1. Let \( a \) and \( b \) be integers which are not relatively prime. Show that there is a prime number \( p \) such that \( p \) divides both \( a \) and \( b \).

**Solution:** Because \( a \) and \( b \) are not relatively prime, there is a positive integer \( n \neq 1 \) which divides both \( a \) and \( b \). Take \( p \) to be a prime factor of \( n \).

2. Show that any two consecutive perfect squares are relatively prime, i.e. that if \( n \) is a positive integer then \( n^2 \) and \((n+1)^2 \) are relatively prime.

**Solution:** Let \( n \) be a positive integer and suppose that \( n^2 \) and \((n+1)^2 \) are not relatively prime. Then, by Question 1 there is a prime number \( p \) which divides both \( n^2 \) and \((n+1)^2 \). Because \( p \) is a prime, it follows that \( p \) divides both \( n \) and \( n+1 \), so that \( p \) divides 1; this is a contradiction, as required.

3. (Cataldi 1548–1626) Let \( n \) be a positive integer with \( n \neq 1 \). Suppose that \( n \) is composite (i.e. that \( n \) is not a prime number). Show that there is a prime number \( p \) such that \( p \) divides \( n \) and \( p \leq \sqrt{n} \).

**Solution:** There are positive integers \( a \) and \( b \) such that \( a \neq 1, b \neq 1 \) and \( ab = n \). We can not have both \( a > \sqrt{n} \) and \( b > \sqrt{n} \), because if we did we would have \( ab > n \). By interchanging \( a \) and \( b \) if necessary we can suppose that \( a \leq \sqrt{n} \). Take \( p \) to be a prime factor of \( a \).

4. Use Question 3 and a calculator to determine whether or not 1763 and 1777 are prime numbers.

**Solution:** We have 1763 = 41 \cdot 43, so that 1763 is not prime. But 1777 is prime because it is not divisible by any of the primes up to and including 41.

5. Find all prime numbers \( p \) such that \( 7p + 4 \) is a perfect square and justify your answer.
Solution: Suppose that $p$ is a prime number and that $7p + 4 = n^2$ for some positive integer $n$. We have $7p = n^2 - 4 = (n + 2)(n - 2)$, so that $n + 2$ is a positive factor of $7p$. Because $n$ is positive we can not have $n + 2 = 1$. Hence we need only consider the three following cases.

Case 1: Suppose that $n + 2 = 7$ and $n - 2 = p$. Then $n = 5$ and $p = n - 2 = 3$.

Case 2: Suppose that $n + 2 = p$ and $n - 2 = 7$. Then $n = 9$ and $p = n + 2 = 11$.

Case 3: Suppose that $n + 2 = 7p$ and $n - 2 = 1$. Then $n = 3$ and $7p = n + 2 = 5$, which is a contradiction.

Therefore the only prime numbers $p$ such that $7p + 4$ is a perfect square are $p = 3$ and $p = 11$.

6. Let $n$ be a positive integer such that $2^n - 1$ is a prime number. Show that $n$ is a prime number. (Hint: Suppose that $a$ and $b$ are positive integers with $ab = n$. Use the identity

$$x^b - 1 = (x - 1)(x^{b-1} + x^{b-2} + \cdots + x^2 + x + 1)$$

to show that $2^n - 1$ divides $2^n - 1$).

Solution: Set $p = 2^n - 1$. Because $p$ is a prime number we have $p \neq 1$, so that $n \neq 1$. Now suppose that $a$ and $b$ are positive integers with $ab = n$. Putting $x = 2^a$ in the identity given in the hint shows that $p = 2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1 = x^b - 1 = (x - 1)(x^{b-1} + \cdots + x + 1)$ where the expression in the last bracket is a positive integer and $x - 1 = 2^a - 1$. Therefore $2^a - 1$ divides $p$. But $p$ is a prime number. Therefore EITHER $2^a - 1 = 1$, i.e. $2^a = 2$, i.e. $a = 1$; OR $2^a - 1 = p = 2^n - 1$, i.e. $a = n$. Thus $n \neq 1$, and whenever $a$ and $b$ are positive integers with $ab = n$ then either $a = 1$ or $a = n$. Therefore $n$ is a prime number.

7. Use a calculator to show that $2^{11} - 1$ is not a prime number. Thus the converse of the result in Question 6 is not true in general, i.e. there is a prime number $p$ such that $2^p - 1$ is not a prime number.

Solution: $2^{11} - 1 = 2047 = 23 \cdot 89$. 

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