# A Proof of Newton's Identities

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The object of this note is to give a proof of Newton's identities, also known as the Newton – Girard formulae. I was unable to find a proof online, so have made this one available. Although my argument is elementary, I suspect it could be much more so. If anyone reading this has a simple proof, or can simplify my argument, do email me at *sean.prendiville@bristol.ac.uk*. I make no claim to originality and suspect the following argument was known to Newton, so is at least 300 years.

Fix some positive integer $N \in \mathbb{N}$. For each integer $n \geq 0$ we define the $n$th power sum in $N$ variables to be the polynomial

$$S_n = S_n(y_1, \ldots, y_N) := \sum_{i=1}^{N} {y_i}^n$$

For $0 \leq n \leq N$ we define the $n$th elementary symmetric polynomial in $N$ variables to be

$$e_n = e_n(y_1, \ldots y_N) := \sum_{S \in [N]^{(n)}} \prod_{i \in S} y_i$$

where by convention $e_0 := 1$. We have indexed the last sum using the following notation:

- For a real number $X$, we denote by $[X]$ the set $\{n \in \mathbb{N} : n \leq X\}$.
- For a set $A$ and non-negative integer $n$ we define

$$A^{(n)} := \{B \subset A : |B| = n\}$$

We're now in a position to state the result we want to prove.

**Newton's Identities:** For each $0 \leq n \leq N$

$$\sum_{r=0}^{n-1} (-1)^r e_r S_{n-r} + n(-1)^n e_n = 0 \tag{1}$$

Since $S_0 = N$ we see that (1) is equivalent to

$$(N-n)(-1)^n e_n = \sum_{r=0}^{n} (-1)^r e_r S_{n-r} \tag{2}$$

$$= \sum_{u+v=n} (-1)^u e_u S_v \tag{3}$$

The appearance of the 'partition sum' (i.e. a sum over partitions of $[n]$ into two sets) in (3) immediately suggests the use of power series.

To maintain rigour in the following fix $y_1, \ldots, y_1 \in \mathbb{C}$ and let

$$R := \min\left\{\frac{1}{|y_1|}, \ldots, \frac{1}{|y_N|}, 1\right\} > 0$$

For $z \in B_R(0)$ define

$$\begin{aligned} F(z) &:= \prod_{i=1}^{N} (1 - y_i z) \\ &= \sum_{r=0}^{N} (-1)^r \left( \sum_{S \in [N]^{(n)}} \prod_{i \in S} y_i \right) z^r \\ &= \sum_{r=0}^{N} (-1)^r e_r z^r \end{aligned}$$

and

$$\begin{aligned} G(z) := \sum_{r=0}^{\infty} S_r z^r &= \sum_{r=0}^{\infty} \left( \sum_{i=1}^{N} {y_i}^r \right) z^r \\ &= \sum_{i=1}^{N} \left( \sum_{n=0}^{\infty} (y_i z)^r \right) \\ &= \sum_{i=1}^{N} \frac{1}{1 - y_i z} \end{aligned}$$

We see from the above that both $F$ and $G$ are analytic on $B_R(0)$. Hence

$$H(z) := F(z)G(z) = \sum_{n=0}^{\infty} C_n z^n$$

is also analytic on $B_R(0)$. Notice that for $0 \leq n \leq N$

$$C_n = \sum_{u+v=n} (-1)^u e_u S_v = \sum_{r=0}^{n} (-1)^r e_r S_{n-r}$$

By the analyticity of $H$ in a neighbourhood of 0 we have that $H^{(n)}(0) = n!\, C_n$. So in order to establish Newton's identities it suffices to show that for $0 \leq n \leq N$:

$$H^{(n)}(0) = n!(-1)^n(N-n)e_n \tag{4}$$

This should in fact be obtainable, since we have the following simple formula for $H$, which allows us to employ identities from elementary calculus:

$$\begin{aligned} H(z) &= \prod_{i=1}^{N}(1-y_i z)\sum_{i=1}^{N}(1-y_i z)^{-1} \\ &= \sum_{i=1}^{N}\prod_{j\neq i}(1-y_i z) \end{aligned}$$

When differentiating $H$ we will adhere to the convention that a product over the empty set is equal to 1. Let $z \in B_R(0)$.

$$\begin{aligned} H'(z) &= \sum_{i=1}^{N}\sum_{i_1\neq i}(-y_{i_1})\prod_{i_2\notin\{i,i_1\}}(1-y_{i_2}z) \\ H''(z) &= \sum_{i=1}^{N}\sum_{i_1\neq i}(-y_{i_1})\sum_{i_2\notin\{i,i_1\}}(-y_{i_2})\prod_{i_3\notin\{i,i_1,i_2\}}(1-y_{i_3}z) \end{aligned}$$

