c_1y^2)f_2(x + c_2y^2) \right| \geq \delta N^{3/2}.
$$

(10)

There there exists an absolute constant $C = O(1)$ such that for $N \geq C\delta^{-8}$ we can find integers $a_i, b_i, M$ with the $a_i$ distinct, $1 \leq a_i \leq C\delta^{-8}$ and $M \geq \frac{1}{C}\delta^{-8}\sqrt{N}$ such that

$$
\left| \sum_x \sum_{y \in [M]} f_2(x)f_2(x + a_1y + b_1) \cdots f_2(x + a_7y + b_7) \right| \geq \frac{1}{C}\delta^8NM.
$$

(11)

Proof. Our assumption on the support of $f_i$ ensures that $f_0(x)f_1(x + c_1y^2)f_2(x + c_2y^2) \neq 0$ only when $|y| < \sqrt{N}$. Let us set

$$
I := (-\sqrt{N}, \sqrt{N}) \cap \mathbb{Z} \quad \text{and} \quad M := |I|.
$$

We can then write (10) as

$$
\delta NM \ll \left| \sum_x \sum_{y \in I} f_0(x)f_1(x + c_1y^2)f_2(x + c_2y^2) \right|,
$$
This inequality is in a form amenable to an application of Lemma 3.3 with \( \mathcal{H}_0 = \{0\} \), provided that \( M \geq C\delta^{-2} \) with \( C = O(1) \). Let \( \epsilon_1 \in \{\pm 1\} \) (to be determined later) and write \( c := c_2 - c_1 \). Applying Lemma 3.3 together with the remark that follows it, we deduce that there exists an integer \( 1 \leq h_1 \ll \delta^{-2} \), an interval \( I_1 \subset I \) and integers \( b_i \) satisfying
\[
\delta^2 NM \ll \sum_{x, y \in I_1} f_1(x)f_1(x + 2c_1\epsilon_1h_1y + b_1)f_2(x + cy^2 + b_2)f_2(x + cy^2 + 2c_2\epsilon_1h_1y + b_3), \tag{12}
\]
Here we have made use of the simple identity \((y + h)^2 - y^2 = 2hy + h^2\).

To save on notation, let us write \( \tilde{f} \) for a function of the form \( x \mapsto f(x + b) \) for some integer \( b \). Different occurrences of \( \tilde{f} \) in the same equation may refer to different values of \( b \), but no confusion should arise. Inequality (12) may then be re-written as
\[
\delta^2 NM \ll \sum_{x, y \in I_1} f_1(x)\tilde{f}_1(x + 2c_1\epsilon_1h_1y)\tilde{f}_2(x + cy^2)\tilde{f}_2(x + cy^2 + 2c_2\epsilon_1h_1y),
\]
Provided that \( N \geq C\delta^{-4} \) with \( C = O(1) \), we can re-apply Lemma 3.3 and conclude that for \( \epsilon_2 \in \{\pm 1\} \) there exists an integer \( 1 \leq h_2 \ll \delta^{-4} \) and an interval \( I_2 \subset I_1 \) satisfying
\[
\delta^4 NM \ll \sum_{x, y \in I_2} f_1(x)\tilde{f}_1(x)\tilde{f}_2(x + cy^2 - 2c_1\epsilon_1h_1y)\tilde{f}_2(x + cy^2 - 2c_1\epsilon_1h_1y + 2c_2\epsilon_2h_2y)\]
\[
\tilde{f}_2(x + cy^2 - 2c_1\epsilon_1h_1y + 2c_2\epsilon_1h_1y)\tilde{f}_2(x + cy^2 - 2c_1\epsilon_1h_1y + 2c_2\epsilon_1h_1y + 2c_2\epsilon_2h_2y).
\]
A function of the form \( x \mapsto f_1(x)\tilde{f}_1(x) \) is 1-bounded, supported on \([N]\) and independent of \( y \). Combining this with the fact that we are assuming that \( N \geq C\delta^{-8} \) with \( C = O(1) \), we can apply Lemma 3.3 once more and conclude that for \( \epsilon_3 \in \{\pm 1\} \) there exists an integer \( 1 \leq h_3 \ll \delta^{-8} \) and an interval \( I_3 \subset I_2 \) satisfying
\[
\delta^8 NM \ll \sum_{x, y \in I_3} f_2(x)\tilde{f}_2(x + 2c_\epsilon_3h_3y)\tilde{f}_2(x + 2c_\epsilon_2h_2y)\tilde{f}_2(x + 2c_\epsilon_2h_2y + 2c_\epsilon_3h_3y)\tilde{f}_2(x + 2c_\epsilon_1h_1y)\]
\[
\tilde{f}_2(x + 2c_\epsilon_2h_1y + 2c_\epsilon_3h_3y)\tilde{f}_2(x + 2c_\epsilon_1h_1y + 2c_\epsilon_2h_2y)\tilde{f}_2(x + 2c_\epsilon_1h_1y + 2c_\epsilon_2h_2y + 2c_\epsilon_3h_3y).
\]
The above can be re-written in the form (11) on noting that \( I_3 \) takes the form \( b + [M'] \) for some integers \( b \) and \( M' \), with \( M' \) satisfying
\[
NM' \geq \sum_{x, y \in I_3} 1 \gg \delta^8 N^{3/2}.
\]
It remains to show that \( \epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\} \) can be chosen to ensure that all of the following coefficients are distinct and positive
\[
2c_2\epsilon_1h_1, \quad 2c_\epsilon_2h_2, \quad 2c_\epsilon_3h_3,
2c_\epsilon_3h_3 + 2c_\epsilon_2h_2, \quad 2c_2\epsilon_1h_1 + 2c_\epsilon_3h_3, \quad 2c_2\epsilon_1h_1 + 2c_\epsilon_2h_2,
2c_2\epsilon_1h_1 + 2c_\epsilon_2h_2 + 2c_\epsilon_3h_3.
\]
This follows if \( c_2\epsilon_1, c_\epsilon_2 \) and \( c_\epsilon_3 \) are positive. Since \( c_2 \) and \( c = c_2 - c_1 \) are both non-zero, the proof is completed on taking \( \epsilon_1 := \text{sgn}(c_2) \) and \( \epsilon_2 = \epsilon_3 := \text{sgn}(c) \). \qed
The above argument required three applications of Lemma \[3.3\] in order to linearise the simplest example of a non-linear \( k \)th power configuration of length greater than two. In general we require many more applications of the linearisation step, and at each stage of the iteration, it is not immediately obvious that we have reduced the ‘degree’ of the configuration at all. To see that we have indeed reduced an invariant associated to the configuration, we require the following definition.

