Sketch of the Selberg sieve method.

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This is a sketch of the Selberg sieve method as coverd in [MV07].

1 Motivation and consequences.

When $x, y \in \mathbb{R}$ and P is a positive integer we define

$$S(x, y; P) := \#\{n \in \mathbb{Z} : x < n \leq x + y \text{ and } (n, P) = 1\}.$$

If p is a prime in the interval $(x, x+y]$ then either $p|P$ or $(p, P) = 1$. Thus

$$\pi(x+y) - \pi(x) \leq \omega(P) + S(x, y; P), \tag{1.1}$$

where $\omega(P)$ is the number of distinct prime factors of P. An upper bound on $S(x, y; P)$ is therefore of patent interest. Using the Selberg sieve method we can obtain the following bound:

Theorem 1.1. *Let $x, y, z \in \mathbb{R}$ with $y \geq 0$ and $z \geq 1$ Then for any $P \in \mathbb{N}$*

$$S(x, y; P) \leq \frac{y}{L_P(z)} + O\left(\frac{z^2}{L_P(z)^2}\right) \tag{1.2}$$

where

$$L_P(z) = \sum_{\substack{n \leq z \\ n|P}} \frac{\mu(n)^2}{\phi(n)}.$$

To utilise (1.2) we must find a lower bound for $L_P(z)$. To this end take $P = \prod_{p \leq z} p$, which ensures

$$L_P(z) = \sum_{n \leq z} \frac{\mu(n)^2}{\phi(n)}.$$

Now for square-free n

$$\frac{1}{\phi(n)} = \frac{1}{n} \prod_{p|n} \frac{1}{1 - \frac{1}{p}} = \frac{1}{n} \prod_{p|n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{\substack{m \\ s(m)=n}} \frac{1}{m},$$

where $s(m)$ is the square-free kernel of m. Hence for this choice of P

$$L_P(z) = \sum_{n\le z} \mu(n)^2 \sum_{\substack{m \\ s(m)=n}} \frac{1}{m} = \sum_{\substack{m \\ s(m)\le z}} \frac{1}{m} \ge \sum_{m\le z} \frac{1}{m} \ge \log z.$$

Setting $z = y^{1/2}$ and $P_y := \prod_{p\le\sqrt{y}} p$ we obtain the following corollary of theorem 1.1:

Corollary 1.2. *For real x, y with $y > 1$*

$$S(x, y; P_y) \le \frac{2y}{\log y}\left(1 + O\left(\frac{1}{\log y}\right)\right) \tag{1.3}$$

where

$$P_y := \begin{cases} 1 & \text{if } y < 4 \\ \prod_{p\le\sqrt{y}} p & \text{otherwise.} \end{cases}$$

Notice

$$\omega(P_y) \le y^{1/2} \ll \frac{y}{(\log y)^2}.$$

So combining (1.1) and (1.3) we have:

Corollary 1.3. *For $x \ge 0$ and $y > 1$*

$$\pi(x+y) - \pi(x) \le \frac{2y}{\log y}\left(1 + O\left(\frac{1}{\log y}\right)\right).$$

2 Proof of Theorem 1.1

Clearly we can assume P to be square-free. We will use the z in the statement of our theorem as a 'cut-off' parameter. To this end, let $(\Lambda_d)_{d\in\mathbb{N}}$ be a sequence of real numbers satisfying $\Lambda_1 = 1$ and $\Lambda_d = 0$ for $d > z$. Then trivially

$$\left(\sum_{d|n} \Lambda_d\right)^2 \ge \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We use this and standard tricks for manipulating divisors sums to derive the following estimate:

$$\begin{aligned} S(x,y;P) &= \sum_{x<n\le x+y} \sum_{\substack{d|n \\ d|P}} \sum_{\substack{e|n \\ e|P}} \Lambda_d \Lambda_e \\ &= \sum_{\substack{d|P \\ e|P}} \Lambda_d \Lambda_e \sum_{\substack{x<n\le x+y \\ [d,e]|n}} 1 \\ &\le \sum_{\substack{d|P \\ e|P}} \Lambda_d \Lambda_e \left(\left[\frac{x+y}{[d,e]} \right] - \left[\frac{x}{[d,e]} \right] \right) \\ &= y \sum_{\substack{d|P \\ e|P}} \frac{\Lambda_d \Lambda_e}{[d,e]} + O\left(\left(\sum_{d|p} |\Lambda_d| \right)^2 \right). \end{aligned}$$

Next we proceed to diagonalise the quadratic form $Q\big((\Lambda_d)_d\big) := \sum_{\substack{d|P \\ e|P}} \frac{\Lambda_d \Lambda_e}{[d,e]}$. It will be clear from the resulting diagonal form how to minimise $|Q - L_P(z)^{-1}|$ (in fact we will be able to attain zero). Because our change of variables turns out to be invertible, we will then be able to extract the optimal choice for the Λ_d.

