A Proof of Newton’s Identities

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The object of this note is to give a proof of Newton’s identities, also known as the Newton – Girard formulae. I was unable to find a proof online, so have made this one available. Although my argument is elementary, I suspect it could be much more so. If anyone reading this has a simple proof, or can simplify my argument, do email me at sean.prendiville@bristol.ac.uk. I make no claim to originality and suspect the following argument was known to Newton, so is at least 300 years.

Fix some positive integer \( N \in \mathbb{N} \). For each integer \( n \geq 0 \) we define the \( n \)th power sum in \( N \) variables to be the polynomial

\[
S_n = S_n(y_1, \ldots, y_N) := \sum_{i=1}^{N} y_i^n
\]

For \( 0 \leq n \leq N \) we define the \( n \)th elementary symmetric polynomial in \( N \) variables to be

\[
e_n = e_n(y_1, \ldots, y_N) := \sum_{S \in \mathbb{N}^{[n]}} \prod_{i \in S} y_i
\]

where by convention \( e_0 := 1 \). We have indexed the last sum using the following notation:

- For a real number \( X \), we denote by \([X]\) the set \( \{n \in \mathbb{N} : n \leq X\} \).
- For a set \( A \) and non-negative integer \( n \) we define

\[
A^{(n)} := \{B \subset A : |B| = n\}
\]

We’re now in a position to state the result we want to prove.

**Newton’s Identities:** For each \( 0 \leq n \leq N \)

\[
\sum_{r=0}^{n-1} (-1)^r e_r S_{n-r} + n(-1)^n e_n = 0 \tag{1}
\]
Since $S_0 = N$ we see that (1) is equivalent to

$$
(N - n)(-1)^n e_n = \sum_{r=0}^{n} (-1)^r e_r S_{n-r} \\
= \sum_{u+v=n} (-1)^u e_u S_v
$$

The appearance of the ‘partition sum’ (i.e. a sum over partitions of $[n]$ into two sets) in (3) immediately suggests the use of power series.

To maintain rigour in the following fix $y_1, \ldots, y_N \in \mathbb{C}$ and let

$$
R := \min \left\{ \frac{1}{|y_1|}, \ldots, \frac{1}{|y_N|}, 1 \right\} > 0
$$

For $z \in B_R(0)$ define

$$
F(z) := \prod_{i=1}^{N} (1 - y_i z) \\
= \sum_{r=0}^{N} (-1)^r \left( \sum_{S \in [N]^{(n)}} \prod_{i \in S} y_i \right) z^r \\
= \sum_{r=0}^{N} (-1)^r e_r z^r
$$

and

$$
G(z) := \sum_{r=0}^{\infty} S_r z^r = \sum_{r=0}^{\infty} \left( \sum_{i=1}^{N} y_i^r \right) z^r \\
= \sum_{i=1}^{N} \left( \sum_{r=0}^{\infty} (y_i z)^r \right) \\
= \sum_{i=1}^{N} \frac{1}{1 - y_i z}
$$

We see from the above that both $F$ and $G$ are analytic on $B_R(0)$. Hence

$$
H(z) := F(z)G(z) = \sum_{n=0}^{\infty} C_n z^n
$$

is also analytic on $B_R(0)$. Notice that for $0 \leq n \leq N$

$$
C_n = \sum_{u+v=n} (-1)^u e_u S_v = \sum_{r=0}^{n} (-1)^r e_r S_{n-r}
$$
By the analyticity of $H$ in a neighbourhood of 0 we have that $H^{(n)}(0) = n! C_n$. So in order to establish Newton’s identities it suffices to show that for $0 \leq n \leq N$:

$$H^{(n)}(0) = n!(-1)^n(N-n)c_n \quad (4)$$

This should in fact be obtainable, since we have the following simple formula for $H$, which allows us to employ identities from elementary calculus:

$$H(z) = \prod_{i=1}^N (1 - y_i z)^{-1} \sum_{i=1}^N (1 - y_i z)^{-1}$$

$$= \sum_{i=1}^N \prod_{j \neq i} (1 - y_i z)$$

When differentiating $H$ we will adhere to the convention that a product over the empty set is equal to 1. Let $z \in B_R(0)$.

$$H'(z) = \sum_{i=1}^N \sum_{i \neq i} (-y_i) \prod_{i \not\in \{i, i\}} (1 - y_i z)$$

$$H''(z) = \sum_{i=1}^N \sum_{i \neq i} (-y_i) \sum_{i \not\in \{i, i\}} (-y_i) \prod_{i \not\in \{i, i, i\}} (1 - y_i z)$$