**Definition** (Degree sequence). Given polynomials \( P_1, \ldots, P_n \in \mathbb{Z}[x] \), let \( L(P_i) \) denote the leading coefficient of \( P_i \) and define
\[
D_r(P_1, \ldots, P_n) := \# \{ L(P_i) : \deg P_i = r \}.
\]
In words, \( D_r(P) \) is the number of of distinct leading coefficients occurring amongst the degree \( r \) polynomials in \( P \).

Let us define the **degree sequence** of \( P = (P_1, \ldots, P_n) \) by
\[
D(P) := (D_1(P), D_2(P), D_3(P), \ldots).
\]

**Definition** (Colex order). We order degree sequences according to the colexicographical ordering, so that \( D(P) < D(Q) \) if there exists \( r \in \mathbb{N} \) such that \( D_r(P) < D_r(Q) \) and for all \( s > r \) we have \( D_s(P) = D_s(Q) \).

**Lemma 3.5.** Let \( S \) denote the set of sequences \( (m_i)_{i \in \mathbb{N}} \) of non-negative integers with all but finitely many entries equal to zero. Then colex induces a well-ordering on \( S \). In particular, if \( P(m) \) is a proposition defined on \( S \) satisfying
\[
[\forall m' < m] P(m') \implies P(m),
\]
then \( P(m) \) is true for all \( m \in S \).

**Proof.** We leave the reader to check that \( \preceq \) is transitive, anti-symmetric and total. We show that every non-empty subset of \( S \) has a least element.

Let \( \mathcal{F} \) be a non-empty subset of \( S \) and fix \( m \in \mathcal{F} \). Write \( k \) for the maximum index satisfying \( m_k \neq 0 \). Let \( \mathcal{F}_{k+1} \) consist of those \( m' \in \mathcal{F} \) such that \( m'_i = 0 \) for all \( i \geq k + 1 \). Suppose that we have constructed a non-empty subset \( \mathcal{F}_{i+1} \subset \mathcal{F}_{k+1} \). Writing \( \pi_i \) for the projection onto the \( i \)th coordinate, \( \pi_i(\mathcal{F}_{i+1}) \) is a non-empty set of non-negative integers, hence contains a least element \( m'_i \). Define \( \mathcal{F}_i \) to be the set of \( m' \in \mathcal{F}_{i+1} \) for which \( m'_i = m^*_i \). Iterating this process we eventually obtain \( \mathcal{F}_1 = \{ m^*_1 \} \) and one can check that \( m' \preceq m' \) for all \( m' \in \mathcal{F} \).

**Lemma 3.6** (Degree sequence inductive step). Let \( f_0, f_1, \ldots, f_n : \mathbb{Z} \rightarrow [-1, 1] \) be \( 1 \)-bounded functions supported on \([N]\). Let \( I \) be an interval of at most \( M \) integers. Let \( P_1, \ldots, P_n \in \mathbb{Z}[x] \) be polynomials of height at most \( H \) such that \( 0, P_1, \ldots, P_n \) have distinct non-constant parts. Suppose that \( k := \max_i \deg P_i > 1 \) and
\[
\delta NM \leq \sum_x \sum_{y \in I} f_0(x)f_1(x + P_1(y)) \cdots f_n(x + P_n(y)).
\]

Then provided that \( M \geq 8n^2 \delta^{-2} \), there exists an interval \( I' \subset I \) along with \( 1 \)-bounded functions \( g_0, \ldots, g_n \) supported on \([N]\) and polynomials \( Q_1, \ldots, Q_n \) of height at most \( H(8n^2 \delta^{-1})^{2k} \) and degree sequence \( D(Q) < D(P) \) such that
\[
\frac{1}{2} \delta^2 NM \leq \sum_{x \in I'} \sum_{y \in I'} g_0(x)g_1(x + Q_1(y)) \cdots g_n(x + Q_n(y)).
\]

Moreover, we can ensure that \( 0, Q_1, \ldots, Q_n \) have distinct non-constant parts, \( g_n = f_n \), \( n' \leq 2n - 1 \) and that for \( r := \min \{ r : D_r(P) \neq 0 \} \) we have
\[
D(Q) = (i_1, \ldots, i_{r-1}, D_r(P) - 1, D_{r+1}(P), D_{r+2}(P), \ldots)
\]
for some $i \in \mathbb{N}_0^{-1}$ with $i_1 + \cdots + i_{r-1} \leq 2n - 1$.

Proof. At the cost of increasing the height $H$ by a factor of 2, we may assume that the polynomial $P_n$ occurring in the argument of the function $f_n$ has degree greater than one. To see why this is so, suppose that $j := \max \{ i : \deg P_i > 1 \} < n$. Then the configuration

$$\overline{P} := (-P_1, P_1 - P_J, \ldots, P_n - P_J)$$

has distinct non-constant parts, height at most $2H$, satisfies an analogous inequality to \([13]\) and $\deg \overline{P} > 1$.