Iterating we have that when $1 \leq n \leq N-1$:

$$H^{(n)}(z) = \sum_{i=1}^{N}\sum_{i_1\notin\{i\}}(-y_{i_1})\ \cdots \sum_{i_n\notin\{i,i_1,\ldots,i_{n-1}\}}(-y_{i_n})\prod_{i_{n+1}\notin\{i,i_1,\ldots,i_n\}}(1-y_{i_{n+1}}z)$$

Hence for $1 \leq n \leq N-1$

$$\begin{aligned} H^{(n)}(0) &= \sum_{i=1}^{N}\sum_{i_1\notin\{i\}}(-y_{i_1})\ \cdots \sum_{i_n\notin\{i,i_1,\ldots,i_{n-1}\}}(-y_{i_n}) \\ &= (-1)^n\sum_{i=1}^{N}\sum_{i_1\notin\{i\}}\cdots\sum_{i_n\notin\{i,i_1,\ldots,i_{n-1}\}} y_{i_1}\cdots y_{i_n} \\ &= (-1)^n\sum_{i=1}^{N}\sum_{(i_1,\ldots,i_n)\in T_i} y_{i_1}\cdots y_{i_n} \end{aligned}$$

where $T_i = \{(i_1,\ldots,i_n) \in ([N]\setminus\{i\})^n : i_1,\ldots,i_n \text{ are distinct}\}$. Notice that the function from $T_i$ to $([N]\setminus\{i\})^{(n)}$ which maps $(i_1,\ldots,i_n)$ to $\{i_1,\ldots i_n\}$ is an $n!$

to 1 map. Thus

$$
\begin{aligned}
H^{(n)}(0) &= (-1)^n n! \sum_{i=1}^{N} \sum_{S\in([N]\setminus\{i\})^{(n)}} \prod_{j\in S} y_j \\
&= (-1)^n n! \sum_{i=1}^{N} \left( \sum_{S\in[N]^{(n)}} \prod_{j\in S} y_j - \sum_{\substack{S\in[N]^{(n)} \\ i\in S}} \prod_{j\in S} y_j \right) \\
&= (-1)^n n! \left( N \sum_{S\in[N]^{(n)}} \prod_{j\in S} y_j - \sum_{i=1}^{N} \sum_{\substack{S\in[N]^{(n)} \\ i\in S}} \prod_{j\in S} y_j \right) \\
&= (-1)^n n! \left( N \sum_{S\in[N]^{(n)}} \prod_{j\in S} y_j - \sum_{S\in[N]^{(n)}} \sum_{\substack{i=1 \\ i\in S}}^{N} \prod_{j\in S} y_j \right) \\
&= (-1)^n n! \left( N \sum_{S\in[N]^{(n)}} \prod_{j\in S} y_j - n \sum_{S\in[N]^{(n)}} \prod_{j\in S} y_j \right) \\
&= n!(-1)^n (N-n) e_n
\end{aligned}
$$

It remains to establish the above identity when $n = 0$ and $n = N$. The case $n = 0$ is trivial, since $H(0) = N = 0!(-1)^0(N-0)e_0$. For the case $n = N$, we first note that the $(n-1)$th derivate of $H$ is constant on $B_R(0)$. Thus $H^{(N)}(0) = 0$, from which our required identity follows.

**A Corollary:** Rearranging (1), we see that for $1 \le n \le N$

$$e_n = -\frac{1}{n} \left( \sum_{r=0}^{n-1} e_r (-1)^{n-r} S_{n-r} \right)$$

Since $e_1 = S_1$, a simple induction shows that for each $e_n$ there exists a polynomial $P_n \in \mathbb{Q}[x_1, \ldots, x_n]$ with

$$e_n = P_n(S_1, \ldots, S_n)$$

Hence if

$$\sum_{i=1}^{N} {x_i}^j - {y_i}^j = 0$$

for $j = 1, \ldots, N$, then for all $x$

$$\begin{aligned}
\prod_{i=1}^{N}(x - x_i) &= \sum_{r=0}^{N}(-1)^r e_r(x_1, \ldots, x_N)x^{N-r} \\
&= \sum_{r=0}^{N}(-1)^r P_r\big(S_1(x_1, \ldots, x_N), \ldots, S_r(x_1, \ldots, x_N)\big)x^{N-r} \\
&= \sum_{r=0}^{N}(-1)^r P_r\big(S_1(y_1, \ldots, y_N), \ldots, S_r(y_1, \ldots, y_N)\big)x^{N-r} \\
&= \prod_{i=1}^{N}(x - y_i)
\end{aligned}$$

It follows that there exists a permutation $\sigma$ of $\{1, \ldots, N\}$ for which $x_i = y_{\sigma(i)}$ $(i = 1, \ldots, N)$.