Since $\phi * \mathbf{1}_{\mathbb{N}} = id_{\mathbb{N}}$, we have

$$\frac{1}{[d,e]} = \frac{(e,d)}{de} = \frac{1}{de} \sum_{\substack{f|d \\ f|e}} \phi(f).$$

Inputting this into our formula for Q and changing the order of summation, we have

$$Q = \sum_{\substack{d|P \\ e|P}} \frac{1}{de} \sum_{\substack{f|d \\ f|e}} \Lambda_d \Lambda_e \phi(f) = \sum_{f|P} \phi(f) \sum_{\substack{f|d|P \\ f|e|P}} \frac{\Lambda_d}{d} \frac{\Lambda_e}{e} = \sum_{f|P} \phi(f) y_f^2,$$

where $y_f = \sum_{f|d|P} \frac{\Lambda_d}{d}$. Using the same change in order of summation found in the proof of Möbius inversion, it can easily be shown that

$$y_f = \sum_{f|d|P} \frac{\Lambda_d}{d} \text{ for all } f \quad \Longleftrightarrow \quad \Lambda_d = d \sum_{d|f|P} y_f \mu(f/d) \text{ for all } d.$$

Consequently

$$\Lambda_1 = 1 \Longleftrightarrow \sum_{f|P} y_f \mu(f) = 1 \qquad \text{and} \qquad \Lambda_d = 0 \text{ for } d > z \Longleftrightarrow y_f = 0 \text{ for } f > 0.$$

Completing the square, it can be checked that

$$\sum_{f|P} \phi(f) y_f^2 - \frac{1}{L_P(z)} = \sum_{\substack{f \leq z \\ f|P}} \phi(f) \left(y_f - \frac{\mu(z)}{\phi(f) L_P(z)} \right)^2 .$$

The right-hand side above can be made to equal zero by setting

$$y_f = \frac{\mu(f)}{\phi(f) L_P(z)} \text{ when } f \leq z, \quad y_f = 0 \text{ when } f > z.$$

Fortuitously, this choice of y_f also satisfies $\sum_{f|P} y_f \mu(f) = 1$. It remains to bound the error term.

Since P is square-free, if $d|P$ and $e|P/d$ then $(d,e) = 1$. So

$$\begin{aligned}
\Lambda_d = d \sum_{d|f|P} y_f \mu(f/d) &= \frac{d}{L_P(z)} \sum_{\substack{d|f|P \\ f \leq z}} \frac{\mu(f)\mu(f/d)}{\phi(f)} \\
&= \frac{d}{L_P(z)} \sum_{\substack{e|P/d \\ e \leq z/d}} \frac{\mu(de)\mu(e)}{\phi(de)} = \frac{d\mu(d)}{L_P(z)\phi(d)} \sum_{\substack{e|P/d \\ e \leq z/d}} \frac{\mu(e)^2}{\phi(e)}.
\end{aligned}$$

Therefore

$$\sum_{d \leq z} |\Lambda_d| \leq \frac{1}{L_P(z)} \sum_{d \leq z} \frac{d}{\phi(d)} \sum_{e \leq z/d} \frac{1}{\phi(e)} = \frac{1}{L_P(z)} \sum_{e \leq z} \frac{1}{\phi(e)} \sum_{d \leq z/e} \frac{d}{\phi(d)}. \tag{2.1}$$

Both $id_{\mathbb{N}}/\phi$ and $\mu^2 * \frac{1}{\phi}$ are multiplicative and equal at prime powers, hence for all d

$$\frac{d}{\phi(d)} = \sum_{r|d} \frac{\mu(r)^2}{\phi(r)}.$$

It follows that

$$\sum_{d \leq z/e} \frac{d}{\phi(d)} = \sum_{d \leq z/e} \sum_{r|d} \frac{\mu(r)^2}{\phi(r)} = \sum_{r \leq z/e} \frac{\mu(r)^2}{\phi(r)} \left[\frac{z}{er} \right] \leq \frac{z}{e} \sum_{r=1}^{\infty} \frac{1}{r\phi(r)} \ll \frac{z}{e}. \tag{2.2}$$

The last step in the above inequality assumes

$$\frac{1}{\phi(r)} \ll \frac{1}{r^{\varepsilon}} \tag{2.3}$$

for some $\varepsilon > 0$. To show this we first note that

$$r \geq \prod_{p|r} p \geq 2.2.4^{\omega(r)-2} = \left(\frac{2^{\omega(r)}}{2} \right)^2 .$$

Thus

$$\frac{1}{\phi(r)} = \frac{1}{r}\prod_{p|r}\frac{p}{p-1} \le \frac{2^{\omega(r)}}{r} \ll \frac{r^{1/2}}{r} = \frac{1}{r^{1/2}},$$

which establishes (2.3).

Inputting (2.2) into (2.1) we have

$$\sum_{d\le z}|\Lambda_d| \le \frac{1}{L_P(z)}\sum_{e\le z}\frac{z}{\phi(e)e} \ll \frac{z}{L_P(z)},$$

which is the required estimate for our error term.

References

[MV07] H. L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory, 1. Classical Theory.* Cambridge University Press, 2007.