Given this assumption, we may re-order indices to ensure the existence of an integer $l \leq n$ such that

$$\deg P_i = 1 \iff i < l.$$

Claim. There are at most $n^2$ choices of $h$ for which the following polynomials have indistinct non-constant parts

$$P_1(y), \ldots, P_n(y), P_i(y + h), \ldots, P_n(y + h) \tag{15}$$

To establish the claim, let us suppose that $h$ is such that two of the polynomials in the list \([15]\) have the same non-constant part. Since the polynomials $P_1, \ldots, P_n$ have distinct non-constant parts, the only possibility is that $P_i(y + h) = P_j(y)$ is constant for some $l \leq i \leq n$ and $1 \leq j \leq n$. Let $P_i(y) = a_d y^d + a_{d-1} y^{d-1} + \ldots$ with $a_d \neq 0$. Then since $d > 1$ we have

$$P_i(y + h) = a_d y^d + (da_d h + a_{d-1}) y^{d-1} + \ldots$$

The expression $da_d h + a_{d-1}$ is equal to the coefficient of $y^{d-1}$ in $P_i$, and this completely determines $h$. Since there are at most $n$ choices for $P_j$ and at most $n$ choices for $P_i$ the claim follows.

Let $\mathcal{H}_0$ denote the set of $h$ for which two of the polynomials in \([15]\) have the same non-constant part. Notice that $\mathcal{H}_0$ contains 0. Applying Lemma 3.3 we deduce that there exists $h \in \mathbb{Z} \setminus \mathcal{H}_0$ with $1 \leq h \leq 1 + 8n^2 \delta^{-2}$ and an interval $I' \subset I$ such that

$$\frac{1}{2} \delta^2 N M \leq \sum_{x} \sum_{y \in I'} \prod_{1 \leq i \leq n} \omega \in \{0,1\} f_i(x + P_i(y + \omega h) - P_i(y)). \tag{16}$$

Given $f : \mathbb{Z} \to \mathbb{R}$ and $a \in \mathbb{Z}$, write $\Delta(f, a)$ for the function

$$\Delta(f, a)(x) := f(x + a) f(x).$$

Then for each $i < l$ there exists an integer $a_i = a_i(h)$ such that

$$f_i(x + P_i(y) - P_i(y)) f_i(x + P_i(y + h) - P_i(y)) = \Delta(f_i, a_i)(x + (P_i - P_i)(y)).$$

We can therefore re-write the right-hand side of \([16]\) as

$$\sum_{x} \sum_{y \in I'} \prod_{i \leq l} \Delta(f_i, a_i)(x + (P_i - P_i)(y)) \prod_{i > l} f_i(x + (P_i - P_i)(y)) f_i(x + P_i(y + h) - P_i(y))$$

$$= \sum_{x} \sum_{y \in I'} g_0(x) g_1(x + Q_1(y)) \cdots g_{n'}(x + Q_{n'}(y)), \text{ say.} \tag{17}$$

From our claim we see that $0, Q_1, \ldots, Q_{n'}$ have distinct non-constant parts, since adding $P_i$ to each polynomial in this sequence gives the sequence \([15]\). From the way we have written \([17]\), we see that $g_{n'} = f A$. From the binomial theorem, one can check that the height of each $Q_i$ is at most $H(8 \delta n^{-1})^{2k}$. 


It remains to show that \( D(Q) \) has the form given in (14), and consequently \( D(Q) \prec D(P) \). From our ordering of indices, we have \( r = \deg P_i \). Hence if \( \deg P_i > r \) then for either choice of \( \omega \in \{0, 1\} \), the polynomial \( P_i(y + \omega h) - P_i(y) \) has the same leading term as \( P_i \). It follows that for \( s > r \) we have \( D_s(Q) = D_s(P) \). Let \( \{a_1, \ldots, a_r\} \) denote the set of leading coefficients which appear in some \( P_i \) with \( \deg P_i = r \). We may assume that \( a_i \) is the leading coefficient of \( P_i \). Then the set of leading coefficients occurring amongst those \( Q_i \) with \( \deg Q_i = r \) is equal to \( \{a_2 - a_1, \ldots, a_r - a_1\} \), which has cardinality one less than \( \{a_1, \ldots, a_r\} \). Moreover, since there are at most \( 2n - 1 \) polynomials \( Q_i \), the number of \( Q_i \) with \( \deg Q_i < r \) is also at most \( 2n - 1 \). This leads to the bound for \( i_1 + \cdots + i_{r-1} \) claimed in the theorem.

**Lemma 3.7** (Degree sequence induction). Let \( f_0, f_1, \ldots, f_n : \mathbb{Z} \to [-1, 1] \) be 1-bounded functions supported on \([N]\). Let \( I \) be an interval of at most \( M \) integers. Let \( P_1, \ldots, P_n \in \mathbb{Z}[x] \) be polynomials of height at most \( H \) such that \( 0, P_1, \ldots, P_n \) have distinct non-constant parts. Suppose that

\[
\delta NM \leq \sum_{x} \sum_{y \in I} f_0(x) f_1(x + P_1(y)) \cdots f_n(x + P_n(y)).
\]

Writing \( m := D(P) \), there exist absolute constants \( C = C(n; m) \) and \( d(n; m) \), dependent only on \( n \) and \( m \), such that for \( M \geq C\delta^{-C} \) one can find:

- an interval \( I' \subset I \);
- 1-bounded functions \( g_0, \ldots, g_d \) supported on \([N]\) with \( d \leq d(n; m) \) and \( g_d = f_n \);
- integers \( a_i, b_i \) with \( a_i \) distinct, non-zero, of magnitude at most \( HC\delta^{-C} \);

and together these satisfy

\[
\frac{1}{\varepsilon}\delta^CNM \leq \sum_{x} \sum_{y \in I'} g_0(x) g_1(x + a_1 y + b_1) \cdots g_d(x + a_d y + b_d). \tag{18}
\]

**Proof.** We proceed by induction along the collex order of \( m := D(P) \).

If \( k := \max_i \deg P_i = 1 \) then we are done on taking \( C(n; m) := 1, d(n; m) := n, I' := I, n' := n, g_i := f_i \) and \( a_i y + b_i := P_i \).