Iterating we have that when $1 \leq n \leq N - 1$:

$$H^{(n)}(z) = \sum_{i=1}^N \sum_{i \neq i} (-y_i) \cdots \sum_{i \neq i} (-y_i) \prod_{i \not\in \{i, i, \ldots, i\}} (1 - y_{i+1} z)$$

Hence for $1 \leq n \leq N - 1$

$$H^{(n)}(0) = \sum_{i=1}^N \sum_{i \neq i} (-y_i) \cdots \sum_{i \neq i} (-y_i)$$

$$= (-1)^n \sum_{i=1}^N \sum_{i \neq i} (-y_i) \cdots \sum_{i \neq i} y_i \cdots y_i$$

$$= (-1)^n \sum_{i=1}^N \sum_{\{i, \ldots, i\} \in T_i} y_i \cdots y_i$$

where $T_i = \{(i_1, \ldots, i_n) \in ([N] \setminus \{i\})^n : i_1, \ldots, i_n \text{ are distinct}\}$. Notice that the function from $T_i$ to $([N] \setminus \{i\})^n$ which maps $(i_1, \ldots, i_n)$ to $(i_1, \ldots, i_n)$ is an $n!$
to 1 map. Thus

\[ H^{(n)}(0) = (-1)^n n! \sum_{i=1}^{N} \sum_{S \in [N \setminus \{i\}]^{(n)}} \prod_{j \in S} y_j \]

\[ = (-1)^n n! \sum_{i=1}^{N} \left( \sum_{S \in [N]^{(n)}} \prod_{j \in S} y_j - \sum_{i \in S} \prod_{j \in S} y_j \right) \]

\[ = (-1)^n n! \left( N \sum_{S \in [N]^{(n)}} \prod_{j \in S} y_j - \sum_{i=1}^{N} \sum_{S \in [N]^{(n)}} \prod_{j \in S} y_j \right) \]

\[ = (-1)^n n! \left( N \sum_{S \in [N]^{(n)}} \prod_{j \in S} y_j - \sum_{i=1}^{N} \sum_{j \in S} \prod_{j \in S} y_j \right) \]

\[ = (-1)^n n! \left( N \sum_{S \in [N]^{(n)}} \prod_{j \in S} y_j - n \sum_{S \in [N]^{(n)}} \prod_{j \in S} y_j \right) \]

\[ = n!(-1)^n(N - n)e_n \]

It remains to establish the above identity when \( n = 0 \) and \( n = N \). The case \( n = 0 \) is trivial, since \( H(0) = N = 0!(−1)^0(N - 0)e_0 \). For the case \( n = N \), we first note that the \( (n - 1) \)th derivative of \( H \) is constant on \( B_R(0) \). Thus \( H^{(N)}(0) = 0 \), from which our required identity follows.

**A Corollary:** Rearranging (1), we see that for \( 1 \leq n \leq N \)

\[ e_n = -\frac{1}{n} \left( \sum_{r=0}^{n-1} e_r (-1)^{n-r} S_{n-r} \right) \]

Since \( e_1 = S_1 \), a simple induction shows that for each \( e_n \) there exists a polynomial \( P_n \in \mathbb{Q}[x_1, \ldots, x_n] \) with

\[ e_n = P_n(S_1, \ldots, S_n) \]

Hence if

\[ \sum_{i=1}^{N} x_i^j - y_i^j = 0 \]
for \( j = 1, \ldots, N \), then for all \( x \)

\[
\prod_{i=1}^{N} (x - x_i) = \sum_{r=0}^{N} (-1)^r e_r(x_1, \ldots, x_N) x^{N-r}
\]

\[
= \sum_{r=0}^{N} (-1)^r P_r(S_1(x_1, \ldots, x_N), \ldots, S_r(x_1, \ldots, x_N)) x^{N-r}
\]

\[
= \sum_{r=0}^{N} (-1)^r P_r(S_1(y_1, \ldots, y_N), \ldots, S_r(y_1, \ldots, y_N)) x^{N-r}
\]

\[
= \prod_{i=1}^{N} (x - y_i)
\]

It follows that there exists a permutation \( \sigma \) of \( \{1, \ldots, N\} \) for which \( x_i = y_{\sigma(i)} \) \((i = 1, \ldots, N)\).