Let us therefore assume that \( k := \max_i \deg P_i > 1 \). Provided that

\[
M \geq 8n^2\delta^{-2}
\]

we may apply Lemma 3.6 and conclude the existence of:

- an interval \( I' \subset I \);
- 1-bounded functions \( g_0, \ldots, g_n \) supported on \([N]\) with \( n' \leq 2n - 1 \) and \( g_n = f_n \);
- polynomials \( 0, Q_1, \ldots, Q_{n'} \) of height at most \( H(8n\delta^{-1})^{2k} \) and distinct non-constant parts,

and together these satisfy the inequality

\[
\frac{1}{\varepsilon}\delta^2 NM \leq \sum_{x} \sum_{y \in I'} g_0(x) g_1(x + Q_1(y)) \cdots g_{n'}(x + Q_{n'}(y)).
\]

Furthermore, writing \( r \) for the smallest index such that \( m_r > 0 \), we have

\[
m' := D(Q) := (i_1, \ldots, i_{r-1}, m_r - 1, m_{r+1}, \ldots) \prec m
\]

for some \( i_1 + \cdots + i_{r-1} \leq 2n - 1 \).

Applying the induction hypothesis, we conclude that there exist absolute constants \( C = C(n'; m') \) and \( d(n'; m') \) such that for

\[
M \geq C(\delta^2/2)^{-C}
\]
there exist the following.
- An interval $I'' \subset I'$.
- 1-bounded functions $\tilde{g}_0, \ldots, \tilde{g}_d$ supported on $[N]$ with $d \leq d(n'; m')$ and $\tilde{g}_d = f_n$.
- Integers $a_i, b_i$ with $a_1, \ldots, a_d$ distinct, non-zero and of magnitude at most $H(8n\delta^{-1})^{2k}C(\delta^2/2)^{-C}$.

Moreover, we have the inequality
$$\frac{1}{C} (\delta^2/2)^C NM \leq \sum_{x \in I''} \sum_{y \in I'} \tilde{g}_0(x) \tilde{g}_1(x + a_1 y + b_1) \cdots \tilde{g}_d(x + a_d y + b_d)$$

The lemma follows provided that we can define appropriate constants $C(n; m)$ and $d(n'; m')$ dependent only on $n$ and $m$. Let us define the set
$$\mathcal{M}(n; m) := \{ m' : m'_j \leq 2n - 1 \text{ for } j < r, \quad m'_{r} = m_r - 1, \quad m'_j = m_j \text{ for } j > r \}.$$  Notice that since $r$ is the minimal index for which $m_r \neq 0$, this set $\mathcal{M}(n; m)$ is finite and completely determined by $n$ and $m$. By induction along colex, for any $n'$ and any $m' \in \mathcal{M}(n; m)$, the constants $C(n'; m')$ and $d(n'; m')$ exist. We can therefore take
$$d(n; m) := \max_{ n' \leq 2n - 1 \atop m' \in \mathcal{M}(n; m) } d(n'; m').$$

Similarly, one can check that we can take
$$C(n; m) := \max_{ n' \leq 2n - 1 \atop m' \in \mathcal{M}(n; m) } (8n)^{2k} 4^{C(n'; m')}.$$  

\vspace{1em}

**Corollary 3.8** (Linearisation for $k$th power configurations). Let $f_0, f_1, \ldots, f_n : \mathbb{Z} \to [-1, 1]$ be 1-bounded functions supported on $[N]$ with $f_n = f_A$ and let $c_1, \ldots, c_n$ be distinct non-zero integers. Suppose that
$$\left| \sum_{x \in I} \sum_{y \in I} f_0(x) f_1(x + c_1 y^k) \cdots f_n(x + c_n y^k) \right| \geq \delta N^{1 + \frac{k}{2}}. \tag{19}$$  

Then there exists $C = C(n, k)$ and $d(n, k)$ such that for $N \geq C\delta^{-C}$ there are 1-bounded functions $g_0, g_1, \ldots, g_d$ supported on $[N]$ with $d \leq d(n, k)$ and $g_d = f_A$, along with integers $a_i, b_i, M$ with $M \geq \frac{1}{C} \delta^{C} N^{1/k}$ and the $a_i$ distinct, non-zero, of order $O_{c,k}(\delta^{-C})$ and such that
$$\left| \sum_{x \in [M]} \sum_{y \in [M]} g_0(x) g_1(x + a_1 y + b_1) \cdots g_d(x + a_d y + b_d) \right| \geq \frac{1}{C} \delta^{C} NM. \tag{20}$$

**Proof.** Define
$$I := (-N^{1/k}, N^{1/k}) \cap \mathbb{Z} \quad \text{and} \quad M := |I|.$$  

We can then write \eqref{eq:19} as
$$\delta NM \ll_k \left| \sum_{x \in I} \sum_{y \in I} f_0(x) f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) \right|.$$  

The conclusion follows on applying Lemma 3.7.  \hfill \square
4. Large local linear count implies many large localised Gowers norms

In this section we show how a function $g_d$ which has a large linear arithmetic average of the form (20) also has large Gowers $U^d$-norm on many short intervals.

Definition (Difference function). Given $f : \mathbb{Z} \to \mathbb{R}$ define the (multiplicative) difference function at $y \in \mathbb{Z}$ by

$$\Delta(f, y)(x) := f(x + y)f(x).$$

Definition (Gowers uniformity norm). Given a function $f : \mathbb{Z} \to \mathbb{R}$ with finite support, define

$$\|f\|_{U^1}^2 := \left| \sum_x f(x) \right|^2$$

and for $d \geq 2$ set

$$\|f\|_{U^d}^{2^d} := \sum_y \|\Delta(f, y)\|_{U^{d-1}}^{2^d-1}.$$

Notation. Let us define the localisation of $f : \mathbb{Z} \to \mathbb{C}$ at $(x, a, M)$ by

$$f_{x,a,M}(y) := f(x + ay)1_{[M]}(y)$$

Lemma 4.1. There exists an absolute constant $C = O(1)$ such that for $M \geq C\delta^{-2}$ the following holds. Suppose that $f_0, \ldots, f_d : \mathbb{Z} \to [-1, 1]$ are functions supported on $[N]$ satisfying

$$\left| \sum_x \sum_{y \in [M]} f_0(x)f_1(x + a_1y + b_1) \cdots f_d(x + a_dy + b_d) \right| \geq \delta NM. \tag{21}$$

where $a_i, b_i$ are integers with the $a_i$ distinct and satisfying $0 < |a_i| \leq H$ for all $i$. Then for any $f \in \{f_0, f_1, \ldots, f_n\}$ there exists a non-zero integer $|a| \ll H$ such that

$$\sum_x \|f_{x,a,M}\|_{U^d} \gg_d \delta NM^{(d+1)/2d+1} H^{-2}.$$

We deduce this from a standard result in which the common difference $y$ is not constrained to lie in a short interval.

Lemma 4.2. Let $g_0, g_1, \ldots, g_d : \mathbb{Z} \to [-1, 1]$ be functions supported on $[-N, N]$ and let $a_2, \ldots, a_d \in \mathbb{Z}^2$ with no $a_i$ colinear to $(1, 0)$. Then we have

$$\left| \sum_{z_0, z_1} g_0(z_0)g_1(z_1)g_2(a_2 \cdot z) \cdots g_d(a_d \cdot z) \right| \ll_d N^{2d-\frac{d+1}{2d}} \|g_0\|_{U^d}.$$

Proof. We proceed by induction on $d \geq 1$. For $d = 1$ we have the following simple deduction.

$$\left| \sum_{z_0, z_1} g_0(z_0)g_1(z_1) \right| = \left| \sum_{z_0} g_0(z_0) \right| \left| \sum_{z_1} g_1(z_1) \right| \leq (2N + 1) \left| \sum_{z_0} g_0(z_0) \right| = (2N + 1) \|g_0\|_{U^1}.$$

For the induction step, when $d \geq 2$, let us write $(a_0, a_1)$ for $a_d$ and

$$G(z) := g_0(z_0)g_1(z_1)g_2(a_2 \cdot z) \cdots g_{d-1}(a_{d-1} \cdot z).$$
Then we have
\[
\left| \sum_{z_0, z_1} g_0(z_0)g_1(z_1)g_2(a_2 \cdot z) \cdots g_d(a_d \cdot z) \right| = \left| \sum_z G(z)g_d(a_0z + a_1z_1) \right| \\
= \left| \int_T \hat{G}(a_0\alpha, a_1\alpha)\bar{g}_d(\alpha)\,d\alpha \right| \\
\leq \|\hat{g}_d\|_2 \left( \int_T |\hat{G}(a_0\alpha, a_1\alpha)^2|\,d\alpha \right)^{1/2}.
\]

Interpreting the underlying equations, we see that
\[
\int_T |\hat{G}(a_0\alpha, a_1\alpha)|^2\,d\alpha = \sum_{a_d \cdot (z - w) = 0} G(z)G(w) \\
= \sum_{h : a_d \cdot h = 0} \sum_z G(z)G(z + h).
\]

Our non-collinearity assumption certainly implies that \( a_d = (a_0, a_1) \neq 0 \). Let \( c = \text{hcf}(a_0, a_1) \) and set \( b = (b_0, b_1) := c^{-1}(a_1, -a_0) \in \mathbb{Z}^2 \). Then we know that the set
\[
\{ h \in \mathbb{Z}^2 : a_d \cdot h = 0 \}
\]
is in bijective correspondence with \( \mathbb{Z} \) via the map \( h \mapsto hb \). Hence on setting \( g_0^{(h)} = \Delta(g_0; hb_0) \), \( g_1^{(h)} = \Delta(g_1; hb_1) \) and \( g_i^{(h)} = \Delta(g_i; ha_i \cdot b) \) for \( i \geq 2 \), we have
\[
\sum_{a_d \cdot h = 0} \sum_z G(z)G(z + h) = \sum_{h} \sum_z g_0^{(h)}(z_0)g_1^{(h)}(z_1)g_2^{(h)}(a_2 \cdot z) \cdots g_d^{(h)}(a_d \cdot z).
\]

Applying the induction hypothesis, we conclude that
\[
\sum_{a_d \cdot h = 0} \sum_z G(z)G(z + h) \ll_d N^{2 - d2^1 - d} \sum_{h} \|\Delta(g_0; hb_0)\|_{U^{d-1}}.
\]

Since \((a_0, a_1)\) is not colinear to \((1, 0)\), we have \( b_0 = c^{-1}a_1 \neq 0 \), so we are legitimate in the assertion that
\[
\sum_{h} \|\Delta(g_0; hb_0)\|_{U^{d-1}} \leq \sum_{h} \|\Delta(g_0; h)\|_{U^{d-1}}.
\]

Given that \( g_0 \) is supported on \([-N, N]\), the set of integers \( h \) for which \( \Delta(g_0, h) \) is not identically zero is contained in the interval \([-2N, 2N]\). Hence by Hölder’s inequality
\[
\left( \sum_{h} \|\Delta(g_0; h)\|_{U^{d-1}} \right)^{2^d-1} \leq (4N + 1)^{2^{d-1} - 1} \sum_{h} \|\Delta(g_0; h)\|_{U^{d-1}}^{2^d-1}
\]
\[
= (4N + 1)^{2^{d-1} - 1} \|g_0\|_{U^{d}}^{2^d}.
\]

Thus
\[
\left| \sum_{z_0, z_1} g_0(z_0)g_1(z_1)g_2(a_2 \cdot z) \cdots g_d(a_d \cdot z) \right| \ll_d N^{\frac{1}{2} + 1 - d2^{-d} + \frac{1}{2} - 2^{-d}} \|g_0\|_{U^{d}}.
\]

\[\square\]

**Proof of Theorem 4.1**

We may assume that \( f \in \{ f_0, \ldots, f_n \} \) is equal to \( f_0 \). If not, we shift the \( x \) variable in (21) by \(-a_iy - b_i\) and re-label indices, at worst increasing \( H \) by a factor of two.
Given \( g : \mathbb{Z} \to [-1, 1] \) one can check that
\[
\sum_{y \in [M]} g(y) = \frac{1}{\sqrt{M}} \sum_{z_0, z_1 \in [\sqrt{M}]} \sum_{y \in [M] - [z_0, z_1]} g(y - z_0 + z_1)
= \frac{1}{M} \sum_{z_0, z_1 \in [\sqrt{M}]} \sum_{y \in [M]} g(y - z_0 + z_1) + O(\sqrt{M}).
\]
Applying this to (21) we deduce that
\[
\delta NM \leq \left| \frac{1}{M} \sum_{x \in \mathbb{Z}} \sum_{z_0, z_1 \in [\sqrt{M}]} \sum_{y \in [M]} f_0(x) f_1(x + a_1(y - z_0 + z_1) + b_1) \cdot \cdots \cdot f_d(x + a_d(y - z_0 + z_1) + b_d) \right|
+ O(N \sqrt{M}).
\]
Our assumption that \( M \geq C\delta^{-2} \) ensures that subtracting the \( O(N \sqrt{M}) \) term above from \( \delta NM \) gives a quantity bounded below by \( \frac{1}{2} \delta NM \). Shifting the \( x \) variable by \( a_1 z_0 \), maximising over \( y \) and setting \( a_i := (a_1 - a_i, a_i) \), we see that there exist integers \( c_i \) such that
\[
\delta NM \leq \sum_{x \in \mathbb{Z}} \left| \sum_{z_0, z_1 \in [\sqrt{M}]} \sum_{y \in [M]} f_0(x + a_1 z_0) f_1(x + a_1 z_1 + c_1) f_2(x + a_2 \cdot z + c_2) \cdot \cdots \cdot f_d(x + a_d \cdot z + c_d) \right|
\]
For each value of \( x \in \mathbb{Z} \) let us define functions \( g_{x,i} : \mathbb{Z} \to [-1, 1] \) by
\[
g_{x,0}(z) := f_0(x + a_1 z) 1_{[\sqrt{M}]}(z), \quad g_{x,1}(z) := f_1(x + a_1 z + c_1) 1_{[\sqrt{M}]}(z)
\]
and for \( i \geq 2 \) set
\[
g_{x,i}(z) := f_i(x + z + c_i) 1_{[-2H \sqrt{M}, 2H \sqrt{M}]}(z).
\]
Notice that if \( z_0, z_1 \in [\sqrt{M}] \) then
\[
a_i \cdot z = (a_1 - a_i) z_0 + a_i z_1 \in [-2H \sqrt{M}, 2H \sqrt{M}].
\]
Hence by Lemma 4.2
\[
\sum_{z_0, z_1 \in [\sqrt{M}]} f_0(x + a_1 z_0 + c_0) f_1(x + a_1 z_1 + c_1) f_2(x + a_2 \cdot z + c_2) \cdot \cdots \cdot f_d(x + a_d \cdot z + c_d)
= \sum_{z_0, z_1 \in [\sqrt{M}]} g_{x,0}(z_0) g_{x,1}(z_1) g_{x,2}(a_2 \cdot z) \cdot \cdots \cdot g_{x,d}(a_d \cdot z)
\ll_d \left( H \sqrt{M} \right)^{2 - \frac{d+1}{2d}} \| g_{x,0} \|_{U^d}.
\]
Therefore
\[
\delta NM \ll_d \left( H \sqrt{M} \right)^{2 - \frac{d+1}{2d}} \sum_x \| f_{x,a_1} \|_{U^d}.
\]
\[
\square
\]
5. A modified local inverse theorem for the $U^d$-norm

Unfortunately we cannot use Gowers’s local inverse theorem as it is presently found in the literature. Our difficulty is that the theorem as stated only gives us information about the correlation of a function on long arithmetic progressions. Our present approach requires information about correlation on progressions of a special form.

**Definition** ($k$th power progression). Let us call an arithmetic progression a $k$th power progression if it has common difference of the form $y^k$ for some positive integer $y$.

**Theorem 5.1** ($k$th power local inverse theorem, see [Pre]). For $d, k \geq 2$ there exist $C = C(d, k) > 0$ such that the following is true. Suppose that $N \geq \exp(C \delta^{-C})$.

and that $f : \mathbb{Z} \to [-1, 1]$ is a function supported on $[N]$ with $\|f\|_{U^d} \geq \delta N^{\frac{d+1}{2d}}$.

Then one can partition $[N]$ into $k$th power progressions $P_i$, each of length at least $N^{\frac{1}{C} \delta^C}$ such that

$$\sum_i \left| \sum_{x \in P_i} f(x) \right| \geq \frac{1}{C} \delta^C N.$$

**Remarks.**

(i) The case $k = 1$ of Theorem 5.1 is due to Gowers [Gow01b].
(ii) A proof of Theorem 5.1 for $k = 2$ and $d = 2, 3$ can be found in Green [Gre02].
(iii) For $d = 2$ and any $k$, the theorem is a nice exercise in Diophantine approximation and Fourier analysis.
(iv) The theorem should follow in a standard way from an adaptation of the methods of Gowers [Gow01b]. Unfortunately, the only way we can think of obtaining such an adaptation is by re-running a large section of the argument of [Gow01b], together with some extra applications of basic facts on small fractional parts of polynomials.
(v) For inhomogeneous polynomial configurations such as $x, x + y, x + y^2$, one requires some control on the size of the $k$th power in the given progression. This requires a further strengthening of the theorem.
(vi) A good estimate for the value of $C(d, k)$ is important if one wishes to determine the nature of the log log $N$ exponent in Theorem 1.1.

We relegate the proof of Theorem 5.1 to a separate note [Pre], where we give a modified exposition of Gowers’s argument [Gow01b].

6. The density increment and final iteration

**Lemma 6.1** (Density increment lemma). There exist absolute constants $B(c, k)$ and $C = C(n, k)$ such that the following is true. Suppose that $N \geq \exp(B \delta^{-C})$.

and that $A$ is a subset of $[N]$ of size at least $\delta N$ lacking a configuration of the form

$$x, x + c_1 y^k, \ldots, x + c_n y^k \quad \text{with} \quad y \in \mathbb{Z} \setminus \{0\}.$$  

Then there exists a $k$th power progression $P$ of length at least $N^{\delta^C/B}$ such that $|A \cap P| \geq (\delta + \frac{1}{B} \delta^C)|P|$.
Proof. In order to be able to apply previous results, let us take $B(c,k)$ and $C(n,k)$ appearing in (22) at least as large as any of the corresponding absolute constants occurring in Lemma 2.1, Corollary 3.8 and Lemma 4.1.

As $A$ lacks configurations of the form (23), we employ Lemma 2.1 and conclude that there exist $f_0, f_1, \ldots, f_n \in \{\delta I_{[N]}, f_A\}$ with $f_n = f_A$ and distinct non-zero integers $\tilde{c}_1, \ldots, \tilde{c}_n$ of order $O_{c}(1)$ such that

$$\left| \sum_{x,y} f_0(x)f_1(x+\tilde{c}_1y^k) \cdots f_n(x+\tilde{c}_ny^k) \right| \gg_{c,k} \delta^{n+1}N^{1+\frac{1}{k}}.$$  

The conditions of Corollary 3.8 having been met, we conclude that there exists $d \leq d(n,k)$, functions $g_0, g_1, \ldots, g_d : \mathbb{Z} \to [-1,1]$ supported on $[N]$ with $g_d = f_A$, and integers $a_i, b_i, M$ with $M \gg_{c,k} \delta^{-C}N^{1/k}$, the $a_i$ distinct, non-zero, of order $O_{c,k}(\delta^{-C})$ and such that

$$\left| \sum_{x,y} g_0(x)g_1(x+a_1y+b_1) \cdots g_d(x+a_dy+b_d) \right| \gg_{c,k} \delta^{C}NM.$$  

Writing $f$ for the balanced function $f_A$, we apply Lemma 4.1 with $H \ll_{c,k} \delta^{-C}$ and conclude that

$$\sum_x \|f_{x,a,\sqrt{M}}\|_{U^d} \gg_{c,d} \delta^{3C}N \left(\sqrt{M}\right)^{(d+1)2^{-d}},$$

where $a$ is a non-zero integer of order $O_{c,k}(\delta^{-C})$.

Notice that if the function $f_{x,a,\sqrt{M}}$ is not identically zero, then there exists $y \in [\sqrt{M}]$ such that $x+ay \in [N]$. Thus $x \in [N] - a \cdot [\sqrt{M}]$. Increasing $B$ and $C$ in (22) if necessary, we can ensure that $aM \leq N$. Hence the set of $x$ for which $f_{x,a,\sqrt{M}}$ is not identically zero is contained in $[-N,N]$.

Increasing $C$ by a factor of 3, it follows that there exists a set $X \subset [-N,N]$ of size $|X| \gg \delta^{C}N$ such that for every $x \in X$ we have

$$\|f_{x,a,\sqrt{M}}\|_{U^d} \gg_{c,k} \delta^{C} \left(\sqrt{M}\right)^{(d+1)2^{-d}}.$$  

Let $C_1 = C_1(d,k)$ denote the absolute constant appearing in Theorem 5.1. In order to apply that result, we require that $\sqrt{M} \geq C_1 \exp(B\delta^{-CC_1})$. Since $M \gg_{c,k} \delta^{-C}N^{1/k}$, we can ensure this using (22), increasing the absolute constants appearing there if necessary. Applying Theorem 5.1 and increasing $C$ further, we see that for each $x \in X$, there exists a partition of $[\sqrt{M}]$ into $k$th power arithmetic progressions $P_{x,i}$ $(i \in I(x))$, each of length at least $N^{\delta C/B}$ and such that

$$\sum_{i \in I(x)} \left| \sum_{y \in P_{x,i}} f_{x,a,\sqrt{M}}(y) \right| \gg_{c,k} \delta^{C} \sqrt{M}.$$  

If $x \notin X$ let us take $I(x) := \{1\}$ and $P_{x,1} := [\sqrt{M}]$. Increasing $C$ by a factor of two, we deduce that

$$\sum_{|x| \leq N} \sum_{i \in I(x)} \left| \sum_{y \in P_{x,i}} f(x+ay) \right| \gg_{c,k} \delta^{C} N \sqrt{M}$$

$$\gg \delta^{C} \sum_{|x| \leq N} \sum_{i \in I(x)} |P_{x,i}|.$$
Since \( f = 1_A - \delta 1_{[N]} \) has mean zero,
\[
\sum_{|x| \leq N} \sum_{i \in I(x)} \sum_{y \in P_{x,i}} f(x + ay) = \sum_x \sum_{y \in [\sqrt{M}]} f(x + ay)
\]
\[
= \sum_{y \in [\sqrt{M}]} \sum_x f(x + ay)
\]
\[
= 0.
\]
Adding the above to (24) we find that
\[
\sum_{|x| \leq N} \max \left\{ \sum_{y \in P_{P_{x,i}}} f(x + ay), 0 \right\} \gg_{c,k} \delta C \sum_{|x| \leq N} \sum_{i \in I(x)} |P_{x,i}|.
\]
Hence there exists a 1th power arithmetic progression \( P \) of length at least \( N^{\delta C / B} \) such that
\[
\sum_{y \in P} f(x + ay) \gg_{c,k} \delta C |P|.
\]
The modulus \( a \) appearing above is of order \( O_{c,k}(\delta^{-C}) \). Thanks to (22), this is small enough to ensure that on partitioning \( P \) into congruence classes mod \( |a^{k-1}| \) each sub-progression has length at least \( N^{\delta C / B} \) (increasing \( B \) and \( C \) if necessary). We therefore conclude that there exists an arithmetic progression \( Q \) with 1th power common difference and length at least \( N^{\delta C / B} \) such that
\[
\sum_{y \in Q} f(y) \gg_{c,k} \delta C |Q|.
\]
This completes the proof of the lemma. \( \square \)

**Proof of Theorem 1.1.** Suppose that \( A \subset [N] \) with \( |A| = \delta N \) lacks a configuration of the form
\[
x, x + c_1 y^k, \ldots, x + c_n y^k \quad \text{with} \quad y \in \mathbb{Z} \setminus \{0\}.
\]
Then by Lemma 6.1, provided that
\[
N \geq \exp(B\delta^{-C}),
\]
there exists a 1th power progression \( P = r + a^k \cdot [N] \) of length at least \( N^{\delta C / B} \) such that
\[
|A \cap P| \geq (\delta + \frac{1}{B} \delta C)|P|.
\]
Let \( A_1 := \{ x \in \mathbb{Z} : r + a^k x \in A \cap P \} \). Then we have obtained a set \( A_1 \subset [N] \) lacking configurations of the form (25) and of density \( \delta_1 := |A_1|/N_1 \) satisfying \( \delta_1 \geq \delta + \frac{1}{B} \delta C \).

Setting
\[
\delta_0 := \delta, \quad N_0 := N, \quad A_0 := A,
\]
and iteratively applying Lemma 6.1, we see that provided
\[
N_i \geq \exp(B\delta_i^{-C}) \quad (0 \leq i < n),
\]
there exists a set \( A_n \subset [N] \) lacking configurations of the form (25) and of density \( \delta_n := |A_n|/N_n \) satisfying
\[
\delta_n \geq \delta_{n-1} + \frac{1}{B} \delta_{n-1}
\]
and
\[
N_n \geq N_{n-1}^{\delta_{n-1}/B}.
\]
(27)
If (26) holds for \( n \geq B\delta^{1-C} \) then \( \delta_n \geq \delta + \frac{n}{-B}\delta^C \geq 2\delta \). Hence if (26) holds for \( n \geq B\delta^{1-C} + B(2\delta)^{1-C} + \cdots + B(2^m\delta) \),
then
\[
\delta_n \geq 2^m\delta.
\]
Taking \( m \geq 2\log(1/\delta) \) leads to the contradiction that \( \delta_n > 1 \). Thus we must have
\[
N_i < \exp(B\delta_i^{-C}) \tag{28}
\]
for some \( i \) satisfying
\[
i \leq B\delta^{1-C}(1 + 2^{1-C} + 2^{2(1-C)} + \ldots) \leq 2B\delta^{1-C} \tag{29}
\]
Combining (29) with (27), we have
\[
N_i \geq N_{i-1}^{(d^C/B)} \geq N_{i-2}^{(d^C/B)^2} \geq \cdots \geq N_i^{(d^C/B)^i} \geq N_i^{(d^C/B)^{2B\delta^{1-C}}}
\]
Incorporating this and the monotonicity estimate \( \delta_i \geq \delta \) into (28) we deduce that
\[
N_i^{(d^C/B)^{2B\delta^{1-C}}} < \exp(B\delta^{-C}).
\]
Taking logarithms gives
\[
\log N < B^{1+2B\delta^{1-C} \delta^{-C-2BC\delta^{1-C}}},
\]
Taking a second logarithm, we obtain
\[
\log \log N < (1 + 2B\delta^{1-C}) \log B + (C + 2BC\delta^{1-C}) \log(\delta^{-1}) \leq c_{\text{a,b}} \delta^{-C}.
\]
This completes the proof of Theorem 1.1. \[\square\]

References


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